Analytical solutions for inherently incremental, similar elastic contact problems with bulk stress

N. Sundaram *, T.N. Farris

School of Aeronautics and Astronautics, Purdue University, 701 W. Stadium Ave., West Lafayette, IN 47907-2045, USA

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ABSTRACT

Analytical solutions are obtained for a class of inherently incremental, similar elastic half-plane frictional contact problems which involve simultaneous application of a changing, remote bulk-stress during oblique loading. The method is based on Abel Integral Equations. The spatial gradient of the slip function is obtained uniquely by the conditions determining the existence of a bounded solution to the shear traction Cauchy SIE and global shear equilibrium, provided there is stick at all points in the contact throughout the loading. Solutions are obtained for both the cylindrical indenter and the flat punch with rounded edges. The method also lends itself easily to generalization to non-linear load paths and a number of such problems are considered. Problems involving the development of slip zones within the contact may be solved in some special cases.

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1. Introduction

Two-dimensional contact mechanics is an important area of solid mechanics, and an extensive body of analytical and numerical work deals with problems in this area. For a review see the paper by Barber and Ciavarella (2000). Assuming that the two contacting bodies can be treated as elastic half-planes allows the formulation of the contact problem as a pair of Cauchy Singular Integral Equations (SIE) for the pressure and shear tractions. The equations are uncoupled if the two bodies are similar. Due to the presence of friction, it is well known that the most general class of shear traction problems is inherently incremental (Barber, 2002). In other words, the slip function between the two bodies is unknown a priori and has to be updated at the end of each increment, and subsequently used to find the shear traction in the next increment. This usually makes it necessary to use a numerical approach for the most general load paths. Known analytical solutions for inherently incremental problems include those for the so-called oblique load problem, which involves simultaneous application of a normal load P and a proportional shear load Q, as well as problems with oscillating normal and shear loads. Solutions for such problems were first obtained by Mindlin and Deresiewicz (1953) in a very influential work. The oblique load solution for similar cylinders was obtained by Hills and Nowell (1994). In a series of papers, Dundurs and Comninou (1981, 1982, 1983) examined the effects of loading history on a partially closed crack in an unbounded solid loaded by a point force using a dislocation density approach. For this simplified problem, they considered loading, unloading and re-loading for the cases of so-called strong friction (in which no slip zone develops) and weak friction (in which slip zones develop). Another approach to inherently incremental problems was developed by Spence using a self-similarity argument for normal indentation of a half-space by a polynomial axisymmetric indenter with full adhesion (Spence, 1968) and partial slip using a somewhat involved eigenvalue approach (Spence, 1973).

In this work analytical solutions are obtained for a class of inherently incremental problems, involving either remotely applied varying bulk-stresses accompanying oblique loading, non-linear load paths or a combination thereof. Essentially, the existence of bounded solutions to the shear traction Cauchy SIE and global shear equilibrium are sufficient conditions to determine the spatial gradient of the slip function uniquely when there is stick at all contact points throughout the loading. Once the slip gradient is known, it can be used to obtain the shear traction.

2. Governing equations for similar contacts

The pressure traction Cauchy Singular Integral Equation for similar materials resulting in a single contact patch (b, a) is (please see Barber (2002) for a derivation)

\[ \frac{dh_b(x)}{dx} - C_1 = A \int_b^a \frac{p(s)}{x - s} \, ds \quad \forall \ x \in (b, a) \] (1)
where \( p(x) \) is the pressure traction, \( h_0(x) \) is the initial separation or initial gap function, \( C_1 \) is a small rigid-body rotation and \( A \) is a term encapsulating elastic behavior in plane strain, \( A = 4(1 - v^2)/\pi E \) for similar, isotropic material contact. For similar materials, the pressure equation is unaffected by the presence of friction. The pressure traction must satisfy the equilibrium equations

\[
\int_b^a p(x) \, dx = P
\]

\[
\int_b^a xp(x) \, dx = M
\]

where \( P \) and \( M \) are, respectively, the applied normal load and moment. Positive pressure is considered compressive in the nomenclature used above. For symmetric profiles with no applied moment \( (M = 0) \), \( C_1 = 0 \) and \( b = -a \). Published analytical solutions for \( p(x) \) are available for a number of indenter profiles, including the cylinder and the flat-punch with rounded edges and these will be used in subsequent sections. The former is the classical Hertzian elliptical contact and the flat-punch with rounded edges and these will be used in the following equation at all points

\[
Y(x, t) = A \int_{-a(t)}^{a(t)} q(s, t) \, ds - \frac{1 - v^2}{E} \sigma_0(t) \quad \forall x
\]

where the remote stress may be replaced by the strain \( \sigma_0(t) = \frac{1}{E} \sigma_0(t) \). For points in stick, since \( Y \) is independent of time,

\[
Y(x) = A \int_{-a(t)}^{a(t)} q(s, t) \, ds - \sigma_0(t) \quad \forall x \in (-a, a)
\]

The effect of the remote-applied stress results in purely anti-symmetric shear, since the applied shear load \( Q' \) is 0 in this case. We also expect the slip function \( s(x, t) \) to be an anti-symmetric (odd) function of \( x \) at each instant of the loading. The first spatial derivative of the slip, \( Y(x, t) \), must then be a symmetric (even) function assuming \( s(x, t) \) is differentiable. Let this be denoted by \( Y_e(x) \). Then Eq. (9) becomes

\[
Y_e(x) = A \int_{-a(t)}^{a(t)} q(s, t) \, ds - \sigma_0(t) \quad \forall x \in (-a, a)
\]

This is a Cauchy Singular Integral Equation for the shear traction, and solutions to such an equation may be sought in one of several classes, namely, (1) bounded everywhere, (2) unbounded at both ends, (3) unbounded at one end. For finite values of \( \mu \) and bounded pressure, the requirement that \( |q(x, t)| < \mu p(x, t) \) in the stick zone implies that \( q(x, t) \) must in this case belong to the class of functions that is everywhere bounded and goes to zero at the ends.\(^1\)

Now, it is known from the theory of singular integral equations that solutions to a Cauchy SIE that are bounded at both ends exist only if a consistency (or side) condition is satisfied (Muskhelishvili, 1992). Thus, the existence of a bounded solution \( q(x, t) \) requires

\[
\frac{1}{\pi a} \int_{-a}^a \frac{Y_e(s) + \sigma_0(t)}{\sqrt{a^2 - s^2}} \, ds = 0
\]

Since \( \sigma_0(t) \) is only a function of time, it can be taken to the right hand side by using the result

\[
\int_{-a}^a \frac{ds}{\sqrt{a^2 - s^2}} = \pi
\]

Also, since both \( Y_e(s) \) and the kernel of the integral equation \( 1/\sqrt{a^2 - s^2} \) are even functions, we can re-write the above equation to change the ends of integration to \((0, a)\)

\[
\int_0^a \frac{Y_e(s)}{\sqrt{a^2 - s^2}} \, ds = -\pi \sigma_0(t)
\]

Since the applied strain \( \sigma_0(t) \) is proportional to the normal load, it may be re-written as

\[
\int_0^a \frac{Y_e(s)}{\sqrt{a^2 - s^2}} \, ds = -\frac{\pi}{2} \sigma_0(t)
\]

\(^1\) Such a variable is often called pseudo-time.
The peak shear traction occurs at a distance symmetric, as expected. Written as needed.

This is a special example of a Volterra Integral Equation of the first kind known as an Abel Integral Equation, of the type encountered in the solution of frictionless axisymmetric contact problems (Barber, 2002). Its non-trivial solution is given by

\[
Y_e(x) = \frac{d}{dx} \int_0^x \frac{s^2 ds}{\sqrt{x^2 - s^2}}
\]

The definite integral

\[
\int_0^x \frac{s^2 ds}{x^2 - s^2} = \frac{2x^3}{3}
\]

Evaluating and simplifying,

\[
Y_e(x) = \frac{\sqrt{2}x^2}{ARp}
\]

Notice that this is an even function of \(x\). Once the slip derivative is known, it is possible to find the inversion to the shear SIE, whose bounded solution is given by

\[
q(x) = -\frac{\sqrt{2\pi}}{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{a^2 - s^2}} ds
\]

Evaluating the singular integrals, and simplifying, the shear traction for the case with stick everywhere is given by

\[
q(x) = \frac{\sqrt{2\pi}}{\pi} \frac{\sqrt{a^2 - x^2}}{A}
\]

where the final contact-size \(a = \sqrt{2ARp}\). Notice that \(q(x)\) is anti-symmetric, as expected.

The solution above is valid only as long as \(|q(x)| < \mu p(x)\) throughout the loading. Using the Hertzian solution \(p(x) = \sqrt{2\pi a^3}/\pi AR\), the condition for validity of the solution may be written as

\[
\frac{|q(x)|}{p(x)} < \mu = \frac{\sqrt{2\pi}}{\pi} \frac{\sqrt{a^3}}{a}
\]

Since the ratio \(g_0/P = g_0/P\) for proportional loading, and the maximum value of \(|x|\) is at most \(a = \sqrt{2AR}\) at any time, we get a limiting value of applied strain \(\sigma_0^{lim}\) (and hence applied remote bulk-stress) as

\[
\sigma_0^{lim} = \mu \frac{\sqrt{p}}{\pi} \frac{\sqrt{2A}}{\sqrt{K}}
\]

for stick everywhere throughout the loading. Fig. 1 shows normalized shear tractions obtained for various fractions of the limiting strain \(\gamma = \frac{\Delta}{\sigma_0}\) applied together with a normal load \(P\) for a cylinder.

The peak shear traction occurs at a distance \(x = \pm \sqrt{a^2}\) along the contact. The dotted lines in this plot (and all other plots) represent \(q(x) = \pm \mu p(x)\).

If the magnitude of the applied remote strain exceeds the limiting value, relative motion between the indenter and the halfspace continues even after a point comes into the contact, with anti-symmetric slip at either end. Unfortunately, the solution in this case may not be obtained analytically and numerical methods are needed.

The definite integral

\[
\int_0^a \frac{Ye(s)}{\sqrt{a^2 - s^2}} ds = -\pi g_0 \frac{g_0}{4ARPp} a^2
\]

4. Indentation of a halfspace by a cylinder with simultaneous normal load, shear load and remote bulk stress

Next, consider the problem of normal load \(P\), shear load \(Q\) and strain \(g_0\) applied together, with \(Q\) and \(g_0\) proportional to \(P\). Again, the solution with stick everywhere throughout the loading will be considered. In contrast to the previous result, the slip gradient \(Y(x)\) is no longer an even function. However, consider the decomposition of the gradient into odd and even parts, as follows:

\[
Y(x) = Y_e(x) + Y_o(x)
\]

Substituting this value into the consistency condition gives

\[
\int_{-a}^{a} \frac{Y_e(s) + Y_o(s) + g_0(t)}{\sqrt{a^2 - s^2}} ds = 0
\]

Since \(Y_o\) is an odd function of space,

\[
\int_{-a}^{a} \frac{Y_o(s)}{\sqrt{a^2 - s^2}} ds \equiv 0
\]

i.e. the integral is identically zero and the consistency condition is unable to provide any information about \(Y_o\); however, the even part of the slip gradient \(Y_e\) is clearly the same as in the case when no shear is applied. It is also clear that in the absence of applied bulk-stress, the shear traction obtained should be the classical oblique load solution, corresponding to a linear slip gradient and it suggests that \(Y_e(x) = kx\), where \(k\) is a constant. To prove this, consider the bounded solution obtained by inverting the shear SIE in the present case

\[
q(x) = \frac{\sqrt{2\pi}}{\pi} \frac{\sqrt{a^2 - x^2}}{\pi AR} \left( \int_{-a}^{a} \frac{Y_e(s) + g_0(t)}{\sqrt{a^2 - s^2}} ds + \int_{-a}^{a} \frac{Y_o(s)}{\sqrt{a^2 - s^2}} ds \right)
\]

The even components integrate to an anti-symmetric shear, which do not contribute to the resultant \(Q\). Thus, the integral over the odd parts should integrate to \(Q\) for equilibrium giving

\[
\frac{1}{\pi A} \int_{-a}^{a} \frac{\sqrt{2\pi}}{\pi} \frac{\sqrt{a^2 - x^2}}{\pi AR} \left( \int_{-a}^{a} \frac{Y_o(s)}{\sqrt{a^2 - s^2}} ds \right) dx = Q
\]

Since only one of the two integrands is singular\(^3\), it is possible to reverse the order of integration
\[
\frac{1}{\pi A} \int_{-a}^{a} \frac{Y_y(s)}{\sqrt{a^2 - s^2}} \left( \int_{-a}^{a} \frac{\sqrt{a^2 - x^2}}{s - x} \, dx \right) \, ds = Q
\]  

(28)

Evaluating the Cauchy principal value of the inner integral
\[
\frac{1}{\pi A} \int_{-a}^{a} s Y_y(s) \, ds = Q
\]  

(29)

Again, the instantaneous shear load \( Q \) may be re-written in terms of the contact size
\[
Q = \frac{Q_f}{p_f} \frac{Q}{\pi A} = \frac{Q}{2AR}
\]  

(30)

Using the above equation and realizing that the product of two odd functions \( sY_y \) is an even function leads to another Abel Integral Equation
\[
\int_{0}^{a} s Y_y(s) \sqrt{a^2 - s^2} \, ds = \frac{\pi Q_f}{4RP \pi a^2}
\]  

(31)

Its solution is
\[
Y_y(x) = \frac{Q_f}{RP} x
\]  

(32)

which is linear, as expected.\(^4\) Adding the shear component resulting from \( Y_y \), the final shear traction is
\[
q(x) = \left( \frac{Q_f - g_o x}{A} \right) \sqrt{a^2 - x^2} \frac{1}{\pi AR} p
\]  

(33)

where \( g_o = \sqrt{2ARp} \). When \( g_o = 0 \), this reduces to the classical oblique load solution.

The plots in Fig. 2 show normalized shear tractions obtained for simultaneous application of \( P, Q \), and \( \sigma_o \). \( g_{o\text{max}} \) is defined as in Eq. (22), but is no longer the limiting strain value. The plot on top is at a value of \( \gamma = \frac{Q_f}{Q} = 0.5 \), and shear load ratios \( \delta = \frac{Q}{P} \) varying from -0.4 to 0.4. Only the plot with \( \delta = 0 \) is anti-symmetric and the rest are skewed. The plot below shows tractions obtained at a value of \( \gamma = \frac{g_{o\text{max}}}{Q} = -0.25 \), and shear load ratios \( \delta = \frac{Q}{P} \) varying from -0.6 to 0.6.

Proceeding as before, it is possible to show that the stick everywhere solution is valid as long as the function \( D \), defined below, is negative.
\[
D = \max(\Delta_q + \Delta_y, |\Delta_q - \Delta_y|) - 1 < 0
\]  

(34)

where
\[
\Delta_q = \frac{Q_f}{RP} \quad \Delta_y = \frac{g_o \sqrt{2R}}{\mu \sqrt{AP}}
\]  

(35)

For \( D > 0 \), there is slip on side and stick on the other beyond some fraction \( 0 < \varepsilon_t < 1 \) of the load-step. Consideration of the existence and equilibrium conditions for the associated SIE in this case leads to a somewhat complicated system of coupled integral equations which appear analytically intractable (at present) although good approximate solutions may be obtained for a wide range of shear and bulk-stress values.

5. Indentation of a halfspace by a flat punch with rounded edges with simultaneous normal load, shear load and remote bulk stress

As in the case of the cylinder, the odd part of the slip gradient contributes only to the oblique load solution; it is sufficient to ob-

\(^4\) This is also an alternate way to obtain the classical oblique load solution.

\(^5\) It seems Eq. (38) was first derived by G. Schubert in 1942, in the German journal Ing. Arch. (13), pp. 132–161.
\[ \int_{c}^{a} \frac{Y_{c}(s)}{\sqrt{a^2 - s^2}} \, ds = \frac{g_{0}}{2P'AR} \left[c \sqrt{a^2 - c^2} - c^2 \cos^{-1} \left( \frac{c}{a} \right) \right] \]  

(39)

Its inversion is

\[ Y_{c}(x) = \frac{g_{0}}{\pi P'AR} d \left[ \int_{c}^{x} \frac{cs}{\sqrt{x^2 - s^2}} \cos^{-1} \left( \frac{s}{x} \right) \, ds \right] \]  

(40)

These definite integrals can be evaluated analytically

\[ \int_{c}^{x} \frac{cs}{\sqrt{x^2 - s^2}} \cos^{-1} \left( \frac{s}{x} \right) \, ds = \frac{\pi}{4} (x^2 - c^2) \]  

(41)

\[ \int_{c}^{x} s^3 \cos^{-1} \left( \frac{s}{x} \right) \, ds = \frac{\pi}{12} (x-c)(4x^2 + cx + c^3) \]  

(42)

Using these results in Eq. (40), differentiating and simplifying,

\[ Y_{c}(x) = \frac{g_{0}}{\pi P'AR} (cx - x^2) \]  

(43)

The presence of the odd term in \( Y_{c}(x) \) is explained by the fact that \( Y_{c} \) is a multi-part function, described by different expressions in different regions, i.e. considering \( x < -c \) would give a complimentary result for \( Y_{c}(x) \).

Thus,

\[ Y_{c}(x) = \begin{cases} \frac{g_{0}}{P'AR}(-cx - x^2) & \text{if } x < -c; \\ 0 & \text{if } -c < x < c; \\ \frac{g_{0}}{P'AR}(cx - x^2) & \text{if } x > c. \end{cases} \]

Notice \( Y_{c}(x) = Y_{c}(-x) \) and the function, while multi-part, is still continuous at \( x = \pm c \). Then, \( q_{e}(x) \), the part of the shear contributed by \( Y_{c}(x) \) is

\[ q_{e}(x) = \frac{\sqrt{a^2 - x^2}}{\pi A} \left[ \int_{c}^{a} Y_{c}(s) + g_{0}(t) \, ds \right] \]  

(44)

Or,

\[ q_{e}(x) = \frac{\sqrt{a^2 - x^2}}{\pi A} \left[ \int_{c}^{a} g_{0}(t) \, ds + \int_{-a}^{-c} Y_{c}(s) \, ds + \int_{-a}^{c} Y_{c}(s) \, ds \right] \]  

(45)

The first integral is 0; substituting for \( Y_{c} \) gives

\[ q_{e}(x) = \frac{\sqrt{a^2 - x^2}}{\pi A} \left[ \frac{g_{0}}{P'AR} \int_{c}^{a} - cs - s^2 \, ds \right] \]  

+ \int_{c}^{a} \frac{cs - s^2}{\sqrt{a^2 - s^2}} \, ds \]  

(46)

Both integrals may be re-written as the sum of two non-singular integrals and another integral which is singular for some values of \( x \)

\[ I_{1} = -(c+x) \int_{-a}^{-c} \frac{ds}{\sqrt{a^2 - s^2}} - \int_{-a}^{c} \frac{s \, ds}{\sqrt{a^2 - s^2}} = (x^2 + cx) \]  

(47)

\[ I_{2} = (c-x) \int_{-a}^{a} \frac{ds}{\sqrt{a^2 - s^2}} - \int_{c}^{a} \frac{s \, ds}{\sqrt{a^2 - s^2}} + (cx - x^2) \]  

(48)

The former are elementary; a technique to evaluate the latter is given in Goryacheva et al. (2002). Denoting these integrals by \( -S_{1} \) and \( -S_{2} \),

\[ q_{e}(x) = \frac{g_{0}\sqrt{a^2 - x^2}}{\pi P'AR} \left[ -2x \cos^{-1} \left( \frac{c}{a} \right) + (x^2 + cx)S_{1} + (x^2 - cx)S_{2} \right] \]  

(49)

where

\[ S_{1} = \frac{1}{\sqrt{a^2 - x^2}} \left[ \log \frac{\beta + w(x)}{\beta - w(x)} \right] \]  

(50)

and

\[ \beta = \frac{a + c}{a - c} = \frac{1}{x} \]  

(51)

The complete shear traction is obtained by adding the oblique shear traction to \( q_{e}(x) \) to give

\[ q(x) = \frac{\sqrt{a^2 - x^2}}{\pi A R} \left[ \frac{g_{0}}{P'AR} \left(-2x \cos^{-1} \left( \frac{c}{a} \right) + (x^2 + cx)S_{1} + (x^2 - cx)S_{2} \right) \right] \]  

+ \frac{Q'^{\prime}}{P} \left( 2 \cos^{-1} \left( \frac{c}{a} \right) - (x + c)S_{1} + (x - c)S_{2} \right) \]  

(52)

where \( S_{1}, S_{2} \) are defined as before and \( a \) is the final contact size. Normalized shear tractions obtained with simultaneous application of \( P, Q \) and \( \sigma_{n} \) to a flat punch with rounded edges are considered in Fig. 3. In the upper plot, the strain \( \gamma = \frac{d}{P} = 0.4 \), and \( \delta = \frac{Q'}{P} \) varies

![Fig. 3. Shear tractions obtained with simultaneous application of P, Q and \( \sigma_{n} \). In the upper plot, c/a = 0.8, \( \gamma = \frac{d}{P} = 0.4 \), and \( \delta = \frac{Q'}{P} \) varies from −0.3 to 0.4. In the lower plot, c/a = 0.6, \( \gamma = \frac{d}{P} = -0.25 \), and \( \delta = \frac{Q'}{P} \) varies from −0.5 to 0.5.](image-url)
from −0.3 to 0.4 and c/a = 0.8. In the lower plot, γ = \frac{R}{R_0} = −0.25, and δ = \frac{b}{R} varies from −0.5 to 0.5 and \(g_{60}^{a\alpha} = μf^2/Aa^3\). In this case, \(c/a = 0.6\). There is stick everywhere throughout the loading.

6. Non-linear load paths

As long as conditions of stick prevail throughout the loading, the methods discussed so far may be easily generalized to non-linear load paths. In this case, either the applied remote stress or shear (or both) may vary non-linearly over the load step. Realizing that the applied shear load contributes only to the odd part of the slip gradient and the remote-bulk stress only to the even part as before provides a powerful method to obtain analytical shear solutions for problems involving general load paths. Two examples are considered here.

Consider a cylindrical indenter. Let the remote bulk stress (and hence strain) and applied shear traction vary as real positive powers of the load-fraction, i.e.,

\[
s_0 = \frac{S_0}{P} \quad \rho > 0
\]

\[
Q = \frac{Q}{P} \quad \rho > 0
\]

In general, \(ρ \neq γ\). Again, considering the Abel Integral Equation for the existence of a bounded solution leads to the following expression for the even part of the slip gradient

\[
Y_e(x) = -\frac{g_0}{(2ARP)^{\gamma}} \frac{d}{dx} \int_0^x \frac{s^{2\gamma-1}}{s^2} ds
\]

and the requirement that the shear traction component obtained from the odd part of the slip gradient integrate instantaneously to \(Q\) leads to

\[
Y_o(x) = \frac{2AQ\rho}{(2ARP)^{\gamma}} \int_0^x \frac{s^{2\rho-1}}{s^2} ds
\]

Using the substitution \(s = x \cos(θ)\) in the integral in \(Y_o(x)\) transforms the integral to

\[
Y_o(x) = \frac{2AQ\rho}{(2ARP)^{\gamma}} \int_0^x \cos^{2\rho-1} \theta dθ
\]

This integral is easily recognized as \(\frac{1}{2} B\left(\frac{1}{2}, \rho\right)\), the beta function, which may finally be expressed in terms of the gamma function (Abramowitz and Stegun, 1972) to give

\[
Y_o(x) = \frac{AQ\rho}{(2ARP)^{\rho}} \Gamma\left(\frac{\rho}{2}\right) x^{2\rho-1}
\]

Similarly,

\[
Y_e(x) = \frac{g_0}{(2ARP)^{\gamma}} \frac{\sqrt{R} \Gamma\left(γ + \frac{1}{2}\right)}{2 \Gamma\left(γ + \frac{3}{2}\right)} x^{2\gamma}
\]

The singular integrals involving \(Y_e\) and \(Y_o\) may be evaluated in terms of elementary functions for \(ρ = m/2, γ = n/2\).

As an example, consider the oblique loading of a cylinder with \(P, Q\) applied simultaneously, with the applied shear load varying quadratically over the load step and no bulk-stress \((g_0 = 0)\). In this case, setting \(ρ = 2\) in Eq. (58)

\[
Y_o(x) = \frac{8}{3} \frac{AQ}{(2ARP)^{2}} x^3
\]

and \(Y_e(x) = 0\). Substituting in the SIE, the shear traction is given by

\[
q(x) = \frac{1}{π^2 A} \frac{2AQ}{3 (2ARP)^{2}} \int_{-a}^{a} \frac{s^3}{\sqrt{a^2 - s^2}(s-x)} ds
\]

(61)

Evaluating the Cauchy principal value and simplifying

\[
q(x) = \frac{Q}{3π ARP} \sqrt{a^2 - x^2} \left[ 2 + \frac{4x^2}{a^2} \right]
\]

(62)

where \(a = \sqrt{2ARP}\). Unlike the classical oblique load solution, this is not proportional to \(p(x)\). Further, it is easy to prove that for this solution to be valid, it is required that \(2Q < μf^2\).

Depending on the load path, \(Y_e(x)\) or \(Y_o(x)\) may be described by a multi part function. Consider \(P, g_o\) applied together to a contact pair consisting of a cylinder and a halfspace, with the applied strain varying as the square-root of the load fraction, i.e., \(γ = \frac{3}{2}\) in Eq. (59).

In this case, the exponent of \(x\) in \(Y_e(x)\) is 1, which is odd. Thus, the function is multi-part\(^6\) and \(Y_e(x)\) is defined as

\[
Y_e(x) = \begin{cases} \frac{1}{2} \frac{g_0}{\sqrt{2AR}} (x) & \text{if } x < 0; \\ \frac{1}{2} \frac{g_0}{\sqrt{2AR}} (-x) & \text{if } x > 0. \end{cases}
\]

\(Y_e(x)\) is continuous at \(x = 0\). The resulting shear traction is

\[
q(x) = \frac{\sqrt{a^2 - x^2}}{πA} \frac{g_0}{2 \sqrt{2AR}} \left[ \int_{-a}^{a} ds \frac{s}{\sqrt{a^2 - s^2}(s-x)} - \int_{0}^{a} ds \frac{s}{\sqrt{a^2 - s^2}(s-x)} \right]
\]

(63)

The integrals may be split up further by using \(s = (s-x)+x\); the resulting elementary integrals cancel out to give

\[
q(x) = \frac{\sqrt{a^2 - x^2}}{πA} \frac{g_0}{2 \sqrt{2AR}} \left[ \int_{-a}^{0} ds \frac{ds}{\sqrt{a^2 - s^2}(s-x)} - \int_{0}^{a} ds \frac{ds}{\sqrt{a^2 - s^2}(s-x)} \right]
\]

(64)

Noticing that these integrals are the same as those for the case of a flat punch with rounded edges with \(c = 0\), the results in Eq. (50) may be used to obtain

\[
q(x) = \frac{1}{πA} \frac{g_0}{\sqrt{2AP}} \left[ \log \left( \frac{1 - w(x)}{1 + w(x)} \right) \right]
\]

(65)

where \(w(x)\) is defined as

\[
w(x) = \frac{a - x}{\sqrt{a + x}}
\]

(66)

It may be shown that this solution is valid if the applied strain satisfies the following condition

\[
|g_0| < μ \frac{\sqrt{2AP}}{\sqrt{R}}
\]

(67)

Shear tractions obtained using various non-linear paths for \(Q\) and \(g_o\) are shown in Fig. 4.

7. Non-linear oblique loading with slip at both ends

In the problems considered so far, the condition that relative motion between corresponding points on the indenter and half-space cease after they come into contact has been required to

\(^6\) In fact, this argument applies whenever the exponent of \(x\) is not an even integer since, strictly speaking, \(Y_e(x)\) is obtained only for \(x > 0\) by this procedure and determined by the appropriate symmetry property for even functions when \(x < 0\).
obtain a solution. While the occurrence of slip usually renders the problem analytically intractable using the present method, it is still possible to obtain shear traction solutions in special cases. As an example, consider the quadratic oblique load problem (i.e., \( \rho = 2 \)) with applied shear load \( Q' \) such that \( 0.5\mu P < Q' < \mu P \). Clearly, the stick-everywhere solution discussed in the previous section is no longer valid. However, the stick-everywhere solution is valid up to a certain fraction \( \phi \) of the load step, beyond which slip sets in at the edges of contact. This is precisely the load fraction at which the load path \( Q = Q_0 \left( \frac{P}{P'} \right) \) intersects the line \( Q = 0.5\mu P \). Solving, the critical load fraction \( P' / P \) beyond which slip occurs inside the contact is

\[
P' / P = \frac{\mu P'}{2Q'}
\]

(68)

At this stage, the slip gradient in the region \((-a_c, a_c)\) is

\[
\nu_s(x) = \frac{8}{3} \frac{AQ'}{(2ARP)^2} x^3
\]

(69)

where the subscript or superscript \( c \) indicates quantities at the critical load fraction. Now, in subsequent loading, if the slip zone grows into regions that are always in stick, it is possible to write a Cattaneo–Mindlin type corrective shear traction SIE using \( q(x) = \mu p(x) - q'(x) \) as follows:

(70)

where \( c \) is the stick-zone half-size. Inverting this SIE for \( q'(x) \) and simplifying, the shear traction \( q(x) = \mu p(x) - q'(x) \) in the stick-zone is given by

\[
q(x) = \frac{\sqrt{x^2 - c^2} - c^2}{\pi AR} - \frac{\sqrt{\mu}}{\pi AR} \frac{2Q' c^2}{3} \left[ 2 + 4 \frac{x^2}{c^2} \right]
\]

(71)

where \( c^2 = 2ARP \) is the square of the contact half-size at the critical load. Obviously, \( q(x) = \mu p(x) \) in the slip zones. The size of the stick zone is obtained from the equilibrium equation

\[
\int_{-c}^{c} q'(x) dx = \mu P' - Q'
\]

(72)

which leads to the following algebraic equation

\[
\frac{Q'}{2c' p} c'^3 - \frac{\mu}{2AR} c'^3 + \mu P' - Q' = 0
\]

(73)

Re-writing \( Q', \alpha_c \) in terms of \( Q', P_l \), dividing throughout by \( \mu P' \) and setting \( \beta = Q' / \mu P' \),

\[
\frac{\beta}{c'^3} = \frac{1}{4\beta} c'^3 + (1 - \beta) = 0
\]

(74)

This is a quadratic in \( c'^3 / \beta \),

\[
\frac{c'^3}{\beta} = \frac{1 \pm \sqrt{1 - 4\beta + 4\beta^2}}{2\beta} = \frac{1 \pm (1 - 2\beta)}{2\beta}
\]

(75)

Ignoring the shear load independent solution \( c = a_c \), the ratio of stick zone size to contact size is given by

\[
\frac{c}{a_c} = \sqrt{1 - \frac{\beta}{2}} \beta > \frac{1}{2}
\]

(76)

Thus, in the quadratic oblique load problem, slip occurs near the edges of the contact when \( Q' > 0.5\mu P' \). The onset of slip in such cases occurs at a load fraction \( \mu P' / 2Q' \) and the slip zones continue to grow till the end of loading. A smaller final slip zone size is obtained for larger \( \beta \) since the slope \( d^2 c / d\beta = -\frac{1}{\beta^2} \) is always negative. Of course, at \( \beta = 1 \), the stick-zone vanishes completely.

8. Discussion

When there is stick everywhere throughout the loading, the shear traction obtained is independent of the coefficient of friction; \( \mu \) only plays a role in determining the range in which the solution is valid. This is in agreement with the results of Dundurs and Comninou (1982) for their simplified problem. Of course, in the case when there is slip near the ends of contact, the solution is no longer independent of \( \mu \). The present method obviates the need for an incremental approach by obtaining the slip gradient directly without, however, relying on self-similarity arguments. It is interesting to note that the solution for the case when \( P \) and \( \sigma_0 \) are applied together may be obtained analytically, while a ‘step-stair’ increment of this solution (a load increment \( \Delta P \) followed by a stress increment \( \Delta \sigma_0 \)) is not obtainable in closed form. It may also be noted that the easy decomposition of the slip field into odd and even components associated, respectively (and exclusively), with the odd and even components of the shear requires symmetric contacts (symmetric profile and no applied moment). Such an association is not possible in the case of asymmetric contacts. In addition, asymmetric contacts may re-
cede on one side while advancing on the other (which is overall still an advancing contact), which complicates the analysis.

Several important generalizations of inherently incremental shear problems remain open. They are (1) problems with slip on one side and stick on the other for some fraction of the loading, (2) problems involving asymmetric contacts (e.g., changing moment applied to a flat punch with rounded edges during oblique loading with or without bulk-stress) and (3) unloading in problems with slip.

9. Conclusions

Analytical solutions were obtained for a subset of similar elastic contact problems in which the contact pair is subjected to combined normal, shear and remote bulk loads. When conditions of stick prevail at all contact points throughout the loading, the slip function (and slip gradient) may be obtained uniquely for a wide variety of problems without recourse to incremental or self-similarity arguments. The conditions that (a) the associated shear Cauchy SIE always have a bounded solution and (b) global shear equilibrium be satisfied lead to a pair of Abel Integral Equations that determine, respectively, the even and odd parts of the slip gradient. Once the slip gradient is obtained, it may be used in the Cauchy SIE to determine the shear traction. Analytical solutions were obtained for both the cylindrical indenter and the flat punch with rounded edges. The shear traction solutions obtained when there is stick everywhere throughout the loading are independent of the co-efficient of friction which does, however, determine the range of validity of the solutions. The method is applicable when the shear load and bulk-stress vary non-linearly over the load step and several examples of such problems were considered. Finally, in some cases, problems with development of slip zones within the contact may also be solved.

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References


