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On new symplectic elasticity approach for exact bending solutions of rectangular thin plates with two opposite sides simply supported

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Abstract

This paper presents a bridging research between a modeling methodology in quantum mechanics/relativity and elasticity. Using the symplectic method commonly applied in quantum mechanics and relativity, a new symplectic elasticity approach is developed for deriving exact analytical solutions to some basic problems in solid mechanics and elasticity which have long been bottlenecks in the history of elasticity. In specific, it is applied to bending of rectangular thin plates where exact solutions are hitherto unavailable. It employs the Hamiltonian principle with Legendre's transformation. Analytical bending solutions could be obtained by eigenvalue analysis and expansion of eigenfunctions. Here, bending analysis requires the solving of an eigenvalue equation unlike in classical mechanics where eigenvalue analysis is only required in vibration and buckling problems. Furthermore, unlike the semi-inverse approaches in classical plate analysis employed by Timoshenko and others such as Navier's solution, Levy's solution, Rayleigh–Ritz method, etc. where a trial deflection function is pre-determined, this new symplectic plate analysis is completely rational without any guess functions and yet it renders exact solutions beyond the scope of applicability of the semi-inverse approaches. In short, the symplectic plate analysis developed in this paper presents a breakthrough in analytical mechanics in which an area previously unaccountable by Timoshenko's plate theory and the likes has been trespassed. Here, examples for plates with selected boundary conditions are solved and the exact solutions discussed. Comparison with the classical solutions shows excellent agreement. As the derivation of this new approach is fundamental, further research can be conducted not only on other types of boundary conditions, but also for thick plates as well as vibration, buckling, wave propagation, etc.

Keywords: Eigenvalue; Eigenvector; Hamiltonian; Legendre; Plate bending; Symplectic elasticity; Timoshenko

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1. Introduction

Symplecticity is a mathematical concept of geometry. A symplectic group is a classical group and it was first used and defined by Weyl (1939) by borrowing a term from the Greek. The theory on symplectic geometry can be referred to Koszul and Zou (1986). Since then, the use of symplectic space has been exploited in a number of fields in physics and mathematics for many years particularly in relativity and gravitation (Kauderer, 1994), and classical and quantum mechanics (De Gosson, 2001) including the famous Yang-Mills field theory (Krauth and Staudacher, 2000), etc. In elasticity and Hamiltonian mechanics, the computational approach for symplectic Hamiltonian systems including fluid dynamics was first developed by Feng and his associates including Feng (1985, 1986a,b); Qin (1990); Feng and Qin (1991). Beginning from 1984, Feng proposed symplectic algorithms based on symplectic geometry for Hamiltonian systems with finite and infinite dimensions, and on dynamical systems with Lie algebraic structures, such as contact systems, source free systems, etc, via the corresponding geometry and Lie group. These algorithms are superior to conventional algorithms in many practical applications, such as celestial mechanics, molecular dynamics, etc. The contribution of Feng (1985, 1986a, 1986b, with Qin 1991) in symplectic algorithm was particularly significant and important as stated in a memorial article dedicated to him by Lax (1993).

Unlike Feng and his associates who emphasized on computational algorithm, Zhong and his associates including Zhong (1991, 1992); Yao and Xu (2001); Yao and Yang (2001); Yao et al. (2007) developed a new analytical symplectic elasticity approach for deriving exact analytical solutions to some basic problems in solid mechanics and elasticity since the early 1990s. These problems have long been bottlenecks in the development of history of elasticity. It is based on Hamiltonian principle with Legendre's transformation and analytical solutions could be obtained by expansion of eigenfunctions. It is rational and systematic with a clearly defined, step-by-step derivation procedure. The advantage of symplectic approach with respect to the classical approach by semi-inverse method is at least threefold. First, the symplectic approach alters the classical practice and concept of solution methodology and many basic problems previously unsolvable or too complicated to be solved can hence be resolved accordingly. For instance, the conventional approach in plate and shell theories by Timoshenko has been based on the semi-inverse method with trial 1D or 2D displacement functions, such as Navier's method and the Levy's method for plates. The trial functions, however, do not always exist except in some very special cases of boundary conditions such as plates with two opposite sides simply supported. Using the symplectic approach, trial functions are not required. Second, it consolidates the many seemingly scattered and unrelated solutions of rigid body movement and elastic deformation by mapping with a series of zero and nonzero eigenvalues. Last but not least, the Saint-Venant problems for plain elasticity and elastic cylinders can be described in a new system of equations and solved. The difficulty of satisfying end boundary conditions in conventional problems which could only be covered using the Saint-Vanent principle can also be solved.

As mentioned above, although research on thin plate is abundant and plate bending has been a subject of study in solid mechanics for more than a century and many exact solutions have been developed (Timoshenko and Woinowsky-krieger, 1970; Leissa, 1969, 1973; Hutchinson, 1984, 2004; Eisenberger and Alexandrov, 2003; Cheng et al., 2005), the analytical solutions are rather incomplete. They are based on the semi-inverse method and limited to only plates with two opposite sides simply supported because a trial function satisfying the boundary conditions is indispensable which does, however, not always exist. Meanwhile, a considerable amount of work on numerical analysis of plate bending problems has been developed such as the finite element method (Zienkiewicz and Cheung, 1964; Shih, 1979), finite strip method (Cheung, 1976), and boundary integral equations (Jaswon and Maiti, 1968). Although these analyses are satisfactory, the numerical methods are only able to provide numerical solutions within a limited range of validity and therefore a bird's eye view on the general behaviour of plate bending cannot be observed.

In this paper, the new symplectic approach is further developed to derived analytical, exact bending solutions for bending of rectangular thin plates. It is based on the Kirchhoff's classical plate theory (CPT) assuming that normals to the mid-plane before deformation remain straight and normal to the plane after deformation so that transverse shear strain can be neglected. New exact bending solutions for rectangular thin plates with two opposite sides simply supported are presented and discussed. To verify the accuracy and validity of this method, comparison with established exact solutions are illustrated. The analysis can be further extended to plates with any other combinations of boundary conditions.

2. Symplectic formulation and Hamiltonian variational principle

The coordinate system of a thin, isotropic plate under consideration is illustrated in Fig. 1 where $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$. Exact solution for bending of such a plate is sought. In general, the plate can be subjected to any arbitrary loading profile and for simplicity, a uniformly distributed load q is treated here. Hence, the governing equation is

$$\nabla^2\nabla^2 w = \frac{q}{D} \tag{1}$$

where D is the flexural rigidity, w is the transverse deflection of plate midplane and ∇^2 is the Laplace operator. The strain energy density in terms of curvature is

$$v_\varepsilon(\boldsymbol{\kappa}) = \tfrac{1}{2}\boldsymbol{\kappa}^{\mathrm{T}} C \boldsymbol{\kappa} = \tfrac{1}{2} D[\kappa_x^2 + \kappa_y^2 + 2v\kappa_x\kappa_y + 2(1-v)\kappa_{xy}^2] \tag{2}$$

where v is Poisson's ratio and

$$\boldsymbol{\kappa} = \begin{Bmatrix} \kappa_y \\ \kappa_x \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \text{ and } \boldsymbol{C} = D\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 2(1-v) \end{bmatrix} \tag{3a, b}$$

are the curvature vector and elasticity coefficient matrix of material, respectively.

In accordance with the Hellinger–Reissner variational principle for plane elasticity, the Pro-Hellinger–Reissner variational principle for thin plate bending is introduced as (Hu, 1981; Yao et al., 2007)

$$\begin{aligned}\delta\Pi_2 = \delta\Big\{\int\int_V \Big[\boldsymbol{\kappa}^{\mathrm{T}}\hat{\boldsymbol{E}}(\nabla)\boldsymbol{\phi} - v_\varepsilon(\boldsymbol{\kappa})\Big]\mathrm{d}x\mathrm{d}y - \int_{\Gamma_u}(\phi_s\bar{\kappa}_{ns} + \phi_n\bar{\kappa}_s)\mathrm{d}s - \int_{\Gamma_\sigma}[\kappa_{ns}(\phi_s - \bar{\phi}_s) + \kappa_s(\phi_n - \bar{\phi}_n)]\mathrm{d}s\Big\} \\ = 0\end{aligned} \tag{4}$$

where

$$\hat{\boldsymbol{E}}(\nabla) = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \quad \boldsymbol{\phi} = \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} \text{ and } \begin{Bmatrix} M_y \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{1}{2}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \end{Bmatrix} \tag{5a–c}$$

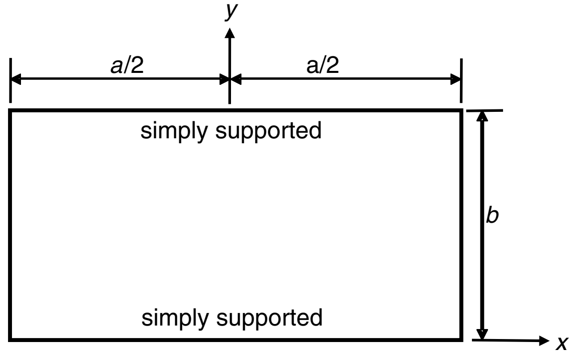

Fig. 1. Coordinate system of a thin plate with dimensions.

are the operator matrix, bending moment function and bending moment, respectively. Subscripts n and s in Eq. (4) indicate directions normal and tangential to the boundary, while Γ_u and Γ_σ are the boundaries with specified geometric conditions (displacements, gradients, etc.) and natural conditions (forces, moments, etc.), respectively. Known constants on the boundaries are denoted by $\bar{\kappa}_s$, $\bar{\kappa}_{ns}$, $\bar{\phi}_s$ and $\bar{\phi}_n$. Substituting Eqs. (2), (3a) and (5a–c) into Eq. (4) and using $M_x = D(\kappa_x + v\kappa_y)$ to eliminate κ_x yields

$$\delta\left\{\int_{x_0}^{x_f}\int_{b_1}^{b_2}\left[\kappa_y\dot{\phi}_x + \kappa_{xy}\dot{\phi}_y - v\kappa_y\frac{\partial\phi_y}{\partial y} + \kappa_{xy}\frac{\partial\phi_x}{\partial y} + \frac{1}{2D}\left(\frac{\partial\phi_y}{\partial y}\right)^2 - D(1-v)\kappa_{xy}^2 - \frac{D(1-v^2)}{2}\kappa_y^2\right]\mathrm{d}y\mathrm{d}x \right.$$
$$\left. - \int_{\Gamma_\sigma}[\kappa_{ns}(\phi_s - \bar{\phi}_s) + \kappa_s(\phi_n - \bar{\phi}_n)]\mathrm{d}s - \int_{\Gamma_u}(\phi_s\bar{\kappa}_{ns} + \phi_n\bar{\kappa}_s)\mathrm{d}s\right\} = 0 \tag{6}$$

where an overdot denotes differentiation with respect to x. The state variables Eq. (6) are ϕ_x, ϕ_y, κ_y and κ_{xy}. The variation of Eq. (6) yields the Hamiltonian dual equation as

$$\dot{v} = \boldsymbol{H}v \tag{7}$$

where the Hamiltonian operator matrix $\boldsymbol{H}$ is defined as

$$\boldsymbol{H} = \begin{bmatrix} 0 & v\frac{\partial}{\partial y} & D(1-v^2) & 0 \\ -\frac{\partial}{\partial y} & 0 & 0 & 2D(1-v) \\ 0 & 0 & 0 & -\frac{\partial}{\partial y} \\ 0 & -\frac{1}{D}\frac{\partial}{\partial y} & v\frac{\partial}{\partial y} & 0 \end{bmatrix} \tag{8}$$

and $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ is the state vector for variables.

Applying the method of separation of variables to v yields

$$v(x,y) = \xi(x)\boldsymbol{\psi}(y) \tag{9}$$

Substituting the above expression into Eq. (2) gives

$$\xi(x) = \mathrm{e}^{\mu x} \tag{10}$$

and the eigenvalue equation

$$\boldsymbol{H}\boldsymbol{\psi}(y) = \mu\boldsymbol{\psi}(y) \tag{11}$$

where μ is the eigenvalue and $\boldsymbol{\psi}(y)$ is the corresponding eigenvector. The eigen-solutions of nonzero eigenvalues in Eq. (11) may be obtained by expanding the eigenvalue equation. First, the eigenvalues λ in the y-direction can be obtained by substituting

$$\phi_x = \mathrm{e}^{\lambda y} \quad \phi_y = \mathrm{e}^{\lambda y} \quad \kappa_y = \mathrm{e}^{\lambda y} \quad \kappa_{xy} = \mathrm{e}^{\lambda y} \tag{12}$$

into Eq. (11). Expanding the determinant yields the eigenvalue equation

$$\left(\lambda^2 + \mu^2\right)^2 = 0 \tag{13}$$

with repeated roots $\lambda = \pm\mu i$ as the eigenvalues. Hence, the general solutions of nonzero eigenvalues are

$$\begin{aligned} \phi_x &= A_1\cos(\mu y) + B_1\sin(\mu y) + C_1 y\sin(\mu y) + D_1 y\cos(\mu y) \\ \phi_y &= A_2\sin(\mu y) + B_2\cos(\mu y) + C_2 y\cos(\mu y) + D_2 y\sin(\mu y) \\ \kappa_y &= A_3\cos(\mu y) + B_3\sin(\mu y) + C_3 y\sin(\mu y) + D_3 y\cos(\mu y) \\ \kappa_{xy} &= A_4\sin(\mu y) + B_4\cos(\mu y) + C_4 y\cos(\mu y) + D_4 y\sin(\mu y) \end{aligned} \tag{14}$$

The constants are not all independent. For convenience, A_2, B_2, C_2 and D_2 may be chosen as the independent constants. Substituting Eq. (14) into Eq. (11) yields the relations between these constants. Further substituting the general solution (14) into the corresponding boundary conditions on both sides $y = b_1$ or b_2 yields the transcendental equation of nonzero eigenvalues and the corresponding eigenvectors. Then method of eigenvector expansion then can be applied.

Eq. (14) is only valid for the basic eigenvectors with nonzero eigenvalues μ. If Jordan form eigen-solution exists, we should solve the following equation

$$\boldsymbol{H}\boldsymbol{\psi}^{(k)} = \mu\boldsymbol{\psi}^{(k)} + \boldsymbol{\psi}^{(k-1)} \quad (k = 1, 2, \ldots) \tag{15}$$

where superscript k denotes the kth order Jordan form eigen-solution. The Jordan form eigen-solution is formed by superposing a particular solution resulted from the inhomogeneous term $\boldsymbol{\psi}^{(k-1)}$ and the solution of Eq. (14).

3. Plates with two opposite sides simply supported

Consider a plate with two opposite sides simply supported at $y = 0$ and $y = b$, the boundary conditions are

$$M_y|_{y=0,b} = 0; \quad w|_{y=0,b} = 0 \tag{16}$$

Knowing that $M_y = \frac{\partial\phi_x}{\partial x}$ and $\kappa_x = \frac{1}{D}\frac{\partial\phi_y}{\partial y} - v\kappa_y = \frac{\partial^2 w}{\partial x^2}$, the boundary conditions in Eq. (16) can be replaced by

$$\begin{aligned} &\phi_x|_{y=0} = 0; \left.\left(\frac{1}{D}\frac{\partial\phi_y}{\partial y} - v\kappa_y\right)\right|_{y=0} = 0 \\ &\phi_x|_{y=b} = a_1; \left.\left(\frac{1}{D}\frac{\partial\phi_y}{\partial y} - v\kappa_y\right)\right|_{y=b} = 0 \end{aligned} \tag{17}$$

The unknown constant a_1 in the boundary conditions should be solved first because it is an inhomogeneous term. After obtaining the expression for deflection w with respect to boundary condition a_1, it appears that this solution does not satisfy the boundary condition $w = 0$ on both sides in Eq. (16). It is a spurious solution and thus should be abandoned. The emergence of this spurious solution of the original problem is due to the replacement of $w = 0$ by $\kappa_x = 0$ in the boundary conditions (17). Therefore with respect to bending of a plate simply supported on two opposite sides, the homogeneous boundary conditions are

$$\phi_x|_{y=0,b} = 0; \quad \left.\left(\frac{1}{D}\frac{\partial\phi_y}{\partial y} - v\kappa_y\right)\right|_{y=0,b} = 0 \tag{18}$$

For a zero eigenvalue, the eigen-solutions are all equal to zero. These are trivial solutions and they do not have physical interpretation. For nonzero eigenvalues, substituting the general eigen-solutions expressed by Eq. (14) into the homogeneous boundary conditions (18), and equating the determinant of coefficient matrix to zero yield the transcendental equation of nonzero eigenvalues for bending of simply supported plate on opposite sides along $y = 0$ and $y = b$ as

$$\sin^2(\mu b) = 0 \tag{19}$$

which gives real repeated double roots as

$$\mu_n = \frac{n\pi}{b} \quad (n = \pm 1, \pm 2, \ldots) \tag{20}$$

The corresponding basic eigenvector is

$$\boldsymbol{\psi}_n^{(0)} = \begin{pmatrix} \phi_x \\ \phi_y \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} = \begin{pmatrix} \frac{D(1-v)}{\mu_n}\sin(\mu_n y) \\ \frac{D(1-v)}{\mu_n}\cos(\mu_n y) \\ \sin(\mu_n y) \\ \cos(\mu_n y) \end{pmatrix} \tag{21}$$

Then the solution to eigenvalue Eq. (7) is

$$v_n^{(0)} = e^{\mu_n x}\boldsymbol{\psi}_n^{(0)} \tag{22}$$

From the curvature–deflection relation (3a), the deflection of plate can be expressed as

$$w_n^{(0)} = -\frac{1}{\mu_n^2} e^{\mu_n x} \sin(\mu_n y) \tag{23}$$

where the constants of integration are determined as zero by imposing the boundary conditions $w = 0$ on both sides.

Because the eigenvalue μ_n is a double root, the first-order Jordan form eigen-solution can be solved via

$$H\boldsymbol{\psi}^{(1)} = \boldsymbol{\psi}^{(0)} + \mu\boldsymbol{\psi}^{(1)} \tag{24}$$

Imposing the boundary conditions (18) yields

$$\boldsymbol{\psi}_n^{(1)} = \begin{pmatrix} \phi_x \\ \phi_y \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} = \begin{pmatrix} -\frac{3+\nu}{2\mu_n^2} D\sin(\mu_n y) \\ \frac{3+\nu}{2\mu_n^2} D\cos(\mu_n y) \\ -\frac{1}{2\mu_n}\sin(\mu_n y) \\ \frac{1}{2\mu_n}\cos(\mu_n y) \end{pmatrix} \tag{25}$$

Hence the solution to Eq. (7) is

$$v_n^{(1)} = e^{\mu_n x}(x\boldsymbol{\psi}_n^{(0)} + \boldsymbol{\psi}_n^{(1)}) \tag{26}$$

Again from the curvature–deflection relation (3a), the deflection of plate can be expressed as

$$w_n^{(1)} = \frac{1 - 2x\mu_n}{2\mu_n^3} e^{\mu_n x} \sin(\mu_n y) \tag{27}$$

The eigenvectors in Eqs. (21) and (25) are adjoint symplectic orthogonal because $\boldsymbol{H}$ is a Hamiltonian operator matrix. The eigenvector symplectic adjoint with $\boldsymbol{\psi}_n^{(0)}$ should be $\boldsymbol{\psi}_{-n}^{(1)}$, i.e.

$$\langle \boldsymbol{\psi}_n^{(0)}, \boldsymbol{\psi}_{-n}^{(1)} \rangle = -\frac{2Db}{\mu_n^2} \neq 0 \quad \text{for} \quad n = \pm 1, \pm 2, \ldots \tag{28}$$

while the other eigenvectors are symplectic orthogonal to each other. The symplectic inner product for any two vectors α, β in a $2n$-dimensional phase space W in a real number field R is denoted as $<\alpha,\beta>$ and it satisfies four basic properties (Yao et al., 2007).

From the eigenvalues and eigenvectors with adjoint symplectic orthogonality property, the general solution for plate bending simply supported on both opposite sides can be expressed as

$$v = \sum_{n=1}^{\infty} \left[f_n^{(0)} v_n^{(0)} + f_n^{(1)} v_n^{(1)} + f_{-n}^{(0)} v_{-n}^{(0)} + f_{-n}^{(1)} v_{-n}^{(1)} \right] \tag{29}$$

according to the expansion theorem. The equation above strictly satisfies the homogeneous differential equation in the domain and the homogeneous boundary conditions (18) while $f_n^{(k)} (k = 0, 1; n = \pm 1, \pm 2, \ldots)$ are unknown constants which can be determined by imposing the remaining two boundary conditions at $x = -a/2$ and $x = a/2$.

After determining the constants $f_n^{(k)}$, the solution of the original problem for bending deflection of a thin plate governed by Eq. (1) is

$$w = \bar{w} + \sum_{n=1}^{\infty} \left[f_n^{(0)} w_n^{(0)} + f_n^{(1)} w_n^{(1)} + f_{-n}^{(0)} w_{-n}^{(0)} + f_{-n}^{(1)} w_{-n}^{(1)} \right] \tag{30}$$

where $\bar{w}$ is a particular solution with respect to the transverse load q.

For example, the particular solution for a plate with two opposite sides simply supported at $y = 0$, $y = b$ and with uniformly distributed load q is

$$\bar{w} = \frac{q}{24D}(y^4 - 2by^3 + b^3 y) \tag{31}$$

and the corresponding curvatures and bending moments are

$$\begin{aligned}&\bar{\kappa}_y = \tfrac{q}{2D} y(y-b); \quad &&\bar{\kappa}_x = 0; \quad &&\bar{\kappa}_{xy} = 0 \\ &\overline{M}_x = \tfrac{1}{2} q v y(y-b); \quad &&\overline{M}_y = \tfrac{1}{2} q y(y-b); \quad &&\overline{M}_{xy} = 0\end{aligned} \tag{32a–f}$$

The expressions of $\overline{M}_x$ and $\bar{\kappa}_y$ above can be represented in Fourier series as

$$\overline{M}_x = \sum_{n=1}^{\infty} \frac{2\sin\left(\frac{n\pi y}{b}\right)}{b} \int_0^b \frac{1}{2} q v y(y-b) \sin\left(\frac{n\pi y}{b}\right) \mathrm{d}y = -\frac{4b^2 q v}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi y}{b}\right) \tag{33a}$$

$$\bar{\kappa}_y = \sum_{n=1}^{\infty} \frac{2\sin\left(\frac{n\pi y}{b}\right)}{b} \int_0^b \frac{q}{2D} y(y-b) \sin\left(\frac{n\pi y}{b}\right) \mathrm{d}y = -\frac{4b^2 q}{\pi^3 D} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi y}{b}\right) \tag{33b}$$

which are required to determine the other four constants when the boundary conditions at the remaining two sides are considered.

4. Exact plate bending solutions and numerical examples

The fundamental presented above is valid for bending of a plate with arbitrary boundary condition. For presentation purpose, the formulation derived in Section 3 is valid for bending of a thin plate with two opposite sides simply supported at $y = 0$ and $y = b$ and no restriction is imposed on the remaining two boundaries. Exact bending solutions for various examples of such plates are presented as follows.

4.1. Fully simply supported plate (SSSS)

A fully simply supported plate denoted as SSSS is solved first because it is a classical problem with well-established exact solution for comparison. The plate is bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$. In addition to the two simply supported boundary conditions at $y = 0$, b expressed in Eq. (18), the additional boundary conditions are

$$M_x|_{x=\pm a/2} = 0; \quad \kappa_y|_{x=\pm a/2} = 0 \tag{34a, b}$$

in which $w|_{x=\pm a/2} = 0$ is replaced by $\kappa_y|_{x=\pm a/2} = 0$. From Eqs. (32) and (34), we obtain

$$M_x|_{x=\pm a/2} = -\overline{M}_x = -\frac{1}{2} q v y(y-b); \quad \kappa_y|_{x=\pm a/2} = -\bar{\kappa}_y = -\frac{q}{2D} y(y-b) \tag{35a, b}$$

However, from Eqs. (29) and (5c), we have

$$\begin{aligned} M_x = \frac{\partial \phi_y}{\partial y} = \sum_{n=1}^{\infty} D\bigg\{ & -f_n^{(0)} \mathrm{e}^{\mu_n x}(1-v) - f_n^{(1)} \mathrm{e}^{\mu_n x}\left[x(1-v) + \frac{3+v}{2\mu_n}\right] \\ & + f_{-n}^{(0)} \mathrm{e}^{-\mu_n x}(1-v) + f_{-n}^{(1)} \mathrm{e}^{-\mu_n x}\left[x(1-v) - \frac{3+v}{2\mu_n}\right]\bigg\} \sin(\mu_n y) \end{aligned} \tag{36a}$$

$$\kappa_y = \sum_{n=1}^{\infty}\left[f_n^{(0)} \mathrm{e}^{\mu_n x} + f_n^{(1)} \mathrm{e}^{\mu_n x}\left(x - \frac{1}{2\mu_n}\right) - f_{-n}^{(0)} \mathrm{e}^{-\mu_n x} - f_{-n}^{(1)} \mathrm{e}^{-\mu_n x}\left(x + \frac{1}{2\mu_n}\right)\right] \sin(\mu_n y) \tag{36b}$$

Substituting $x = \pm a/2$ into Eqs. (36a) and (36b) for the left-hand-side of Eqs. (35a-b) and using the Fourier series representations of $\overline{M}_x$ and $\bar{\kappa}_y$ in Eqs. (33a) and (33b) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}$, $f_n^{(1)}$, $f_{-n}^{(0)}$, $f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$
\begin{aligned}
&f_n^{(0)} = f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0 \quad \text{for} \quad n = 2,4,6,\ldots \\
&f_n^{(0)} = -f_{-n}^{(0)} = \frac{q(3+2\alpha_n \tanh \alpha_n)}{2Db\mu_n^3 \cosh \alpha_n} \quad \text{for} \quad n = 1,3,5,\ldots \\
&f_n^{(1)} = f_{-n}^{(1)} = -\frac{q}{Db\mu_n^2 \cosh \alpha_n} \quad \text{for} \quad n = 1,3,5,\ldots
\end{aligned} \tag{37}
$$

where

$$
\alpha_n = \frac{\alpha n\pi}{2b} \quad \text{for} \quad n = 1,3,5,\ldots \tag{38}
$$

From Eqs. (23), (27), (30), (31) and (37), the bending deflection of a thin plate under uniformly distributed load is

$$
w = \frac{q}{24D}\left(y^4 - 2by^3 + b^3 y\right) + \frac{2q}{Db}\sum_{n=1}^{\infty} \frac{[\mu_n x \sinh(\mu_n x) - \cosh(\mu_n x)(2+\alpha_n \tanh \alpha_n)]\sin(\mu_n y)}{\mu_n^5 \cosh \alpha_n} \tag{39}
$$

In addition, the general solutions for bending moments and stress resultants of a SSSS plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.2. Plate with two opposite sides simply supported and the others free (SFSF)

A SFSF plate bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$ is considered here. In addition to the two simply supported boundary conditions at $y = 0, b$ expressed in Eq. (18), the additional boundary conditions are

$$
M_x|_{x=\pm a/2} = 0; \quad \phi_x|_{x=\pm a/2} = 0 \tag{40a, b}
$$

where the free shear force condition $F_{Vx}|_{x=\pm a/2} = 0$ is replaced by $\phi_x|_{x=\pm a/2} = 0$. From Eqs. (32) and (40), we obtain

$$
M_x|_{x=\pm a/2} = -\overline{M}_x = -\frac{1}{2}qvy(y-b); \quad \phi_x|_{x=\pm a/2} = -\bar{\phi}_x = 0 \tag{41a, b}
$$

However, from Eqs. (29) and (5c), we have

$$
\begin{aligned}
M_x = \frac{\partial \phi_y}{\partial y} = \sum_{n=1}^{\infty} D\Bigg\{ &-f_n^{(0)} e^{\mu_n x}(1-v) - f_n^{(1)} e^{\mu_n x}\left[x(1-v) + \frac{3+v}{2\mu_n}\right] \\
&+ f_{-n}^{(0)} e^{-\mu_n x}(1-v) + f_{-n}^{(1)} e^{-\mu_n x}\left[x(1-v) - \frac{3+v}{2\mu_n}\right]\Bigg\} \sin(\mu_n y)
\end{aligned} \tag{42a}
$$

$$
\begin{aligned}
\phi_x = \sum\nolimits_{n=1}^{\infty} D\Bigg\{ &f_n^{(0)} e^{\mu_n x}(1-v) + f_n^{(1)} e^{\mu_n x}\left[x(1-v) - \frac{(3+v)}{2\mu_n}\right] \\
&+ f_{-n}^{(0)} e^{-\mu_n x}(1-v) + f_{-n}^{(1)} e^{-\mu_n x}\left[x(1-v) + \frac{(3+v)}{2\mu_n}\right]\Bigg\} \frac{\sin(\mu_n y)}{\mu_n}
\end{aligned} \tag{42b}
$$

Substituting $x = \pm a/2$ into Eqs. (42a) and (42b) for the left-hand-side of Eqs. (41a-b) and using the Fourier series representations of $\overline{M}_x$ in Eqs. (33a) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}$, $f_n^{(1)}$, $f_{-n}^{(0)}$, $f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$
\begin{aligned}
&f_n^{(0)} = f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0 \quad \text{for } n = 2,4,6\ldots \\
&f_n^{(0)} = -f_{-n}^{(0)} = -\frac{2qv[(3+v)\sinh \alpha_n - 2(1-v)\alpha_n \cosh \alpha_n]}{Db\mu_n^3(1-v)[(3+v)\sinh(2\alpha_n) - 2(1-v)\alpha_n]} \quad \text{for } n = 1,3,5\ldots \\
&f_n^{(1)} = f_{-n}^{(1)} = -\frac{4qv \sinh \alpha_n}{Db\mu_n^2[(3+v)\sinh(2\alpha_n) - 2(1-v)\alpha_n]} \quad \text{for } n = 1,3,5\ldots
\end{aligned} \tag{43}
$$

where α_n is given in Eq. (38).

From Eqs. (23), (27), (30), (31) and (43), the bending deflection of a thin plate under uniformly distributed load is

$$w = \frac{q}{24D}(y^4 - 2by^3 + b^3y) + \frac{4q}{(1-v)bD}\sum_{n=1}^{\infty}$$
$$\times \frac{\{v\cosh(\mu_n x)[\alpha_n(1-v)\cosh\alpha_n - (1+v)\sinh\alpha_n] - \mu_n x(1-v)v\sinh\alpha_n\sinh(\mu_n x)\}\sin(\mu_n y)}{\mu_n^5[\alpha_n(1-v) - (3+v)\cosh\alpha_n\sinh\alpha_n]} \tag{44}$$

In addition, the general solutions for bending moments and stress resultants of a SFSF plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.3. Plate with two opposite sides simply supported and the others clamped (SCSC)

A SCSC plate bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$ is considered here. In addition to the two opposite sides simply supported, the additional boundary conditions are

$$\kappa_y|_{x=\pm a/2} = 0; \quad \kappa_{xy}|_{x=\pm a/2} = 0 \tag{45a, b}$$

where $w|_{x=\pm a/2} = 0$ is replaced by $\kappa_y|_{x=\pm a/2} = 0$ and $\frac{\partial w}{\partial x}|_{x=\pm a/2} = 0$ is replaced by $\kappa_{xy}|_{x=\pm a/2} = 0$.

From Eqs. (32) and (45), we obtain

$$\kappa_y|_{x=\pm a/2} = -\bar{\kappa}_y = -\frac{q}{2D}y(y-b); \quad \kappa_{xy}|_{x=\pm a/2} = -\bar{\kappa}_{xy} = 0 \tag{46a, b}$$

However, from Eq. (29), we have

$$\kappa_y = \sum_{n=1}^{\infty}\left[f_n^{(0)}e^{\mu_n x} + f_n^{(1)}e^{\mu_n x}\left(x - \frac{1}{2\mu_n}\right) - f_{-n}^{(0)}e^{-\mu_n x} - f_{-n}^{(1)}e^{-\mu_n x}\left(x + \frac{1}{2\mu_n}\right)\right]\sin(\mu_n y) \tag{47a}$$

$$\kappa_{xy} = \sum_{n=1}^{\infty}\left[f_n^{(0)}e^{\mu_n x} + f_n^{(1)}e^{\mu_n x}\left(x + \frac{1}{2\mu_n}\right) - f_{-n}^{(0)}e^{-\mu_n x} + f_{-n}^{(1)}e^{-\mu_n x}\left(x - \frac{1}{2\mu_n}\right)\right]\cos(\mu_n y) \tag{47b}$$

Substituting $x = \pm a/2$ into Eqs. (47a) and (47b) for the left-hand-side of Eqs. (46a,b) and using the Fourier series representations of $\bar{\kappa}_y$ in Eqs. (33a) and (33b) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}$, $f_n^{(1)}$, $f_{-n}^{(0)}$, $f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$f_n^{(0)} = f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0 \quad \text{for } n = 2, 4, 6\ldots$$
$$f_n^{(0)} = -f_{-n}^{(0)} = -\frac{2q(2\alpha_n\cosh\alpha_n + \sinh\alpha_n)}{Db\mu_n^3[2\alpha_n + \sinh(2\alpha_n)]} \quad \text{for } n = 1, 3, 5\ldots$$
$$f_n^{(1)} = f_{-n}^{(1)} = -\frac{4q\sinh\alpha_n}{Db\mu_n^2[2\alpha_n + \sinh(2\alpha_n)]} \quad \text{for} \quad n = 1, 3, 5\ldots \tag{48}$$

where α_n is given in Eq. (38).

From Eqs. (23), (27), (30), (31) and (48), the bending deflection of a thin plate under uniformly distributed load is

$$w = \frac{q}{24D}\left(y^4 - 2by^3 + b^3y\right) + \frac{4q}{bD}$$
$$\times \sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)[-2(3\alpha_n + \mu_n x)\cosh(\alpha_n - \mu_n x) + (\alpha_n + \mu_n x)\cosh(3\alpha_n - \mu_n x) - 5\alpha_n$$
$$\times\cosh(\alpha_n + \mu_n x) + 3\mu_n x\cosh(\alpha_n + \mu_n x) + 2\alpha_n\cosh(3\alpha_n + \mu_n x) - 2\mu_n x\cosh(3\alpha_n + \mu_n x)$$
$$+ 8\cosh(2\alpha_n)\cosh(\mu_n x)\sinh\alpha_n - 4\alpha_n(\alpha_n - \mu_n x)\sinh(\alpha_n + \mu_n x)]\}/[\mu_n^5\left(1 + 8\alpha_n e^{4\alpha_n} - e^{8\alpha_n}\right)]\} \tag{49}$$

In addition, the general solutions for bending moments and stress resultants of a SCSC plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.4. Plate with two opposite sides simply supported, one clamped and one free (SFSC)

A SFSC plate bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$ is considered here. In addition to the two opposite sides simply supported, the additional boundary conditions are

$$M_x|_{x=-a/2} = 0; \ \ \phi_x|_{x=-a/2} = 0$$
$$\kappa_y|_{x=a/2} = 0; \ \ \kappa_{xy}|_{x=a/2} = 0 \tag{50a–d}$$

where the free shear force condition $F_{Vx}|_{x=\pm a/2} = 0$ is replaced by $\phi_x|_{x=\pm a/2} = 0$, $w|_{x=\pm a/2} = 0$ is replaced by $\kappa_y|_{x=\pm a/2} = 0$ and $\frac{\partial w}{\partial x}|_{x=\pm a/2} = 0$ is replaced by $\kappa_{xy}|_{x=\pm a/2} = 0$. From Eqs. (32) and (50), we obtain

$$M_x|_{x=-a/2} = -\overline{M}_x = -\frac{1}{2}qvy(y-b); \quad \phi_x|_{x=-a/2} = -\bar{\phi}_x = 0$$
$$\kappa_y|_{x=a/2} = -\bar{\kappa}_y = -\frac{q}{2D}y(y-b); \quad \kappa_{xy}|_{x=a/2} = -\bar{\kappa}_{xy} = 0 \tag{51a–d}$$

However, from Eqs. (29) and (5c), we have

$$\phi_x = \sum_{n=1}^{\infty} D\left\{f_n^{(0)}\mathrm{e}^{\mu_n x}(1-v) + f_n^{(1)}\mathrm{e}^{\mu_n x}\left[x(1-v) - \frac{(3+v)}{2\mu_n}\right]\right.$$
$$\left. + f_{-n}^{(0)}\mathrm{e}^{-\mu_n x}(1-v) + f_{-n}^{(1)}\mathrm{e}^{-\mu_n x}\left[x(1-v) + \frac{(3+v)}{2\mu_n}\right]\right\}\frac{\sin(\mu_n y)}{\mu_n} \tag{52a}$$

$$\kappa_y = \sum_{n=1}^{\infty}\left[f_n^{(0)}\mathrm{e}^{\mu_n x} + f_n^{(1)}\mathrm{e}^{\mu_n x}\left(x - \frac{1}{2\mu_n}\right) - f_{-n}^{(0)}\mathrm{e}^{-\mu_n x} - f_{-n}^{(1)}\mathrm{e}^{-\mu_n x}\left(x + \frac{1}{2\mu_n}\right)\right]\sin(\mu_n y) \tag{52b}$$

$$\kappa_{xy} = \sum_{n=1}^{\infty}\left[f_n^{(0)}\mathrm{e}^{\mu_n x} + f_n^{(1)}\mathrm{e}^{\mu_n x}\left(x + \frac{1}{2\mu_n}\right) - f_{-n}^{(0)}\mathrm{e}^{-\mu_n x} + f_{-n}^{(1)}\mathrm{e}^{-\mu_n x}\left(x - \frac{1}{2\mu_n}\right)\right]\cos(\mu_n y) \tag{52c}$$

Substituting $x = \pm a/2$ into Eqs. (52a), (52b) and (52c) for the left-hand-side of Eqs. (51a–d) and using the Fourier series representations of $\bar{\kappa}_y$ in Eqs. (33b) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}$, $f_n^{(1)}$, $f_{-n}^{(0)}$, $f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$f_n^{(0)} = f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0 \quad \text{for} \ \ n = 2, 4, 6, \ldots$$
$$f_n^{(0)} = \left\{2q\mathrm{e}^{\alpha_n}\left\{\mathrm{e}^{6\alpha_n}(1+2\alpha_n)(v-1)(3+v) - \mathrm{e}^{2\alpha_n}\left[-2\alpha_n(v-1)^2 + 8\alpha_n^2(-1+v)^2 + (3+v)^2\right]\right.\right.$$
$$\left.\left. - \mathrm{e}^{4\alpha_n}v\{-1+v+2\alpha_n[3+4\alpha_n(v-1)+v]\}\right\}\right\}$$
$$/Db\mu_n^3\left\{(v-1)(3+v) + \mathrm{e}^{8\alpha_n}(v-1)(3+v) - 2\mathrm{e}^{4\alpha_n}\left[5+8\alpha_n^2(v-1)^2 + v(2+v)\right]\right\} \quad \text{for} \ n = 1,3,5\ldots$$
$$f_{-n}^{(0)} = -\left\{2\mathrm{e}^{\alpha_n}q\{-(-1+2\alpha_n)(-1+v)(3+v) + \mathrm{e}^{6\alpha_n}v[3+v+2\alpha_n(-1+v)]\right.$$
$$\left. - \mathrm{e}^{4\alpha_n}\left[2\alpha_n(-1+v)^2 + 8\alpha_n^2(-1+v)^2 + (3+v)^2\right] - \mathrm{e}^{2\alpha_n}v\{-1+v+2\alpha_n[-3+4\alpha_n(v-1)-v]\}\}\right\}$$
$$/Db\mu_n^3\left\{(v-1)(3+v) + \mathrm{e}^{8\alpha_n}(v-1)(3+v) - 2\mathrm{e}^{4\alpha_n}\left[5+8\alpha_n^2(v-1)^2 + v(2+v)\right]\right\} \quad \text{for} \ n = 1,3,5\ldots$$
$$f_n^{(1)} = \left\{4q\mathrm{e}^{\alpha_n}\left\{-\mathrm{e}^{2\alpha_n}(-1+4\alpha_n)(-1+v)^2 - (-1+v)v - \mathrm{e}^{6\alpha_n}(-1+v)(3+v) + \mathrm{e}^{4\alpha_n}v[3+4\alpha_n(-1+v)+v]\right\}\right\}$$
$$/Db\mu_n^2\left\{(v-1)(3+v) + \mathrm{e}^{8\alpha_n}(v-1)(3+v) - 2\mathrm{e}^{4\alpha_n}\left[5+8\alpha_n^2(v-1)^2 + v(2+v)\right]\right\} \quad \text{for} \ n = 1,3,5\ldots$$
$$f_{-n}^{(1)} = -\left\{4\mathrm{e}^{\alpha_n}q\left\{-\mathrm{e}^{4\alpha_n}(1+4\alpha_n)(-1+v)^2 + \mathrm{e}^{2\alpha_n}[-3+4\alpha_n(-1+v)-v]v + \mathrm{e}^{6\alpha_n}(-1+v)v + (-1+v)(3+v)\right\}\right\}$$
$$/Db\mu_n^2\left\{(v-1)(3+v) + \mathrm{e}^{8\alpha_n}(v-1)(3+v) - 2\mathrm{e}^{4\alpha_n}\left[5+8\alpha_n^2(v-1)^2 + v(2+v)\right]\right\} \quad \text{for} \ n = 1,3,5\ldots \tag{53}$$

where α_n is given in Eq. (38).

From Eqs. (23), (27), (30), (31) and (53), the bending deflection of a thin plate under uniformly distributed load is

$$
\begin{aligned}
w = {} & \frac{q}{24D}(y^4 - 2by^3 + b^3y) + \frac{8q}{bD}\sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)\{[5 + 4\alpha_n^2(v-1)^2 + v(2+v)]\cosh(\alpha_n - \mu_n x) \\
& + 3\cosh(3\alpha_n + \mu_n x) + v\cosh(\mu_n x)\{[1 + 4\alpha_n^2(v-1) + v]\cosh\alpha_n + (-3 + \alpha_n - 2v)\cosh(3\alpha_n) \\
& + \mu_n x[-5\sinh\alpha_n - (v-1)\sinh(3\alpha_n)]\} + v\{-\mu_n x(1+2v)\cosh\alpha_n + \mu_n x(v-1)\cosh(3\alpha_n) \\
& + [1 + 4\alpha_n^2(v-1) + v]\sinh\alpha_n + (\alpha_n - 1)\sinh(3\alpha_n)\}\sinh(\mu_n x) - \alpha_n\{\cosh(\mu_n x)\{4\mu_n x(v-1)\cosh\alpha_n \\
& - 3\sinh\alpha_n + v\{\cosh(3\alpha_n) + [5 - 4v + 4v\cosh(2\alpha_n)]\sinh\alpha_n\}\} + [(3-7v)\cosh\alpha_n + 2v\cosh(3\alpha_n) \\
& + 4\mu_n x(v-1)(2v-1)\sinh\alpha_n + v\sinh(3\alpha_n)]\sinh(\mu_n x)\} + \mu_n x\sinh(\alpha_n - \mu_n x) \\
& - [\alpha_n(v-3) - \mu_n x(v-1)(v+3)]\sinh(3\alpha_n + \mu_n x)\}\}/\{\mu_n^5\{(v-1)(v+3) + e^{8\alpha_n}(v-1)(v+3) \\
& - 2e^{4\alpha_n}[5 + 8\alpha_n^2(v-1)^2 + v(2+v)]\}\}\}
\end{aligned} \tag{54}
$$

In addition, the general solutions for bending moments and stress resultants of a SFSC plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.5. Plate with three sides simply supported and the other free (SSSF)

A SSSF plate bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$ is considered here. In addition to the two opposite sides simply supported, the additional boundary conditions are

$$
\begin{aligned}
& M_x|_{x=-a/2} = 0; \quad \kappa_y|_{x=-a/2} = 0 \\
& M_x|_{x=a/2} = 0; \quad \phi_x|_{x=a/2} = 0
\end{aligned} \tag{55a–d}
$$

in which $w|_{x=\pm a/2} = 0$ is replaced by $\kappa_y\,|_{x=\pm a/2} = 0$ and the free shear force condition $F_{\mathrm{V}x}|_{x=\pm a/2} = 0$ is replaced by $\phi_x|_{x=\pm a/2} = 0$. From Eqs. (32) and (55), we obtain

$$
\begin{aligned}
& M_x|_{x=-a/2} = -\overline{M}_x = -\frac{1}{2}qvy(y-b); \quad \kappa_y\big|_{x=-a/2} = -\bar{\kappa}_y = -\frac{q}{2D}y(y-b) \\
& M_x|_{x=a/2} = -\overline{M}_x = -\frac{1}{2}qvy(y-b); \quad \phi_x|_{x=a/2} = -\bar{\phi}_x = 0
\end{aligned} \tag{56a–d}
$$

However, from Eqs. (29) and (5c), we have

$$
\begin{aligned}
M_x = \frac{\partial\phi_y}{\partial y} = \sum_{n=1}^{\infty} D\Big\{ & -f_n^{(0)}e^{\mu_n x}(1-v) - f_n^{(1)}e^{\mu_n x}\left[x(1-v) + \frac{3+v}{2\mu_n}\right] \\
& + f_{-n}^{(0)}e^{-\mu_n x}(1-v) + f_{-n}^{(1)}e^{-\mu_n x}\left[x(1-v) - \frac{3+v}{2\mu_n}\right]\Big\}\sin(\mu_n y)
\end{aligned} \tag{57a}
$$

$$
\begin{aligned}
\phi_x = \sum\nolimits_{n=1}^{\infty} D\Big\{ & f_n^{(0)}e^{\mu_n x}(1-v) + f_n^{(1)}e^{\mu_n x}\left[x(1-v) - \frac{(3+v)}{2\mu_n}\right] + f_{-n}^{(0)}e^{-\mu_n x}(1-v) \\
& + f_{-n}^{(1)}e^{-\mu_n x}\left[x(1-v) + \frac{(3+v)}{2\mu_n}\right]\Big\}\frac{\sin(\mu_n y)}{\mu_n}
\end{aligned} \tag{57b}
$$

$$
\kappa_y = \sum\nolimits_{n=1}^{\infty}\left[f_n^{(0)}e^{\mu_n x} + f_n^{(1)}e^{\mu_n x}\left(x - \frac{1}{2\mu_n}\right) - f_{-n}^{(0)}e^{-\mu_n x} - f_{-n}^{(1)}e^{-\mu_n x}\left(x + \frac{1}{2\mu_n}\right)\right]\sin(\mu_n y) \tag{57c}
$$

Substituting $x = \pm a/2$ into Eqs. (57a), (57b) and (57c) for the left-hand-side of Eqs. (56a-d) and using the Fourier series representations of $\overline{M}_x$ and $\bar{\kappa}_y$ in Eqs. (33a) and (33b) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}, f_n^{(1)}, f_{-n}^{(0)}, f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$
\begin{aligned}
f_n^{(0)} &= f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0; \quad \text{for } n = 2,4,6\ldots \\
f_n^{(0)} &= \{e^{-3\alpha_n} q\{-2e^{2\alpha_n}[-3 + 6\alpha_n(-1+v) - v]v + (-3+2\alpha_n)(-1+v)(3+v) \\
&\quad + 2e^{6\alpha_n} v[3 + 2\alpha_n(-1+v) + v] - e^{4\alpha_n}[6\alpha_n(-1+v)^2 + 8\alpha_n^2(-1+v)^2 + (3+v)^2]\}\} \\
&\quad / 2Db\mu_n^3(-1+v)[-4\alpha_n(-1+v) + (3+v)\sinh(4\alpha_n)]; \quad \text{for } n = 1,3,5\ldots \\
f_{-n}^{(0)} &= \{e^{-3\alpha_n} q\{-e^{6\alpha_n}(-1+v)(3+v)(3+2\alpha_n) + 2v[3 + 2\alpha_n(-1+v) + v] \\
&\quad + 2e^{4\alpha_n} v[3 + 6\alpha_n(-1+v) + v] - e^{2\alpha_n}[-6\alpha_n(-1+v)^2 + 8\alpha_n^2(-1+v)^2 + (3+v)^2]\}\} \\
&\quad / 2Db\mu_n^3(-1+v)[-4\alpha_n(-1+v) + (3+v)\sinh(4\alpha_n)]; \text{for } n = 1,3,5\ldots \\
f_n^{(1)} &= \frac{2qe^{\alpha_n}\{3 + v + e^{4\alpha_n}[(3+4\alpha_n)(-1+v) - 4v\cosh(2\alpha_n)]\}}{Db\mu_n^2[3 + 8\alpha_n e^{4\alpha_n}(-1+v) + v - e^{8\alpha_n}(3+v)]}; \quad \text{for } n = 1,3,5\ldots \\
f_{-n}^{(1)} &= \frac{2e^{\alpha_n} q[-e^{2\alpha_n}(-3+4\alpha_n)(-1+v) - 2v - 2e^{4\alpha_n} v + e^{6\alpha_n}(3+v)]}{Db\mu_n^2[3 + 8\alpha_n e^{4\alpha_n}(-1+v) + v - e^{8\alpha_n}(3+v)]}; \quad \text{for } n = 1,3,5\ldots
\end{aligned} \tag{58}
$$

where α_n is given in Eq. (38).

From Eqs. (23), (27), (30), (31) and (58), the bending deflection of a thin plate under uniformly distributed load is

$$
\begin{aligned}
w = {} & \frac{q}{24D}\left(y^4 - 2by^3 + b^3 y\right) + \frac{2q}{bD} \\
& \times \sum_{n=1}^{\infty} \Bigg\{ \Bigg\{ e^{\alpha_n - \mu_n x} \Bigg\{ -2e^{2(\alpha_n + \mu_n x)}[-1 + 3\alpha_n(v-1) + \mu_n x(v-1) - v]v + e^{2\mu_n x}(-2 + \alpha_n + \mu_n x)(v-1) \\
& \times (3+v) - 2e^{4\alpha_n} v[1 + 3\alpha_n(v-1) + \mu_n x(v-1) + v] + 2v[-1 - \alpha_n + \mu_n x + (-1 + \alpha_n - \mu_n x)v] \\
& + e^{2\alpha_n}\left[6 + 4\alpha_n^2(v-1)^2 + 3\mu_n x(v-1)^2 - \alpha_n(5 + 4\mu_n x)(v-1)^2 + 2v^2\right] \\
& + 2e^{6\alpha_n + 2\mu_n x} v[1 + \alpha_n(v-1) + v - \mu_n x(v-1)] + e^{4\alpha_n + 2\mu_n x}\Big[-4\alpha_n^2(v-1)^2 + 3\mu_n x(v-1)^2 \\
& + \alpha_n(-5 + 4\mu_n x)(v-1)^2 - 2(3+v)^2\Big]\Bigg\} \sin(\mu_n y)\Bigg\} / \left\{\mu_n^5(v-1)\left[8\alpha_n e^{4\alpha_n}(v-1) + \left(1 - e^{8\alpha_n}\right)(3+v)\right]\right\}\Bigg\}
\end{aligned} \tag{59}
$$

In addition, the general solutions for bending moments and stress resultants of a SSSF plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.6. Plate with three sides simply supported and the other clamped (SSSC)

A SSSC plate bounded within a domain $-a/2 \leqslant x \leqslant a/2$ and $0 \leqslant y \leqslant b$ is considered here. In addition to the two opposite sides simply supported, the additional boundary conditions are

$$
\begin{aligned}
& M_x|_{x=-a/2} = 0; \quad \kappa_y|_{x=-a/2} = 0 \\
& \kappa_y|_{x=a/2} = 0; \quad \kappa_{xy}|_{x=a/2} = 0
\end{aligned} \tag{60a–d}
$$

in which $w|_{x=\pm a/2} = 0$is replaced by $\kappa_y|_{x=\pm a/2} = 0$ and $\frac{\partial w}{\partial x}\big|_{x=\pm a/2} = 0$ is replaced by $\kappa_{xy}|_{x=\pm a/2} = 0$. From Eqs. (32) and (60), we obtain

$$
\begin{aligned}
& M_x|_{x=-a/2} = -\overline{M}_x = -\frac{1}{2}qvy(y-b); \quad \kappa_y|_{x=-a/2} = -\bar{\kappa}_y = -\frac{q}{2D}y(y-b) \\
& \kappa_y|_{x=a/2} = -\bar{\kappa}_y = -\frac{q}{2D}y(y-b); \quad \kappa_{xy}|_{x=a/2} = -\bar{\kappa}_{xy} = 0
\end{aligned} \tag{61a–d}
$$

However, from Eqs. (29), we have

$$M_x = \frac{\partial \phi_y}{\partial y} = \sum_{n=1}^{\infty} D\left\{ -f_n^{(0)} e^{\mu_n x}(1-v) - f_n^{(1)} e^{\mu_n x}\left[x(1-v) + \frac{(3+v)}{2\mu_n}\right] + f_{-n}^{(0)} e^{-\mu_n x}(1-v) + f_{-n}^{(1)} e^{-\mu_n x}\left[x(1-v) - \frac{(3+v)}{2\mu_n}\right]\right\} \sin(\mu_n y) \tag{62a}$$

$$\kappa_y = \sum\nolimits_{n=1}^{\infty}\left[f_n^{(0)} e^{\mu_n x} + f_n^{(1)} e^{\mu_n x}\left(x - \frac{1}{2\mu_n}\right) - f_{-n}^{(0)} e^{-\mu_n x} - f_{-n}^{(1)} e^{-\mu_n x}\left(x + \frac{1}{2\mu_n}\right)\right]\sin(\mu_n y) \tag{62b}$$

$$\kappa_{xy} = \sum\nolimits_{n=1}^{\infty}\left[f_n^{(0)} e^{\mu_n x} + f_n^{(1)} e^{\mu_n x}\left(x + \frac{1}{2\mu_n}\right) - f_{-n}^{(0)} e^{-\mu_n x} + f_{-n}^{(1)} e^{-\mu_n x}\left(x - \frac{1}{2\mu_n}\right)\right]\cos(\mu_n y) \tag{62c}$$

Substituting $x = \pm a/2$ into Eqs. (62a), (62b), and (62c) for the left-hand-side of Eqs. (61a-d) and using the Fourier series representations of $\overline{M}_x$ and $\bar{\kappa}_y$ in Eqs. (33a) and (33b) on the right-hand-side, four set of equations can be derived. The constants $f_n^{(0)}, f_n^{(1)}, f_{-n}^{(0)}, f_{-n}^{(1)}$ can be solved by comparing the coefficients of $\sin(\mu_n y)$, which are

$$\begin{aligned}
&f_n^{(0)} = f_{-n}^{(0)} = f_n^{(1)} = f_{-n}^{(1)} = 0 \quad \text{for } n = 2,4,6\ldots \\
&f_n^{(0)} = \frac{e^{\alpha_n} q\{-3 + 2\alpha_n + e^{2\alpha_n}[2 - 12\alpha_n + e^{2\alpha_n}(-1 + 2e^{2\alpha_n} - 4\alpha_n)(1 + 2\alpha_n)]\}}{Db\mu_n^3(-1 + e^{8\alpha_n} - 8\alpha_n e^{4\alpha_n})} \quad \text{for } n = 1,3,5\ldots \\
&f_{-n}^{(0)} = \frac{e^{\alpha_n} q[2 - 4\alpha_n - e^{6\alpha_n}(3 + 2\alpha_n) - e^{2\alpha_n}(-1 + 2\alpha_n)(-1 + 4\alpha_n) + e^{4\alpha_n}(2 + 12\alpha_n)]}{Db\mu_n^3(-1 + e^{8\alpha_n} - 8\alpha_n e^{4\alpha_n})} \quad \text{for } n = 1,3,5\ldots \\
&f_n^{(1)} = \frac{2e^{\alpha_n} q\{1 + e^{4\alpha_n}[3 + 4\alpha_n - 4\cosh(2\alpha_n)]\}}{Db\mu_n^2(-1 + e^{8\alpha_n} - 8\alpha_n e^{4\alpha_n})} \quad \text{for } n = 1,3,5\ldots \\
&f_{-n}^{(1)} = -\frac{2e^{\alpha_n} q[-2 + e^{2\alpha_n}(3 - 2e^{2\alpha_n} + e^{4\alpha_n} - 4\alpha_n)]}{Db\mu_n^2(-1 + e^{8\alpha_n} - 8\alpha_n e^{4\alpha_n})} \quad \text{for } n = 1,3,5\ldots
\end{aligned} \tag{63}$$

where α_n is given in Eq. (38).

From Eqs. (23), (27), (30), (31) and (63), the bending deflection of a thin plate under uniformly distributed load is

$$\begin{aligned}
w = {} & \frac{q}{24D}\left(y^4 - 2by^3 + b^3 y\right) + \frac{4q}{bD}\sum_{n=1}^{\infty}\{\{e^{4\alpha_n}\sin(\mu_n y)[-2(3\alpha_n + \mu_n x)\cosh(\alpha_n - \mu_n x) \\
& + (\alpha_n + \mu_n x)\cosh(3\alpha_n - \mu_n x) - 5\alpha_n\cosh(\alpha_n + \mu_n x) + 3\mu_n x\cosh(\alpha_n + \mu_n x) \\
& + 2\alpha_n\cosh(3\alpha_n + \mu_n x) - 2\mu_n x\cosh(3\alpha_n + \mu_n x) + 8\cosh(2\alpha_n)\cosh(\mu_n x)\sinh(\alpha_n) \\
& - 4\alpha_n(\alpha_n - \mu_n x)\sinh(\alpha_n + \mu_n x)]\}/\left[\left(1 + 8e^{4\alpha_n} - e^{8\alpha_n}\right)\mu_n^5\right]\}
\end{aligned} \tag{64}$$

In addition, the general solutions for bending moments and stress resultants of a SSSC plate related to the state vector $v = \{\phi_x, \phi_y, \kappa_y, \kappa_{xy}\}^{\mathrm{T}}$ can be derived accordingly.

4.7. Comparison and discussion

It is noted that exact analytical solutions for many of the cases above are not presented in (Timoshenko and Woinowsky-krieger, 1970) only the maximum deflection $w_{\max}$ or deflection at specific locations are given. Using the approach presented here, the exact deflection solutions for the cases are expressed in Eqs. (39), (44), (49), (54) (59) and (64) for SSSS, SFSF, SCSC, SFSC, SSSF and SSSC, respectively. In addition, exact expressions for bending moments and stress resultants can be easily derived using basic relation of elasticity.

Comparison with the results of (Timoshenko and Woinowsky-krieger, 1970) for the six cases above is presented in Tables 1–6, respectively. It is obvious that excellent comparison is observed in all cases thus indicating applicability and validity of the symplectic approach for solving exact plate bending solutions. The

Table 1
Deflection and bending moment factors α, β, γ for a uniformly loaded SSSS rectangular plate with $\upsilon = 0.3$ at centre of plate $x = 0$, $y = b/2$

Aspect ratio $\frac{a}{b}$	Deflection factor α where $w_{\max} = \alpha qb^4/D$		Bending moment factor β where $(M_y)_{\max} = \beta qb^2$		Bending moment factor γ where $(M_x)_{\max} = \gamma qb^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
1.0	0.00406	0.00406235	0.0479	0.0478864	0.0479	0.0478864
1.2	0.00564	0.00565053	0.0627	0.0626818	0.0501	0.0500809
1.5	0.00772	0.00772402	0.0812	0.0811601	0.0498	0.0498427
1.7	0.00883	0.00883800	0.0908	0.0907799	0.0486	0.0486149
2.0	0.01013	0.0101287	0.1017	0.101683	0.0464	0.0463503

Table 2
Deflection and bending moment factors α, β, γ for a uniformly loaded SFSF rectangular plate with $\upsilon = 0.3$ at centre of plate $x = 0$, $y = b/2$

Aspect ratio $\frac{a}{b}$	Deflection factor α where $w_{\max} = \alpha qb^4/D$		Bending moment factor β where $(M_y)_{\max} = \beta qb^2$		Bending moment factor γ where $(M_x)_{\max} = \gamma qb^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
0.5	0.01377	0.0137131	0.1235	0.123642	0.0102*	0.0121476
1.0	0.01309	0.0130937	0.1225	0.122545	0.0271	0.0270782
2.0	0.01289	0.0128873	0.1235	0.123468	0.0364	0.0363888

* This is a possible typing mistake in Timoshenko and Woinowsky-krieger (1970) because the expression for deflection w in Eq. (44) is identical to that in the reference and hence M_x.

Table 3
Deflection and bending moment factors α, β, γ for a uniformly loaded SCSC rectangular plate with $\upsilon = 0.3$ at centre of plate $x = 0$, $y = b/2$ where $l = b$ for $a \geqslant b$ and $l = a$ for $a < b$

Aspect ratio $\frac{a}{b}$	Deflection factor α where $w_{\max} = \alpha ql^4/D$		Bending moment factor β where $(M_y)_{\max} = \beta ql^2$		Bending moment factor γ where $(M_x)_{\max} = \gamma ql^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
1/2	0.00260	0.00261079	0.0142	0.0141716	0.0420	0.0420629
2/3	0.00247	0.0024757	0.0179	0.0178003	0.0406	0.0406276
1	0.00192	0.00191714	0.0244	0.0243874	0.0332	0.0332449
3/2	0.00531	0.00532645	0.0585	0.0584803	0.0460	0.0459444
2	0.00844	0.00844500	0.0869	0.0868681	0.0474	0.0473622

Table 4
Deflection and bending moment factors α, β, γ for a uniformly loaded SCSF rectangular plate with $\upsilon = 0.3$ where $l = b$ for $a \geqslant b$ and $l = a$ for $a < b$

Aspect ratio $\frac{a}{b}$	Deflection factor α at $x = a/2$, $y = b/2$ where $w_{\max} = \alpha ql^4/D$		Bending moment factor β at $x = a/2$, $y = b/2$ where $M_y = \beta qb^2$		Bending moment factor γ at $x = 0$, $y = b/2$ where $M_x = \gamma qb^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
1/3	0.094	0.0939792	0.0078	0.00782003	0.428	0.427944
1/2	0.0582	0.0582267	0.0293	0.0292631	0.319	0.318975
1	0.0113	0.0112359	0.0972	0.0971846	0.119	0.118407
2	0.0150	0.0149491	0.131	0.130529	0.125	0.124666
3	0.0152	0.0152035	0.133	0.132814	0.125	0.124975

Table 5
Deflection and bending moment factors α, β, γ for a uniformly loaded SSSF rectangular plate with $\upsilon = 0.3$

Aspect ratio $\frac{a}{b}$	Deflection factor α at $x = 0$, $y = b$ where $w_{\max} = \alpha qb^4/D$		Bending moment factor β at $x = 0$, $y = b$ where $(M_y)_{\max} = \beta qb^2$		Bending moment factor γ at centre $x = 0$, $y = b/2$ $M_x = \gamma qb^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
1/2	0.00710	0.00709414	0.060	0.0601585	0.022	0.0223242
2/3	0.00968	0.00967944	0.083	0.0832446	0.030	0.0302317
1	0.01286	0.0128524	0.112	0.111701	0.039	0.0389809
2	0.01507	0.0150692	0.132	0.131608	0.041	0.0414129
3	0.01520	0.0152107	0.133	0.132878	0.039	0.0390640

Table 6
Deflection and bending moment factors α, β, γ for a uniformly loaded SSSC rectangular plate with $\upsilon = 0.3$ where $l = b$ for $a \geqslant b$ and $l = a$ for $a < b$

Aspect ratio $\frac{a}{b}$	Deflection factor α at centre $x = 0$, $y = b/2$ $w = \alpha ql^4/D$		Bending moment factor β at $x = a/2$, $y = b/2$ $M_y = \beta qb^2$		Bending moment factor γ at centre $x = 0$, $y = b/2$ $M_x = \gamma qb^2$	
	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present	Timoshenko and Woinowsky-krieger (1970)	Present
1/2	0.0093	0.00927022	0.122	0.121513	0.047	0.0468662
2/3	0.0064	0.00644513	0.112	0.112132	0.048	0.0477637
1	0.0028	0.00278549	0.084	0.0838752	0.039	0.0391781
1.5	0.0042	0.00424944	0.111	0.111212	0.054	0.0543760
2	0.0049	0.00487850	0.122	0.0121190	0.060	0.0601393

analysis can be easily extended to bending of plates with other types of boundary conditions and it will constitute the scope of analysis in Part II of this paper.

5. Conclusions

This paper has presented a new breakthrough in applied mechanics in which a bottleneck previously prohibiting availability of exact solutions for bending of plates with arbitrary boundary conditions has been trespassed. It is based on a symplectic elasticity approach which has been used previously in quantum mechanics as well as relativity and other theoretical physics disciplines. The strain energy functional in accordance with the Pro-Hellinger–Reissner variational principle is derived in a geometrical symplectic space. Using the essential Hamiltonian principle with Legendre's transformation, an eigenvalue is obtained and thus solved. It should be emphasized here that bending analysis here requires solving of an eigenvalue equation which is only required in vibration and buckling analyses in classical mechanics.

Analytical exact solutions for some cases with two opposite sides simply supported have been presented and excellent agreement with established solutions has been illustrated in all cases. Exact solutions for other combinations of boundary conditions where exact solutions are hitherto unavailable will be reported in the future. Further exact solutions for vibration, buckling and wave propagation in plates can also be obtained based on a similar symplectic approach. The extension will be explored in due course.

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