On convex bodies of constant width

L.E. Bazylevych\textsuperscript{a}, M.M. Zarichnyi\textsuperscript{b,c,*}

\textsuperscript{a} National University “Lviv Polytechnica”, 12 Bandery Str., 79013 Lviv, Ukraine
\textsuperscript{b} Lviv National University, 1 Universytetska Str., 79000 Lviv, Ukraine
\textsuperscript{c} Institute of Mathematics, University of Rzeszów, 16A Rejtana Str., 35-310 Rzeszów, Poland

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Dedicated to Professor E.D. Tymchatyn on the occasion of his 60th anniversary

Abstract

We present an alternative proof of the following fact: the hyperspace of compact closed subsets of constant width in $\mathbb{R}^n$ is a contractible Hilbert cube manifold. The proof also works for certain subspaces of compact convex sets of constant width as well as for the pairs of compact convex sets of constant relative width. Besides, it is proved that the projection map of compact closed subsets of constant width is not 0-soft in the sense of Shchepin, in particular, is not open.

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The topology of the hyperspace of compact convex sets in euclidean spaces was investigated by different authors; see, e.g. [8,7,2,1].

In this note we consider some topological properties of (the maps of) compact convex bodies of constant width. A convex set in euclidean space is said to be of constant width $d$ if the distance between two supporting hyperplanes equals $d$ in every direction. To be more formal, denote by $h_K : S^{n-1} \to \mathbb{R}$ the support function of a convex body $K$ in $\mathbb{R}^n$ defined as follows: $h_K(u) = \max\{ \langle x, u \rangle \mid x \in K \}$. Here, as usual, $\langle \cdot, \cdot \rangle$ stands for the standard inner product.

* Corresponding author.
E-mail address: mzar@litech.lviv.ua (M.M. Zarichnyi).

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product in $\mathbb{R}^n$ and $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. The widths function of $K$ is the function $w_K : S^{n-1} \to \mathbb{R}$ defined by the formula $w_K(u) = h_K(u) - h_K(-u)$. A convex body $K$ is of constant width $d > 0$ provided $w_K$ is a constant function taking the value $d$.

It was proved by the first-named author [2] that the hyperspace of compact convex bodies of constant (non-specified) width in euclidean space of dimension $\geq 2$ is homeomorphic to the punctured Hilbert cube. In Section 1 we present a more direct proof of this result. The technique allows also to prove that some subspaces of the above mentioned hyperspace as well as the hyperspace of pairs of compact convex sets of constant relative width are manifolds modeled on the Hilbert cube ($Q$-manifolds).

In connection to the topology of the hyperspace of compact convex sets of nonmetrizable compact subsets in locally convex spaces it was proved in [10] that, for any affine continuous onto map of convex subsets in metrizable locally convex spaces, the natural topology of the hyperspaces of compact convex subsets is soft in the sense of Shchepin [9]. In Section 2 we demonstrate that this is not the case if we restrict ourselves with the compact $\mathbb{R}$-manifolds.

1. Hyperspaces of compact convex bodies of constant width

By $cc(\mathbb{R}^n)$ we denote the hyperspace of compact convex subsets in $\mathbb{R}^n$. We equip $cc(\mathbb{R}^n)$ with the topology generated by the Hausdorff metric.

By $cw_d(\mathbb{R}^n)$ we denote the subset of $cc(\mathbb{R}^n)$ consisting of convex bodies of constant width $d > 0$ in $\mathbb{R}^n$. We also put $cw_0(\mathbb{R}^n) = \{x \mid x \in \mathbb{R}^n\}$. Note that, for every $A_i \in cw_d(\mathbb{R}^n)$, $i = 1, 2$, and every $t \in [0, 1]$ we have $tA_1 + (1 - t)A_2 \in cw_d(\mathbb{R}^n)$, where $d = td_1 + (1 - t)d_2$. In [2] it is proved that the hyperspace $cw(\mathbb{R}^n) = \bigcup_{d > 0} cw_d(\mathbb{R}^n)$, $n \geq 2$, is a contractible Hilbert cube manifold. Here we essentially simplify the proof of this result.

**Theorem 1.1.** Let $D \subset [0, \infty)$ be a convex subset such that $D \cap (0, \infty) \neq \emptyset$. The hyperspace $\bigcup\{cw_d(\mathbb{R}^n) \mid d \in D\}$ is a contractible Hilbert cube manifold.

**Proof.** First, we embed $cw(\mathbb{R}^n)$ as a convex subset of a Banach space. By $C(S^{n-1})$ we denote, as usual, the Banach space of continuous real-valued functions endowed with the sup-norm.

Define a map $\varphi : cw(\mathbb{R}^n) \to C(S^{n-1})$ by the formula $\varphi(K)(x) = h_K(x)$, $K \in cw(\mathbb{R}^n)$. It is a well-known fact that $\varphi$ is a continuous map. Moreover, it is obvious that $\varphi$ is an embedding which is an affine map in the sense that $\varphi(tA + (1 - t)B) = t\varphi(A) + (1 - t)\varphi(B)$ for every $A, B \in cw(\mathbb{R}^n)$ and $t \in [0, 1]$. The image of $\varphi$ is a locally compact convex subset of $C(S^{n-1})$.

We are going to prove that, for any $d > 0$, the space $cw_d(\mathbb{R}^n)$ is infinite-dimensional.

First consider the case $n = 2$. Let $K$ denote the Reuleaux triangle in $\mathbb{R}^2$ that is the intersection of the closed balls of radius $d$ centered at $(0, 0)$, $(d, 0)$, and $(d/2, d\sqrt{3}/2)$. For any $\alpha \in [0, 2\pi]$, denote by $K_\alpha$ the convex body obtained by rotation of $K$ by angle $\alpha$ counterclockwise around the origin. Note that the set $\{K_\alpha \mid \alpha \in [0, 2\pi]\}$ is contained in $cw_d(\mathbb{R}^2)$ and we are going to show that this set contains a linearly independent subset of
arbitrary finite cardinality. To this end, one has to demonstrate that the family of the support functions \( h_{K_\alpha} : S^1 \rightarrow \mathbb{R} \) contains linearly independent subset of arbitrary finite cardinality.

We identify \( S^1 \) with the subset \( \{ e^{it} \mid t \in [0, 2\pi] \} \) of the complex plane. It is easy to see that \( h_{K_\alpha}(e^{it}) = h_K(e^{i(\theta-t)}) \). Elementary geometric arguments demonstrate that

\[
h_K([0, \pi/3]) = 1, \quad h_K([\pi, 4\pi/3]) = 0, \quad h_K((\pi/3, \pi) \cup (4\pi/3, 2\pi)) \subset (0, 1).
\]

Fix a natural number \( k \). For each \( j = 0, 1, \ldots, k - 1 \), let \( h_j = h_{K_{(j\pi/3)}/(3k)} \). In order to demonstrate that the functions \( h_j \), \( j = 0, 1, \ldots, k \), are linearly independent, consider a linear combination \( g = \sum_{j=0}^{k} \lambda_j h_j \). Suppose that \( g = 0 \), then

\[
g(\pi/3) = \sum_{j=0}^{k} \lambda_j h_j(\pi/3) = \sum_{j=0}^{k} \lambda_j = 0,
\]

\[
g((\pi/3) + (\pi/(3k))) = \lambda_0 h_0((\pi/3) + (\pi/(3k)))
\]

\[
+ \sum_{j=1}^{k} \lambda_j h_j((\pi/3) + (\pi/(3k))) = 0,
\]

whence \( \lambda_0 = 0 \). Consequently evaluating the function \( g \) at the points \((\pi/3) + (j\pi/(3k)), \)

\( j = 2, \ldots, k \), we conclude that \( \lambda_i = 0 \) for every \( i = 0, 1, \ldots, k \).

In the case \( n > 2 \), consider the family of \( n \)-dimensional simplices in \( \mathbb{R}^n \) all whose edges are of length \( d \). For every such a simplex, \( \Delta \), consider the intersection of all closed balls of radius \( d \) centered at the vertices of \( \Delta \) and containing \( \Delta \). The obtained set, \( L \), is obviously a compact convex subset in \( \mathbb{R}^n \) of constant width \( d \). Denote by \( L \) the family of all compact convex subsets in \( \mathbb{R}^n \) that can be obtained in this way. Let \( \text{pr} : \mathbb{R}^n \rightarrow \mathbb{R}^2 \) denote the projection. This projection generates the map \( \text{cc}(\text{pr}) : \text{cc}(\mathbb{R}^n) \rightarrow \text{cc} \mathbb{R}^m, f(A) = \text{pr}(A) \). The map \( \text{cc}(\text{pr}) \) is an affine map.

Then \( \{\text{cc}(\text{pr})(L') \mid L' \in L \} \supset \{K_\alpha \mid \alpha \in [0, 2\pi]\} \) and we conclude that the space \( L \) and, therefore \( \text{cw}_d(\mathbb{R}^n) \) is infinite-dimensional.

To finish the proof, apply the results on topology of metrizable locally compact convex subsets in locally convex spaces \([3]\). It follows from what we have already proved that the space \( X = \omega(\bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \in D\}) \) is infinite-dimensional. Since the space \( X \) is easily shown to be locally compact and convex, the Keller theorem (see \([3]\)) implies that \( X \) (which is homeomorphic to \( \bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \in D\} \)) is a contractible \( Q \)-manifold. \( \square \)

**Corollary 1.2.** Let \( d_0 \geq 0 \). The hyperspace \( \bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \geq d_0\} \) is homeomorphic to a punctured Hilbert cube \( Q \setminus \{*\} \).

**Proof.** By a result of Chapman \([5]\), it is sufficient to prove that there exists a proper homotopy

\[
H : \bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \geq d_0\} \times [1, \infty) \rightarrow \bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \geq d_0\}.
\]

This homotopy can be defined in an obvious way, \( H(K, t) = t K, (K, t) \in \bigcup \{\text{cw}_d(\mathbb{R}^n) \mid d \geq d_0\} \times [1, \infty) \). \( \square \)
Theorem 1.3. Let $X$ be a convex subset in $\mathbb{R}^n$ that contains a closed cube of side $d$. Then for any convex subset $D$ of $[0, d]$ with $D \cap (0, d) \neq \emptyset$ the set $\text{cw}_D(X) = \text{cc}(X) \cap \text{cw}_D(\mathbb{R}^n)$ is a Hilbert cube manifold.

Proof. Note that the family $\{L - \{x\} \mid L \in \text{cw}_D(X), x \in X\}$ contains the family $\{K_\alpha \mid \alpha \in [0, 2\pi)\}$ defined in the proof of Theorem 1.1 (for fixed $d$). Since $X$ is a subset of a finite-dimensional linear space and $\{K_\alpha \mid \alpha \in [0, 2\pi)\}$ contains an infinite linearly independent family, we conclude that $\text{cw}_D(X)$ also contains an infinite linearly independent family. Therefore, the space $\text{cw}_D(X)$ is infinite-dimensional. Then we apply the arguments of the proof of Theorem 1.1.

Remark that it follows, in particular, from Theorem 1.3 that the hyperspace of convex bodies that can be rotated inside a square is homeomorphic to the Hilbert cube, because this hyperspace is a compact contractible $Q$-manifold (see [4]).

2. On softness of the projection map

A map $f : X \to Y$ is soft (respectively $n$-soft) if for every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & X \\
\downarrow i & & \downarrow f \\
Z & \xrightarrow{\varphi} & Y
\end{array}
\]

where $i : A \to Z$ is a closed embedding into a paracompact space $Z$ (respectively a paracompact space $Z$ of covering dimension $\leq n$), there exists a map $\Phi : Z \to X$ such that $\Phi|A = \psi$ and $f \Phi = \varphi$. The notion of $(n)$-soft map was introduced by Shchepin [9].

Let $\text{pr} : \mathbb{R}^n \to \mathbb{R}^m$ denote the projection, $n \geq m$. As we already remarked, this projection generates the map $\text{cc}(\text{pr}) : \text{cc}(\mathbb{R}^n) \to \text{cc}(\mathbb{R}^m)$ and by

\[p = \text{cw}(\text{pr}) : \text{cw}(\mathbb{R}^n) \to \text{cw}(\mathbb{R}^m)\]

we denote its restriction $\text{cc}(\text{pr})|\text{cw}(\mathbb{R}^n)$. It is proved in [10] that the map $\text{cc}(\text{pr})$ is soft.

The images of the fibers of the map $p$ under the embedding $\varphi$ are obviously convex. It is natural to ask whether the map $p$ is also soft. As the following result shows, the answer turns out to be negative. The idea of the proof is suggested by S. Ivanov.

Theorem 2.1. The map $p : \text{cw}(\mathbb{R}^3) \to \text{cw}(\mathbb{R}^2)$ is not 0-soft.

Proof. Consider the compactum $L'$ in $\mathbb{R}^3$ which is the intersection of the closed balls of radius 2 centered at the points $(0, 1, 0), (0, -1, 0), (-1, 0, \sqrt{2})$, and $(1, 0, \sqrt{2})$ (a Reuleaux tetrahedron). It is well known that $L'$ contains a compactum $L$ of constant width such that $p(L) = K$, where $K$ denotes the disc of radius 1 centered at the origin in $\mathbb{R}^2$. For every $i$, denote by $K_i$ the compactum in $\mathbb{R}^2$ described as follows. Let

\[x_j = (\cos(2\pi j/(2i + 1)), \sin(2\pi j/(2i + 1))), \quad j = 0, 1, \ldots, 2i.\]
The compactum $K_i$ is the intersection of the discs of radius $\|x_0 - x_i\|$ centered at the points $x_j$, $j = 0, 1, \ldots, 2i$. Obviously, $K_i \in \text{cw}(\mathbb{R}^2)$ and $\lim_{i \to \infty} K_i = K$ in the Hausdorff metric.

Let $S = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, $f : \{0\} \to \text{cw}(\mathbb{R}^3)$ be the map that sends 0 into $L$, and $F : S \to \text{cw}(\mathbb{R}^2)$ be the map such that $F(0) = K$ and $F(1/n) = K_n$, $n \in \mathbb{N}$.

Suppose now that the map $p$ is 0-soft. Then there exist a map $G : S \to \text{cw}(\mathbb{R}^3)$ such that $G(0) = f(0) = L$ and $pG = F$. Denoting $G(1/i)$ by $L_i$, we see that $p(L_i) = K_i$ and $\lim_{i \to \infty} L_i = L$. Since every diameter of a convex body (i.e., a segment that connects the points at which two parallel supporting hyperplanes touch the body) of constant width is orthogonal to the supporting planes, for every $i$, there exists a plane in $\mathbb{R}^3$ containing the set $\text{pr}^{-1}(\partial K_i) \cap L_i$. From this we conclude that there exists a plane in $\mathbb{R}^3$ containing the set

$$\text{pr}^{-1}(\partial K) \cap L \supset \{(0, 1, 0), (0, -1, 0), (-1, 0, \sqrt{2}), (1, 0, \sqrt{2})\},$$

a contradiction. □

**Corollary 2.2.** The map $p : \text{cw}(\mathbb{R}^3) \to \text{cw}(\mathbb{R}^2)$ is not open.

### 3. Pairs of compact convex bodies of constant relative width

Two convex bodies $K_1, K_2 \subset \mathbb{R}^n$ are said to be a pair of constant relative width if $K_1 - K_2$ is a ball (Maehara [6]). We denote by $\text{crw}(\mathbb{R}^n)$ the set of all pairs of compact convex bodies of constant relative width. The set $\text{crw}(\mathbb{R}^n)$ is topologized with the subspace topology of $\text{cc}(\mathbb{R}^n) \times \text{cc}(\mathbb{R}^n)$. Note that the space $\text{cw}(\mathbb{R}^n)$ can be naturally embedded into $\text{crw}(\mathbb{R}^n)$ by means of the mapping $A \mapsto (A, A)$.

**Theorem 3.1.** The space $\text{crw}(\mathbb{R}^n)$ is a contractible $Q$-manifold.

**Proof.** First, note that the map $\beta : (A, B) \mapsto (h_A, h_B)$ affinely embeds the space $\text{crw}(\mathbb{R}^n)$ into $C(S^{n-1}) \times C(S^{n-1})$ and the image of this embedding is a convex subset of $C(S^{n-1}) \times C(S^{n-1})$. Since the space $\text{cw}(\mathbb{R}^n)$ is infinite-dimensional, so is $\text{crw}(\mathbb{R}^n)$. It is easy to see that $\text{cw}(\mathbb{R}^n)$ is locally compact. Arguing like in the proof of Theorem 1.1, we conclude that $\text{cw}(\mathbb{R}^n)$ is a contractible $Q$-manifold. □

### 4. Remarks and open questions

The set $\text{cw}_d(\mathbb{R}^n)$ is the preimage of the set of closed balls of radius $d$ under the central symmetry map $c : \text{cc}(\mathbb{R}^n) \to \text{cc}(\mathbb{R}^n)$, $c(A) = (A - A)/2$. Since $\text{cw}_d(\mathbb{R}^n)$ is an absolute retract, this directly leads to the following question.

**Question 4.1.** Is the central symmetry map soft as a map onto its image.

Let $L$ be a Minkowski space (i.e. a finite-dimensional Banach space) of dimension $\geq 2$. Given a compact convex body $K \subset L$, one says that $K$ is of constant width if the
set $K - K$ is a closed ball centered at the origin. A natural question arises: for which spaces $L$ the results of this paper can be extended over the hyperspaces of convex bodies of constant width in $L$? Note that for the space $(\mathbb{R}^n, \| \cdot \|_\infty)$ the hyperspace of convex bodies of constant width coincides with that of closed balls in it and therefore is finite-dimensional. In addition, a counterpart of Theorem 2.1 does not hold for this space.

**Question 4.2.** Describe the topology of the hyperspace of smooth convex bodies of constant width.

A *rotor* of a convex polyhedron is a convex body that can be rotated inside this polyhedron touching all its faces in the process of rotation. Every compact convex body of constant width is a rotor in cubes as well as in some other polyhedra.

**Question 4.3.** Find counterparts of the results of this paper for rotors in another polyhedra (e.g., equilateral triangles).

**References**