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Note

Upper bounds for the k-subdomination number of graphs

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Abstract

For a positive integer k, a k-subdominating function of $G=(V,E)$ is a function $f:V \to \{-1,1\}$ such that the sum of the function values, taken over closed neighborhoods of vertices, is at least one for at least k vertices of G. The sum of the function values taken over all vertices is called the aggregate of f and the minimum aggregate among all k-subdominating functions of G is the k-subdomination number $\gamma_{ks}(G)$. In this paper, we solve a conjecture proposed in (Ars. Combin 43 (1996) 235), which determines a sharp upper bound on $\gamma_{ks}(G)$ for trees if $k > |V|/2$ and give an upper bound on γ_{ks} for connected graphs. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs under consideration are simple. For a graph $G=(V,E)$ and vertex $v \in V$, let $N(v)=\{u \in V : uv \in E\}$ and $N[v]=\{v\} \cup N(v)$ be the *open* and *closed neighborhoods* of v in G, respectively. For a subset A of V, we set $N_A(v)=N(v) \cap A$ and $d_A(v)=|N_A(v)|$. For $k \in Z^+$, a *k-subdominating function* (kSF) of G is a function $f:V \to \{-1,1\}$ such that $f[v]=\sum_{u \in N[v]} f(u) \geqslant 1$ for at least k vertices v of G. The *aggregate* $\mathrm{ag}(f)$ of such a function is defined by $\mathrm{ag}(f)=\sum_{v \in V} f(v)$ and the *k-subdomination number* $\gamma_{ks}(G)$ by $\gamma_{ks}(G)=\min\{\mathrm{ag}(f)\colon f \text{ is } k\text{SF of } G\}$. The concept of k-subdominatoin number was introduced and first studied by Cockayne and Mynhardt [2]. In the special cases where $k=|V|$ and $k=\lceil |V|/2 \rceil$, γ_{ks} is respectively the signed domination number γ_s [3] and the majority domination number γ_{maj} [1].

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In [2], Cockayne et al. established a sharp lower bound on γ_{ks} for trees. Moreover, they also gave a sharp upper bound on γ_{ks} for trees if $k \leqslant |V|/2$, and proposed the following two conjectures:

Conjecture 1. For any n-vertex tree T and any k with $n/2 < k \leqslant n$, $\gamma_{ks}(T) \leqslant 2k - n$.

Conjecture 2. For any connected graph G of order n and any k with $n/2 < k \leqslant n$, $\gamma_{ks}(G) \leqslant 2k - n$.

In this paper, we show that Conjecture 1 is true and Conjecture 2 is incorrect, and give an upper bound for the k-subdomination number of graphs.

2. An upper bound on the k-subdomination number for trees

Alon (mentioned in [2]) established the following upper bounds on γ_{ks} for a connected graph.

Theorem A (Cockayne and Mynhardt [2]).
For any connected graph G of order n,

$$\gamma_{\mathrm{maj}}(G) \leqslant \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

And Henning and Hind [4] proved the following:

Theorem B (Henning and Hind [4]).
If T is a tree of order n, $k = \lceil (n+1)/2 \rceil$, then $\gamma_{ks}(T) \leqslant 2$.

In order to prove Conjecture 1, we need some definitions from [2]. A *leaf* of a tree is a vertex of degree one and a *remote vertex* of a tree is a vertex having exactly one non-leaf neighbor. We write L and R for the sets of leaves and remote vertices of T, respectively.

Theorem 1. *For any n-vertex tree T and k with $n/2 < k \leqslant n$, $\gamma_{ks}(T) \leqslant 2k - n$.*

Proof. To prove the theorem, by the definition of γ_{ks}, it suffices to show that there exists a k-subdominating function f with $\mathrm{ag}(f) \leqslant 2k - n$.

First we may suppose $k > \lceil (n+1)/2 \rceil$ by Theorem B and $k < n$ for $f(v) = 1$ for all $v \in V$ is an n-subdominating function with $\mathrm{ag}(f) = n$.

Now we apply induction on the number of vertices of T. Note that the assertion is true for $n \leqslant 4$, so suppose that $n \geqslant 5$ and the theorem holds for smaller values of n. Suppose furthermore T is not a star since for star T with leaves $v_1, v_2, \ldots, v_{n-1}$ there exists a k-subdominating function

$$f(x)=\begin{cases}-1 & \text{for } x=v_i,\ i=1,\dots,n-k,\\ 1 & \text{otherwise}\end{cases}$$

with $\mathrm{ag}(f)=2k-n$. Thus $|R|\geqslant 2$.

Suppose that there exists a vertex $u\in R$ such that $d(u)$ is even, and that v' is a leaf adjacent to u. Then for the subtree $T_1=T-v'$ and k with $(n-1)/2<k\leqslant n-1$, by the induction hypothesis, there exists a k-subdominating function f_1 on T_1 with $\mathrm{ag}(f_1)\leqslant 2k-(n-1)=2k-n+1$. We define

$$f(x)=\begin{cases}-1 & \text{if } x=v',\\ f_1(x) & \text{otherwise.}\end{cases}$$

Then f is a k-subdominating function on T. Indeed, if $f_1[u]<1$ in T_1, f is a k-subdominating function on T. And if $f_1[u]\geqslant 1$ in T_1, then $f_1[u]\geqslant 2$ as $d(u)$ is even, hence f is also a k-subdominating function on T. Clearly, $\mathrm{ag}(f)=\mathrm{ag}(f_1)-1\leqslant 2k-n$.

Hence we may suppose that $d(u)$ is odd for all $u\in R$, hence $d(u)\geqslant 3$ as u is not a leaf of T. Take a $u'\in R$, write $d(u')=2s+1$ and $N(u')=\{v_1,v_2,\dots,v_{2s},v_{2s+1}\}$, where v_{2s+1} is the unique non-leaf neighbor of u'. We separate three cases according to the values of k.

Case 1: $n-s\leqslant k\leqslant n-1$.

Define

$$f(x)=\begin{cases}-1 & \text{if } x=v_i,\ i=1,2,\dots,n-k,\\ 1 & \text{otherwise.}\end{cases}$$

Then it is easily seen that $f[x]\geqslant 1$ if $x\neq v_i$, $i=1,2,\dots,n-k$, thus f is a k-subdominating function on T with $\mathrm{ag}(f)=2k-n$.

Case 2: $(n+3)/2<k\leqslant n-s-1$ ($\Rightarrow n\geqslant 8$ as $s\geqslant 1$).

Put $k_1=k-s-2$ and $n_1=n-2s-1$, then $\frac{1}{2}n_1<k_1<n_1$. Now consider the subtree $T_1=T-(N[u']\setminus\{v_{2s+1}\})$ of order $n_1<n$. By the induction hypothesis, there exists a k_1-subdominating function f_1 on T_1 with $\mathrm{ag}(f_1)\leqslant 2k_1-n_1=2k-n-3$. Define

$$f(x)=\begin{cases}f(x) & \text{if } x\in V(T_1),\\ -1 & \text{if } x=v_i,\ i=1,2,\dots,s-1,\\ 1 & \text{otherwise.}\end{cases}$$

Clearly, f is a k-subdominating function on T with $\mathrm{ag}(f)=\mathrm{ag}(f_1)+3\leqslant 2k-n$.

Case 3: $\lceil (n+1)/2\rceil<k\leqslant (n+3)/2$.

Then $n=2k-3$. To complete the proof, it suffices to show that there exists a k-subdominating function f on T with $\mathrm{ag}(f)\leqslant 2k-n=3$. For this purpose, among all partitions $\{W_1,W_2\}$ of V with $||W_1|-|W_2||\leqslant 1$, called equipartitions, choose one such that the number of edges between W_1 and W_2 is minimum, assume $|W_2|=k-1$ and $|W_1|=k-2$. Define a function $\delta(v)=d_{W_i}(v)-d_{W_{3-i}}(v)$ for every $v\in W_i$, let G_i denote the subgraph induced by W_i, and let L_i and S_i denote the sets of vertices $v\in W_i$ satisfying $d_{W_i}(v)=1$ and $|N(v)\cap L_i|\geqslant\lceil\delta(v)/2\rceil$, respectively, $i=1,2$.

Claim 1. $\delta(v) > 0$ *for all* $v \in V$ *except at most one* $v^* \in W_2$ *with* $\delta(v^*) = 0$.

First $\delta(v) \geqslant 0$ for all $v \in W_2$. Otherwise, moving a $v \in W_2$ with $\delta(v) < 0$ to W_1, we obtain a new equipartition with fewer edges between its parts. Also, $\delta(v) > 0$ for all $v \in W_1$. Otherwise, taking $u \in W_1$ with $\delta(u) \leqslant 0$, we obtain a k-subdominating function of $\mathrm{ag}(f) = 2k - n$ by making u and all of W_2 positive, all remaining vertices negative, as $f[u] = 1 - \delta(u) \geqslant 1$.

Furthermore, if there exist two distinct vertices $v_1, v_2 \in W_2$ with $\delta(v_1) = \delta(v_2) = 0$, then we have a k-subdominating function of $\mathrm{ag}(f) = 3$ by letting the positive set of f consist of v_1, v_2 and all of W_1.

Claim 2. (a) $d_{W_i}(v) \geqslant 1$ *for all* $v \in W_i$, $i = 1, 2$.

(b) $v \in L$ *for all* $v \in L_i$, $i = 1, 2$, *except at most one* $v^* \in W_2$ *with* $\delta(v^*) = 0$ $(d_{W_2}(v^*) = 1,\ d(v^*) = 2)$.

(c) $|L_i| \geqslant 2$, $i = 1, 2$.

Indeed, $d_{W_i}(v) \geqslant \lceil d(v)/2 \rceil \geqslant 1$ and by Claim 1, for all $v \in W_i$ with $d_{W_i}(v) = 1$ except v^*, $d(v) = 2d_{W_i}(v) - \delta(v) \leqslant 1$, yielding (a) and (b). (c) follows from G_i being acyclic.

Claim 3. $S_i \neq \emptyset$, $i = 1, 2$.

To see this, let $P = v_1 v_2 \cdots v_{l+1}$ be a longest path in G_i. Then obviously $l \geqslant 1$ by Claim 2(a). Moreover, $v_l \in S_i$. Otherwise, there exists a path $v_l v' v''$ in G_i with $v' \neq v_{l-1}$, and $P' = v_1 v_2 \cdots v_l v' v''$ is a path longer than P.

If $\lceil \delta(u)/2 \rceil \leqslant \lfloor \delta(v)/2 \rfloor$ for some $u \in S_1$ and some $v \in S_2$, then $\lceil \delta(u)/2 \rceil \leqslant |N(u) \cap L_1|$ and $\lceil \delta(u)/2 \rceil \leqslant |N(v) \cap L_2|$ by the definition of S_i. Let $Q_1 \subseteq N(u) \cap L_1$ and $Q_2 \subseteq N(v) \cap L_2$ be sets of $\lceil \delta(u)/2 \rceil$ vertices, respectively. By Claim 2(b), $w \in L$ for all vertices $w \in Q_1 \cup (Q_2 - \{v^*\})$. Define

$$f(x) = \begin{cases} -1 & \text{if } x \in Q_2 \cup W_1 \backslash (\{u\} \cup Q_1), \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, f is a k-subdominating function on T with $\mathrm{ag}(f) = 3$ if $f[u] \geqslant 1$. And if $f[u] \leqslant 0$, then the exceptional vertex $v^* \in N(u) \cap Q_2$, implying $f[v^*] = f(u) + f(v) - 1 = 1$ by Claim 2(b), which guarantees that f is still a k-subdominating function with $\mathrm{ag}(f) = 3$.

So, suppose $\lceil \delta(u)/2 \rceil > \lfloor \delta(v)/2 \rfloor$ for all $u \in S_1$ and all $v \in S_2$. Thus, for all $u \in S_1$ and all $v \in S_2$, $\lceil \delta(u)/2 \rceil \geqslant \lfloor \delta(v)/2 \rfloor + 1$, so that $\lceil \delta(u)/2 \rceil \geqslant \lceil \delta(v)/2 \rceil$. Let $u \in S_1$ and let $v \in S_2$. Then $|N(u) \cap L_1| \geqslant \lceil \delta(v)/2 \rceil > \lceil \delta(v)/2 \rceil - 1$ and $|N(v) \cap L_2| \geqslant \lceil \delta(v)/2 \rceil$. Let $Q_1 \subseteq N(u) \cap L_1$ and $Q_2 \subseteq N(v) \cap L_2$ be sets of $\lceil \delta(v)/2 \rceil - 1$ and $\lceil \delta(v)/2 \rceil$ vertices, respectively, and define

$$f(x) = \begin{cases} -1 & \text{if } x \in Q_1 \cup W_2 \backslash (\{v\} \cup Q_2), \\ 1 & \text{otherwise.} \end{cases}$$

As before, it follows that f is a k-subdominating function with $\mathrm{ag}(f)=3$. Theorem 1 is proved. □

Note that $\gamma_{ks}(K_{1,n-1})=2k-n$ if $k>\frac{1}{2}n$. The bound established in Theorem 1 is sharp indeed.

3. An upper bound on the k-subdomination number for graphs

Conjecture 2 is shown in [4] to be false in the special case when $k=\lceil (n+1)/2\rceil$. The conjecture has yet to be settled when $\lceil (n+1)/2\rceil < k \leqslant n$. In this section, we prove the conjecture in the special case when $n-k+1$ divides k. For this purpose, we shall need the following result.

Theorem 2. *For any connected graph G of order n and any k with $\frac{1}{2}n<k\leqslant n$,*

$$\gamma_{ks}(G)\leqslant 2\left\lceil \frac{k}{n-k+1}\right\rceil (n-k+1)-n.$$

Proof. Among all partitions $\{A'_{11},A'_{12}\}$ of $V(G)$ with $|A'_{11}|=k$ and $|A'_{12}|=n-k$, let $\{A_{11},A_{12}\}$ be one such that the number of edges between A_{11} and A_{12} is minimum. Note that for any $u\in A_{11}$ and $v\in A_{12}$, if $uv\notin E(G)$, then

$$d_{A_{11}}(u)+d_{A_{12}}(v)\geqslant d_{A_{12}}(u)+d_{A_{11}}(v). \tag{1}$$

And if $uv\in E(G)$, then

$$d_{A_{11}}(u)+d_{A_{12}}(v)\geqslant d_{A_{12}}(u)+d_{A_{11}}(v)-2. \tag{2}$$

Otherwise the exchange of u and v yields a partition with fewer edges between its parts.

If $d_{A_{11}}(u)\geqslant d_{A_{12}}(u)$ for each $u\in A_{11}$, we define

$$f(x)=\begin{cases} 1 & \text{if } x\in A_{11},\\ -1 & \text{if } x\in A_{12}.\end{cases}$$

Then clearly f is a k-subdominating function on G with $\mathrm{ag}(f)\leqslant 2k-n$. Thus we may assume there exists a vertex $u_1\in A_{11}$ with $d_{A_{11}}(u_1)<d_{A_{12}}(u_1)$. Then for any $v\in A_{12}$, using (1) and (2), we have

$$\begin{aligned} d_{A_{12}}(v)&>d_{A_{11}}(v) && \text{if } v\notin N(u_1),\\ d_{A_{12}}(v)&\geqslant d_{A_{11}}(v)-1 && \text{if } v\in N(u_1).\end{aligned}$$

Among all partitions $\{A'_{21},A'_{22}\}$ of $A_{11}-\{u_1\}$ with $|A'_{21}|=2k-n-1$ and $|A'_{22}|=n-k$, let $\{A_{21},A_{22}\}$ be one such that the number of edges joining vertices in A_{21} to vertices in A_{22} is minimum. If $d_{A_{21}}(u)\geqslant d_{A_{22}}(u)$ for each $u\in A_{21}$, define

$$f(x)=\begin{cases} 1 & \text{if } x\in A_{21}\cup A_{12}\cup\{u_1\},\\ -1 & \text{if } x\in A_{22}.\end{cases}$$

It is easily seen that f is a k-subdominating function of $\mathrm{ag}(f) \leqslant 2k - n$, hence $\gamma_{ks}(G) \leqslant 2k-n$. So we may assume there exists $u_2 \in A_{21}$ such that $d_{A_{21}}(u_2) < d_{A_{22}}(u_2)$. For any $v \in A_{22}$, by the choice of $\{A_{21}, A_{22}\}$, similarly we have

$$d_{A_{22}}(v) > d_{A_{21}}(v) \quad \text{if } v \notin N(u_2),$$
$$d_{A_{22}}(v) \geqslant d_{A_{21}}(v) - 1 \quad \text{if } v \in N(u_2).$$

For $A_{21} - \{u_2\}$, a similar argument shows that either $\gamma_{ks}(G) \leqslant 2k - n$ or there exists $u_i \in A_{i1},\ i = 1, 2, \ldots, \lceil k/(n-k+1) \rceil$, such that $d_{A_{i1}}(u_i) < d_{A_{i2}}(u_i)$ and

$$d_{A_{i2}}(v) > d_{A_{i1}}(v) \quad \text{if } v \notin N(u_i),$$
$$d_{A_{i2}}(v) \geqslant d_{A_{i1}}(v) - 1 \quad \text{if } v \in N(u_i).$$

Define

$$f(u) = \begin{cases} 1 & \text{if } u \in A_{12} \cup A_{22} \cup \ldots \cup A_{\lceil k/(n-k+1) \rceil 2} \cup \{u_1, u_2, \ldots, u_{\lceil k/(n-k+1) \rceil}\}, \\ -1 & \text{otherwise.} \end{cases}$$

f is a k-subdominating function on G with

$$\mathrm{ag}(f) \leqslant 2\lceil k/(n-k+1) \rceil (n-k+1) - n.$$

The proof of Theorem 2 is complete. □

Corollary 1. *Let G be a connected graph of order n and k an integer with* $n/2 < k \leqslant n$. *If* $n - k + 1 | k$, *then* $\gamma_{ks}(G) \leqslant 2k - n$.

Thus Conjecture 2 *is true if* $n - k + 1 | k$.

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