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Total interval numbers of complete r-partite graphs ☆

Mingjang Chen, Gerard J. Chang*

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan

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Abstract

A multiple-interval representation of a graph G is a mapping f which assigns to each vertex of G a union of intervals on the real line so that two distinct vertices u and v are adjacent if and only if $f(u) \cap f(v) \neq \emptyset$. We study the total interval number of G, defined as

$$I(G) = \min\left\{\sum_{v\in V} \#f(v)\colon\ f \text{ is a multiple-interval representation of } G\right\},$$

where $\#f(v)$ is the minimum number of intervals whose union is $f(v)$. We give bounds on the total interval numbers of complete r-partite graphs. Exact values are also determined for several cases. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The *intersection graph* of a family $\mathscr{F}$ of sets is the graph obtained by representing each set of $\mathscr{F}$ as a vertex and joining two vertices with an edge if their corresponding sets intersect. The family of sets is called an *intersection representation* of its intersection graph. For an intersection representation $\mathscr{F}$ of a graph $G=(V,E)$, we often use a bijection f from V to $\mathscr{F}$ to represent $\mathscr{F}$, where $f(x)$ is the set in $\mathscr{F}$ corresponding to the vertex x for any $x \in V$. It is well-known that any graph is the intersection graph of some family of sets. The problem of characterizing intersection graphs of families

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* Corresponding author. Department of Mathematics, National Taiwan University, Taipei 106, Taiwan.
E-mail address: gjchang@math.ntu.edu.tw (G.J. Chang).

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of sets having some specific topology or other pattern is often very interesting and frequently has applications in the real world. A typical example is the class of interval graphs. An *interval graph* is the intersection graph of intervals on the real line. They play important roles in many applications, see [2].

More generally, we allow a representation f to assign each vertex a union of intervals on the real line. In this case, f is called a *multiple-interval representation* of the intersection graph of this family of sets. Let $\#f(v)$ denote the minimum number of intervals whose union is $f(v)$; note that these intervals are disjoint. For any subset S of V, we use $\#f(S)$ to denote $\sum_{v\in S}\#f(v)$.

Multiple-interval representations can measure how far a graph is from being an interval graph in two nature ways. The *interval number* of a graph $G=(V,E)$ is

$$i(G)=\min\left\{\max_{v\in V}\#f(v)\text{: } f \text{ is a multiple-interval representation of } G\right\}.$$

Note that a graph is an interval graph if and only if its interval number is one. The concept of interval graph was initiated by Trotter and Harray [9] and Griggs and West [3], and then extensively studied in the literature. The *total interval number* of a graph $G=(V,E)$ is

$$I(G)=\min\{\#f(V)\text{: } f \text{ is a multiple-interval representation of} G\}.$$

This number was proposed by Griggs and West [3] and formally studied by Aigner and Andreae [1] who have found upper bounds on $I(G)$, where G is a tree, a triangle-free planar or outerplanar graph, or a triangle-free graph. For further studies on the total interval numbers of graphs, see [5–8].

The purpose of this paper is to study the total interval numbers of complete r-partite graphs. For any positive integer r, a *complete r-partite graph* is a graph $G=(V,E)$ whose vertex set V can be partitioned into r non-empty *partite sets* $V_1,V_2,\ldots,V_r$ such that for any two vertices $u\in V_i$ and $v\in V_j$, vertex u is adjacent to vertex v if and only if $i\neq j$. We use $K_{n_1,n_2,\ldots,n_r}$ to denote the complete r-partite graph in which $|V_i|=n_i$ for $1\leqslant i\leqslant r$. We use $K_{[r_1]*n_1,[r_2]*2_2,\ldots,[r_k]*n_k}$ as a short notation for $K_{\underbrace{n_1,n_1,\ldots,n_1}_{r_1},\underbrace{n_2,n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,n_k,\ldots,n_k}_{r_k}}$. In this paper, we give bounds for the total interval numbers of complete r-partite graphs. Exact values are also determined for several cases.

2. Upper bound

This section investigates some basic results frequently used in this paper. The first one is the exact values for the total interval numbers of complete bipartite graphs, which were obtained by Andreae and Aigner [1].

Theorem 1. *If $m\geqslant 1$ and $n\geqslant 1$, then $I(K_{m,n})=mn+1$.*

Another useful fact is

Lemma 2. *Suppose $G=(V,E)$ is a graph and G' is a subgraph of G induced by $U\subseteq V$. If f is a multiple-interval representation of G, then $I(G')\leqslant \#f(U)$.*

Finally, we establish an upper bound for the total interval number of a general complete r-partite graph in terms of the sizes of their partite sets.

Theorem 3. *If $r\geqslant 2$ and $n_1\geqslant n_2\geqslant\cdots\geqslant n_r\geqslant 1$, then*

$$I(K_{n_1,n_2,\dots,n_r})\leqslant n_1n_2+1+\sum_{t=3}^{r}n_t(n_t+1)/2.$$

Proof. Suppose the complete r-partite graph $K_{n_1,n_2,\dots,n_r}$ has vertex set $V=\bigcup_{t=1}^{r}V_t$ where $V_t=\{v_{t+kr}\colon 1\leqslant k\leqslant n_t\}$ for $1\leqslant t\leqslant r$, and edge set $E=\{v_iv_j\colon v_i,v_j\in V$ and $i\not\equiv j \pmod r\}$.

To establish the upper bound, we define a multiple-interval representation f of the graph as follows: for any $v_i\in V$, where $i=t+kr$ with $1\leqslant t\leqslant r$ and $1\leqslant k\leqslant n_t$,

$$f(v_i)=J_i\cup\{D_{t+\ell r,i}\colon 1\leqslant \ell<k\},$$

where $J_i=[i+1,i+r]$ and $D_{j,i}=[j+1/i,j+1/i]$.

To show that f is a multiple-interval representation of $K_{n_1,n_2,\dots,n_r}$, we first observe the following properties for the intervals J_i's and $D_{j,i}$'s:
(1) $J_i\cap J_{i'}\neq\emptyset$ whenever $|i-i'|\leqslant r-1$, and $J_i\cap J_{i'}=\emptyset$ otherwise.
(2) $D_{j,i}\cap D_{j',i'}=\emptyset$ whenever $j\neq j'$ or $i\neq i'$.
(3) $J_i\supseteq D_{j',i'}$ whenever $1\leqslant j'-i\leqslant r-1$, and $J_i\cap D_{j',i'}=\emptyset$ otherwise.
(4) If $v_i\in V$, where $i=t+kr$ with $1\leqslant t\leqslant r$ and $1\leqslant k\leqslant n_t$, then $\#f(v_i)=k$.

Consider any two distinct vertices v_i and $v_{i'}$ in V. Without loss of generality, assume that $i<i'$. Let $i=t+kr$ with $1\leqslant t\leqslant r$ and $1\leqslant k\leqslant n_t$; and $i'=t'+k'r$ with $1\leqslant t'\leqslant r$ and $1\leqslant k'\leqslant n_{t'}$. Then $v_i\in V_t$ and $v_{i'}\in V_{t'}$.

Suppose $t=t'$, i.e., $i\equiv i' \pmod r$. Since $i'-i\geqslant r$, by (1), $J_i\cap J_{i'}=\emptyset$; since $i\equiv t'+\ell' r \pmod r$, by (3), $J_i\cap D_{t'+\ell' r,i'}=\emptyset$ for $1\leqslant \ell'<k'$; since $i'\equiv t+\ell r \pmod r$, by (3), $J_{i'}\cap D_{t+\ell r,i}=\emptyset$ for $1\leqslant\ell<k$; since $i\neq i'$, by (2), $D_{t+\ell r,i}\cap D_{t'+\ell' r,i'}=\emptyset$ for $1\leqslant \ell<k$ and $1\leqslant \ell'<k'$. Therefore, $f(v_i)\cap f(v_{i'})=\emptyset$.

Next, consider the case of $t\neq t'$, i.e., $i\not\equiv i' \pmod r$. If $i'-i\leqslant r-1$, then, by (1), we have $J_i\cap J_{i'}\neq\emptyset$ which implies $f(v_i)\cap f(v_{i'})\neq\emptyset$. If $i'-i>r$, then $1\leqslant (i'-\lfloor (i'-i)/r\rfloor r)-i=(i'-i)-\lfloor (i'-i)/r\rfloor r\leqslant r-1$ and so by (3), $D_{i'-\lfloor (i'-i)/r\rfloor r,i'}\subseteq J_i$ which implies $f(v_i)\cap f(v_{i'})\neq\emptyset$.

Therefore, f is a multiple-interval representation of $K_{n_1,n_2,\dots,n_r}$ with

$$\#f(V)=\sum_{t=1}^{r}\sum_{k=1}^{n_t}\#f(v_{t+kr})=\sum_{t=1}^{r}\sum_{k=1}^{n_t}k=\sum_{t=1}^{r}n_t(n_t+1)/2.$$

Note that the intervals $D_{1+r,1+\ell r}$ (for $2\leqslant \ell\leqslant n_1$), J_{1+kr} (for $n_2+2\leqslant k\leqslant n_1$) and $D_{1+kr,1+\ell r}$ (for $n_2+2\leqslant \ell<k\leqslant n_1$) intersect with no other intervals in f. Removing

these intervals from f resulting a multiple-interval representation f' of $K_{n_1,n_2,\ldots,n_r}$ with

$$\#f(V) - \#f'(V) = (n_1 - 1) + (n_1 - n_2 - 1) + \sum_{\ell=n_2+2}^{n_1-1} (n_1 - \ell)$$

$$= (n_1 - 1) + (n_1 - n_2)(n_1 - n_2 - 1)/2.$$

Therefore,

$$I(K_{n_1,n_2,\ldots,n_r}) \leqslant \#f'(V)$$

$$= \sum_{t=1}^{r} n_t(n_t + 1)/2 - (n_1 - 1) - (n_1 - n_2)(n_1 - n_2 - 1)/2$$

$$= n_1 n_2 + 1 + \sum_{t=3}^{r} n_t(n_t + 1)/2. \qquad \square$$

Corollary 4. *If $r \geqslant 2$ and $n \geqslant 1$, then $I(K_{[r]*n}) \leqslant (rn^2 + (r-2)n + 2)/2$.*

Note that the result $i(K_{[r]*n}) = \lceil (nn+1)/(n+n) \rceil = \lceil (n+1)/2 \rceil$ given in [4] implies that $I(K_{[r]*n}) \leqslant rn\lceil (n+1)/2 \rceil$ which is asymptotically equal to, but slightly larger than, the upper bound in Corollary 4.

3. The graphs $K_{n,[s]*2}$ and $K_{[s]*2}$

We first consider the graphs $K_{n,[s]*2}$ and $K_{[s]*2}$.

Theorem 5. *If $n \geqslant 1$ and $s \geqslant 1$, then $I(K_{n,[s]*2}) = 2n + 3s - 2$.*

Proof. Suppose $V(K_{n,[s]*2}) = V_0 \cup V_1 \cup \cdots \cup V_s$, where $V_0, V_1, \ldots, V_s$ are the partite sets of $K_{n,[s]*2}$ with $|V_0| = n$ and $|V_i| = 2$ for $1 \leqslant i \leqslant$. Choose an optimal multiple-interval representation f of $K_{n,[s]*2}$. Without loss of generality, we may assume that $\#f(V_1) \leqslant \#f(V_2) \leqslant \cdots \leqslant \#f(V_s)$. According to Theorem 1, we have $I(K_{n,2}) = 2n + 1$; so according to Lemma 2, we have $\#f(V_0) + \#f(V_1) \geqslant 2n + 1$. Similarly, $I(K_{2,2}) = 5$ leads to $\#f(V_1) + \#f(V_2) \geqslant 5$, which implies $\#f(V_i) \geqslant \#f(V_2) \geqslant 3$ for $2 \leqslant i \leqslant s$. Thus,

$$I(K_{n,[s]*2}) = \#f(V_0) + \#f(V_1) + \#f(V_2) + \cdots + \#f(V_s)$$

$$\geqslant 2n + 1 + 3(s-1) = 2n + 3s - 2.$$

On the other hand, according to Theorem 3, we have $I(K_{n,[s]*2}) \leqslant 2n + 3s - 2$. (Note that we need to consider the cases of $n = 1$ and $n \geqslant 2$ separately.) Therefore, $I(K_{n,[s]*2}) = 2n + 3s - 2$. $\square$

Corollary 6. *If $s \geqslant 2$, then $I(K_{[s]*2}) = 3s - 1$.*

Proof. $I(K_{[s]*2}) = I(K_{2,[s-1]*2}) = 2 \times 2 + 3(s-1) - 2 = 3s - 1$. $\square$

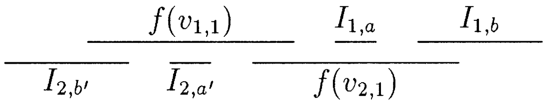

Fig. 1. Relative positions of $f(v_{1,1}), I_{1,a}, I_{1,b}, f(v_{2,1}), I_{2,a'}$ and $I_{2,b'}$.

4. The graphs $K_{n,[r]*3}$ and $K_{[r]*3}$

This section studies the graphs $K_{n,[r]*3}$ and $K_{[r]*3}$.

Theorem 7. *If $n \geqslant 2$ and $r \geqslant 1$, then $I(K_{n,[r]*3}) = 3n + 6r - 5$.*

Proof. Suppose $V(K_{n,[r]*3}) = V_0 \cup V_1 \cup \cdots \cup V_r$, where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n_i}\}$ $(0 \leqslant i \leqslant r)$ are the partite sets of $K_{n,[r]*3}$ with $n_0 = n$ and $n_i = 3$ for $1 \leqslant i \leqslant r$. Choose an optimal multiple-interval representation f of $K_{n,[r]*3}$. Without loss of generality, we may assume that $\#f(V_1) \leqslant \#f(V_2) \leqslant \cdots \leqslant \#f(V_r)$ and $\#f(v_{i,1}) \leqslant \#f(v_{i,2}) \leqslant \cdots \leqslant \#f(v_{i,n_i})$ for $0 \leqslant i \leqslant r$.

According to Theorem 1, we have $I(K_{n,3}) = 3n+1$; and so Lemma 2 implies $\#f(V_0) + \#f(V_1) \geqslant 3n + 1$. If $\#f(V_2) \geqslant 6$, then

$$I(K_{n,[r]*3}) = \sum_{i=0}^{r} \#f(V_i) \geqslant (3n + 1) + 6(r - 1) = 3n + 6r - 5.$$

We now consider the case when $\#f(V_2) \leqslant 5$. Since $I(K_{3,3}) = 10$, according to Lemma 2, we have $\#f(V_1) + \#f(V_2) \geqslant 10$ and so $\#f(V_1) = \#f(V_2) = 5$, which imply $\#f(v_{1,1}) = \#f(v_{2,1}) = 1$, i.e., $f(v_{1,1})$ and $f(v_{2,1})$ are intervals. Since $v_{1,1}v_{2,1}, v_{1,1}v_{2,2} \in E$ but $v_{2,1}v_{2,2} \notin E$, the interval $f(v_{1,1})$ is not properly contained in $f(v_{2,1})$. Similarly, the interval $f(v_{2,1})$ is not properly contained in $f(v_{1,1})$. Hence, there exists an interval $I_{1,a}$ in $f(v_{1,a})$ properly contained in $f(v_{2,1})$ and an interval $I_{1,b}$ in $f(v_{1,b})$ intersecting $f(v_{2,1})$, where $\{a, b\} = \{2, 3\}$. Similarly, there exists an interval $I_{2,a'}$ in $f(v_{2,a'})$ properly contained in $f(v_{1,1})$ and an interval $I_{2,b'}$ in $f(v_{2,b'})$ intersecting $f(v_{1,1})$, where $\{a', b'\} = \{2, 3\}$. Without loss of generality, we may assume that the relative positions of these intervals are shown as in Fig. 1.

Since $v_{1,a}v_{2,a'} \in E$, we have that $f(v_{1,a})$ contains an interval $J_{1,a}$ (other than $I_{1,a}$) intersecting an interval $J_{2,a'}$ (other than $I_{2,a'}$) of $f(v_{2,a'})$. We may assume that $J_{1,a}$ and $J_{2,a'}$ are on the right to $f(v_{2,1})$. (The case when $J_{1,a}$ and $J_{2,a'}$ are on the left to $f(v_{1,1})$ is similar.) Then, $f(v_{2,b'})$ contains an interval $J_{2,b'}$ (other than $I_{2,b'}$) intersecting $J_{1,a}$; so, $J_{2,b'}$ is on the right to $f(v_{2,1})$. Note that the fifth interval of $f(V_1)$ is $J_{1,b}$ in $f(v_{1,b})$ that could be on the right to $I_{1,a}$ (see Cases 1 and 2 of Fig. 2) or on the left to $f(v_{1,1})$ (see Case 3 of Fig. 2).

Therefore, we have

Claim 1. *If $f(V_1) = f(V_2) = 5$, then either $f(v_{1,1})$ is at the middle of the five intervals of $f(V_1)$ or $f(v_{2,1})$ is at the middle of the five intervals of $f(V_2)$; but not both.*

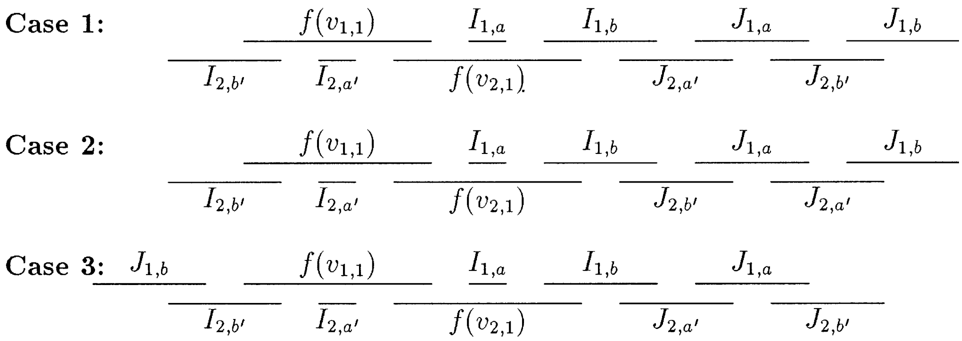

Fig. 2.

Note that we are now considering the second case of Claim 1. Next, we establish that $\#f(V_0) \geqslant 3n-3$ by showing the following three claims.

Claim 2. *If* $n \geqslant 2$, *then* $\#f(v_{0,1}) + \#f(v_{0,2}) \geqslant 3$.

Otherwise, suppose $\#f(v_{0,1}) = \#f(v_{0,2}) = 1$. Note that $I_{1,a}$ and $J_{1,a}$ are both on the right to $f(v_{1,1})$. As intervals $f(v_{0,1})$ and $f(v_{0,2})$ intersect $f(v_{1,1})$, one of them must not intersect $f(v_{1,a})$, a contradiction. This proves Claim 2.

Claim 3. *If* $n \geqslant 3$, *then* $\#f(v_{0,1}) + \#f(v_{0,2}) + \#f(v_{0,3}) \geqslant 6$.

Suppose to the contrary that $\#f(V_0') \leqslant 5$, where $V_0' = \{v_{0,1}, v_{0,2}, v_{0,3}\}$. Then, in fact $\#f(V_0') = \#f(V_1) = \#f(V_2) = 5$. By the assumption above, $f(v_{2,1})$ is at the middle of $f(V_2)$. By Claim 1, $f(v_{1,1})$ is not at the middle of $f(V_1)$ and $f(v_{0,1})$ is not at the middle of $f(V_0')$. These then violate Claim 1 if we consider the parts V_0' and V_1. Thus, Claim 3 holds.

Claim 4. *If* $n \geqslant 4$, *then* $\#f(v_{0,4}) \geqslant 3$.

Suppose $\#f(v_{0,1}) \leqslant \#f(v_{0,2}) \leqslant \#f(v_{0,3}) \leqslant \#f(v_{0,4}) \leqslant 2$. Then, by Claim 3, $\#f(v_{0,1}) = \#f(v_{0,2}) = \#f(v_{0,3}) = \#f(v_{0,4}) = 2$. Since $\#f(v_{2,1}) = 1$, there exists an interval $I_{0,i}$ in $f(v_{0,i})$ such that $f(v_{2,1}) \cap I_{0,i} \neq \emptyset$ for $1 \leqslant i \leqslant 4$ (see labels ① in Fig. 3). We may assume that $I_{0,1}, I_{0,2}, I_{0,3}, I_{0,4}$ are from left to right in this order. Then $I_{0,2}$ and $I_{0,3}$ are properly contained in $f(v_{2,1})$. Therefore, $f(v_{0,2})$ has another interval $J_{0,2}$ and $f(v_{0,3})$ has another interval $J_{0,3}$ such that $J_{0,a''}$ intersects $I_{2,a'}$ and $I_{2,b'}$, and $J_{0,b''}$ intersects $J_{2,a'}$ and $J_{2,b'}$, where $\{a'', b''\} = \{2, 3\}$. If $a'' = 2$ and $b'' = 3$, then $J_{0,2}$ does not intersect $f(v_{1,a})$. In this case, $I_{0,2}$ must intersect $I_{1,a}$, which imply that $f(v_{0,3}) = I_{0,3} \cup J_{0,3}$ does not intersect $f(v_{1,1})$, a contradiction. Thus $a'' = 3$ and $b'' = 2$, see labels ② in Fig. 3. (We only draw $I_{0,i}$'s and $J_{0,j}$'s for Case 2. Other cases are similar.)

As $J_{0,2}$ does not intersect $f(v_{1,1})$, the interval $I_{0,2}$ must intersect $f(v_{1,1})$ (see label ③ in Fig. 3). Also, as $J_{0,3}$ does not intersect $f(v_{1,a})$, the interval $I_{0,3}$ must intersect

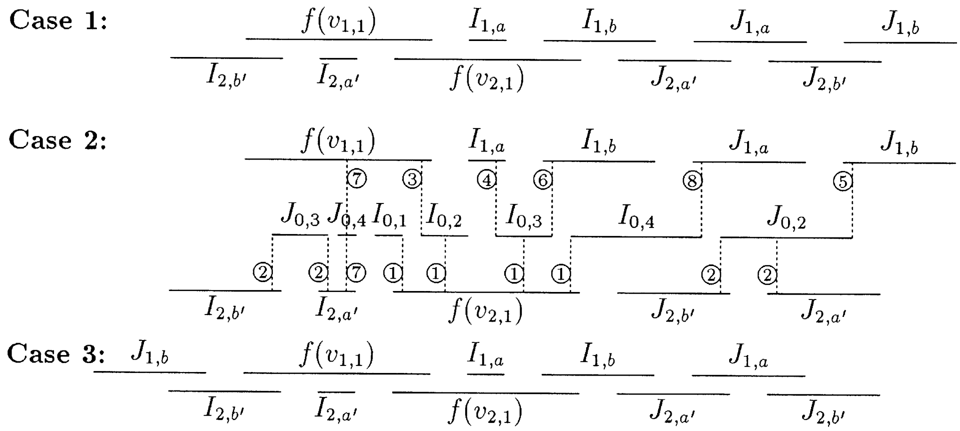

Fig. 3. Dotted lines with labels show the sequence of necessary intersections.

$I_{1,a}$ (see label ④ in Fig. 3). Then, as $I_{0,2}$ does not intersect $I_{1,b}$ and $J_{1,b}$, $J_{0,2}$ must intersect $J_{1,b}$ as shown in Fig. 3 (see label ⑤). Note that this is possible only for Cases 1 and 2. Also, as $J_{0,3}$ does not intersect $f(v_{1,b})$, the interval $I_{0,3}$ must intersect $I_{1,b}$ (see label ⑥ in Fig. 3). Next, since $I_{0,4}$ does not intersect $f(v_{1,1})$, the other interval $J_{0,4}$ of $f(v_{0,4})$ must intersect $f(v_{1,1})$ and $f(v_{2,a'})$ (see labels ⑦ in Fig. 3) and so not intersect $f(v_{1,a})$. However, $I_{0,4}$ does not intersect $I_{1,a}$. So, $I_{0,4}$ must intersect $J_{1,a}$ as shown in Fig. 3 (see label ⑧). Finally, as $I_{0,1}$ does not intersect $f(v_{1,a})$, $f(v_{1,b})$ and $f(v_{2,b'})$, the set $f(v_{0,1})$ must has another interval $J_{0,1}$ intersecting $f(v_{1,a})$, $f(v_{1,b})$ and $f(v_{2,b'})$. But this is impossible as Cases 1 and 2 of Fig. 3 show. This completes the proof of Claim 4.

According to Claims 2–4, we have $\#f(V_0)=\#f(v_{0,1})+\#f(v_{0,2})+\cdots+\#f(v_{0,n}) \geqslant 3n-3$. By the same arguments as proving Claim 3, we have $\#f(V_3) \geqslant 6$. Therefore,

$$f(V_{n,[r]*3}) = \sum_{i=0}^{r} \#f(V_i) \geqslant (3n-3)+5+5+6(r-2) = 3n+6r-5.$$

On the other hand, according to Theorem 3, $I(K_{n,[r]*3}) \leqslant 3n+6r-5$. (Note that we need to consider the cases of $n=2$ and $n \geqslant 3$ separately.) Therefore, $I(K_{n,[r]*3})=3n+6r-5$. □

Corollary 8. *If $r \geqslant 2$, then $I(K_{[r]*3}) = 6r-2$.*

Proof. $I(K_{[r]*3}) = (K_{3,[r-1]*3}) = 3\times 3+6(r-1)-5 = 6r-2$. □

5. The graphs $K_{[r]*3,[s]*2}$ and $K_{4,[r]*3,[s]*2}$

We now investigate the graphs $K_{[r]*3,[s]*2}$ and $K_{4,[r]*3,[s]*2}$.

Theorem 9. *If $r \geqslant 1$ and $s \geqslant 0$ and $r+s \geqslant 2$, then $I(K_{[r]*3,[s]*2}) = 6r+3s-2$.*

Proof. The theorem follows from Theorem 5 when $r = 1$. So, we may assume that $r \geqslant 2$. Suppose $V(K_{[r]*3,[s]*2}) = V_1 \cup V_2 \cup \cdots \cup V_{r+s}$, where $V_1, V_2, \ldots, V_{r+s}$ are the partite sets of $K_{[r]*3,[s]*2}$ such that $|V_i| = 3$ for $1 \leqslant i \leqslant r$ and $|V_j| = 2$ for $r+1 \leqslant j \leqslant r+s$. Choose an optimal multiple-interval representation f of $K_{[r]*3,[s]*2}$. Without loss of generality, we may assume that $\#f(V_1) \leqslant \#f(V_2) \leqslant \cdots \leqslant \#f(V_r)$ and $\#f(V_{r+1}) \leqslant \#f(V_{r+2}) \leqslant \cdots \leqslant \#f(V_{r+s})$.

We first consider the case when $\#f(V_2) \geqslant 6$. According to Lemma 2 and Theorem 5, $\#f(V_1) + \sum_{i=r+1}^{r+s} \#f(V_i) \geqslant I(K_{3,[s]*2}) \geqslant 2 \times 3 + 3s - 2 = 3s + 4$. Then,

$$I(K_{[r]*3,[s]*2}) = \sum_{i=1}^{r+s} \#f(V_i) \geqslant (3s + 4) + 6(r - 1) = 6r + 3s - 2.$$

We now may assume that $\#f(V_2) \leqslant 5$. According to Lemma 2 and Theorem 1, we have $\#f(V_1) + \#f(V_2) \geqslant I(K_{3,3}) = 10$. Then, $\#f(V_1) = \#f(V_2) = 5$. The same arguments as in the proof for Claim 2 in Theorem 7 lead to $\#f(V_{r+1}) \geqslant 3$. According to Lemma 2 and Corollary 8, $\sum_{i=1}^{r} \#f(V_i) \geqslant I(K_{[r]*3}) = 6r - 2$. Then,

$$I(K_{[r]*3,[s]*2}) = \sum_{i=1}^{r+s} \#f(V_i) \geqslant (6r - 2) + 3s = 6r + 3s - 2.$$

On the other hand, according to Theorem 3, $I(K_{[r]*3,[s]*2}) \leqslant 6r + 3s - 2$. Thus, $I(K_{[r]*3,[s]*2}) = 6r + 3s - 2$. □

Lemma 10. *If* $r \geqslant 1$ *and* $s \geqslant 1$, *then* $I(K_{4,[r]*3,[s]*2}) \leqslant 6r + 3s + 6$.

Proof. Suppose $V(K_{4,[r]*3,[s]*2}) = V_0 \cup V_1 \cup \cdots \cup V_{r+s}$, where $V_0, V_1, \ldots, V_{r+s}$ are the partite sets of $K_{4,[r]*3,[s]*2}$ with $V_0 = \{v_{0,1}, v_{0,2}, v_{0,3}, v_{0,4}\}$, $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ for $1 \leqslant i \leqslant r$ and $V_j = \{v_{j,1}, v_{j,2}\}$ for $r + 1 \leqslant j \leqslant r + s$. Define function f by

$$f(v_{0,1}) = [-(4 + 2r), -3] \cup [2r + 2s + 3, 2r + 2s + 4],$$

$$f(v_{0,2}) = [-2, -1] \cup [2r + 2s + 5, 4r + 2s + 6],$$

$$f(v_{0,3}) = [-(6r + 6), -(7 + 2r)] \cup [2s - 1, 2s],$$

$$f(v_{0,4}) = [-(6 + 2r), -(5 + 2r)] \cup [2s + 1, 2s + 2r + 2],$$

$$f(v_{i,1}) = \begin{cases} [-(6r + 6), -(2r + 4i + 7)] \cup [-(2 + 2i), 2s + 2i] \\ \quad \text{if } 1 \leqslant i \leqslant r - 1, \\ [-(2 + 2i), 2s + 2i] \\ \quad \text{if } i = r, \end{cases}$$

$$f(v_{i,2}) = [-(2r + 4i + 4), -(3 + 2i)] \cup [2r + 2s + 2i + 5, 4r + 2s + 6],$$

$$f(v_{i,3}) = [-(2r + 4i + 6), -(2r + 4i + 5)] \cup [2s + 2i + 1, 2r + 2s + 2i + 4],$$

$$f(v_{j,1}) = \begin{cases} [-(6r+6), 2(j-r-1)] \cup [2r+2s+2j+5, 2r+2s+2j+6] \\ \quad \text{if } r+1 \leqslant j \leqslant r+s-1, \\ [-(6r+6), 2(j-r-1)] \\ \quad \text{if } j = r+s, \end{cases}$$

$$f(v_{j,2}) = [2j-2r-1, 2r+2s+2j+4]$$

for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant r+s$. It is straightforward to verify that f is a multiple-interval representation of $K_{4,[r]*3,[s]*2}$ with $\#f(K_{4,[r]*3,[s]*2}) = 6r + 3s + 6$. Hence, $I(K_{4,[r]*3,[s]*2}) \leqslant 6r + 3s + 6$. □

Note that the upper bound in Lemma 10 improves the upper bound given in Theorem 3 by 1.

Corollary 11. *If $r \geqslant 1$ and $s \geqslant 1$, then $I(K_{4,[r]*3,[s]*2}) = 6r + 3s + 6$.*

Proof. Suppose $V(K_{4,[r]*3,[s]*2}) = V_0 \cup V_1 \cup \cdots \cup V_{r+s}$, where $V_0, V_1, \ldots, V_{r+s}$ are the partite sets of $K_{4,[r]*3,[s]*2}$ with $|V_0| = 4, |V_i| = 3$ for $1 \leqslant i \leqslant r$ and $|V_j| = 2$ for $r+1 \leqslant j \leqslant r+s$. Suppose f is an optimal multiple-interval representation of $K_{4,[r]*3,[s]*2}$. According to Lemma 2 and Theorem 5, we have

$$\#f(V_0) + \sum_{i=r+1}^{r+s} \#f(V_i) \geqslant I(K_{4,[s]*2}) = 2 \times 4 + 3s - 2 = 3s + 6.$$

According to Lemma 2 and Theorem 7, we have

$$\#f(V_0) + \sum_{i=1}^{r} \#f(V_i) \geqslant I(K_{4,[r]*3}) = 3 \times 4 + 6r - 5 = 6r + 7.$$

According to Lemma 2 and Theorem 9, we have

$$\sum_{i=1}^{r+s} \#f(V_i) \geqslant I(K_{[r]*3,[s]*2}) = 6r + 3s - 2.$$

Summing up these three inequalities, we have

$$\sum_{i=0}^{r+s} \#f(V_i) \geqslant 6r + 3s + 5.5$$

and so $I(K_{4,[r]*3,[s]*2}) \geqslant 6r + 3s + 6$. This, together with Lemma 10, implies that $I(K_{4,[r]*3,[s]*2}) = 6r + 3s + 6$. □

6. Discussions

In this paper, we establish an upper for the total interval numbers of complete r-partite graphs. In fact, our main concern is on the balanced complete r-partite graphs

$K_{[r]*n}$. By using an argument similar to that in the proof of Theorem 5, we may get the lower bound

$$n^2 + 1 + (r-2)\left\lceil \frac{n^2+1}{2} \right\rceil \leqslant I(K_{[r]*n}).$$

The lower bound has a gap $(r-2)\lfloor (n-1)/2 \rfloor$ from the upper bound in Corollary 4. When $r = 2$ or $n \leqslant 2$, the lower bound is in fact equals to the upper bound. The case when $r = n = 3$ has a gap of 1. The long proof in Theorem 7 establishes that $I(K_{3,3,3})$ is equal to the upper bound 16. In general, we believe that $I(K_{[r]*n})$ attains the upper bound although we are still far from a proof.

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