Existence and Uniqueness Results for the Bending of an Elastic Beam Equation at Resonance

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1. INTRODUCTION

The bending of an elastic beam with simply-supported ends under an external force $e(x)$ is described by the boundary-value problem

$$\frac{d^4u}{dx^4} = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{1.1}$$

Usmani [6] proved that the linear boundary-value problem

$$\frac{d^4u}{dx^4} + g(x)u = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $g(x), e(x)$ are given real-valued continuous functions on $[0, 1]$, has exactly one solution provided $\inf_x f(x) = -\eta > -\pi^4$. This result has now been extended by the author [1] to the nonlinear boundary-value problem

$$\frac{d^4u}{dx^4} + f(u)u' + g(x, u, u', u'') = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{1.3}$$

where $f: \mathbb{R} \to \mathbb{R}$ is a given continuous function, $g: [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a given function satisfying Caratheodory's conditions and $e(x) \in L'[0, 1]$. 

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Usmani [6] noted that the boundary-value problem
\[
\frac{d^4 u}{dx^4} - \pi^4 u = 0, \quad 0 < x < 1,
\]
\[u(0) = u(1) = u''(0) = u''(1) = 0,
\]
(1.4)
does not have a unique solution since \(u = C \sin \pi x\), for \(C\) an arbitrary constant, is a solution of (1.4). Now, the eigenvalues of the linear-eigenvalue problem,
\[
\frac{d^4 u}{dx^4} - \lambda u = 0, \quad 0 < x < 1,
\]
\[u(0) = u(1) = u''(0) = u''(1) = 0,
\]
(1.5)
are given by \(\lambda = n^4 \pi^4\), \(n = 1, 2, \ldots\). The purpose of this paper is to study the following nonlinear analogue of the boundary-value problem for the bending of an elastic beam which is simply supported at both ends and is at resonance:
\[
\frac{d^4 u}{dx^4} - \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,
\]
\[u(0) = u(1) = u''(0) = u''(1) = 0,
\]
(1.6)
where \(g : [0, 1] \times \mathbb{R} \to \mathbb{R}\) satisfies Caratheodory's conditions and \(e(x) \in L^1[0, 1]\) with \(\int_0^1 e(x) \sin \pi x \, dx = 0\). We show that (1.6) has at least one solution if \(g(x, u)u \geq 0\) for all \(x\) in \([0, 1]\) and \(u\) in \(\mathbb{R}\). We also show that (1.6) has a unique solution if \(g(x, u)\) is strictly increasing in \(u\) for every \(x\) in \([0, 1]\) and \(\int_0^1 g(x, 0) \sin \pi x \, dx = 0\).

We also study the boundary value problem
\[
-\frac{d^4 u}{dx^4} + \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,
\]
\[u(0) = u(1) = u''(0) = u''(1) = 0.
\]
(1.7)
We obtain the existence of a solution for (1.7) when \(e(x) \in L^1[0, 1]\) with \(\int_0^1 e(x) \sin \pi x \, dx = 0\) and \(g(x, u)\) satisfies the supplementary conditions
\[
g(x, u)u \geq 0 \quad \text{for all} \quad x \in [0, 1] \quad \text{and} \quad u \in \mathbb{R},
\]
(1.8)
and there is a constant \(\beta \geq 0\) such that
\[
\limsup_{|u| \to \infty} \frac{g(x, u)}{u} = \beta < 15\pi^4,
\]
(1.9)
uniformly for \(x \in [0, 1]\).
We show that (1.7) has a unique solution when \( e(x) \in L^1[0, 1] \) with \( \int_0^1 e(x) \sin \pi x \, dx = 0 \) and \( g \) satisfies the following conditions:

(i) \( g(x, u) \) is strictly increasing in \( u \) for every \( x \) in \([0, 1]\) and

(ii) there is a constant \( 0 \leq \beta < 15\pi^2 \) such that

\[
( g(x, u_1) - g(x, u_2))(u_1 - u_2) \geq \frac{1}{\beta} (g(x, u_1) - g(x, u_2))^2,
\]

for all \( x \) in \([0, 1]\) and \( u_1, u_2 \) in \( \mathbb{R} \).

2. Existence Theorems

Let \( X, Y \) denote the Banach spaces \( X = C[0, 1] \), \( Y = L^1[0, 1] \) with their usual norms and let \( H \) denote the Hilbert space \( L^2[0, 1] \). Let \( Y_2 \) be the subspace of \( Y \) spanned by the function \( \sin \pi x \); i.e.,

\[
Y_2 = \{ u \in Y | u(x) = a \sin \pi x \text{ a.e. for some } a \in \mathbb{R} \},
\]

and let \( Y_1 \) be the subspace of \( Y \) such that \( Y = Y_1 \oplus Y_2 \). (Here and in the following \( \oplus \) denotes direct sum.) We note that for \( u \in Y \) we can write

\[
u(x) = u(x) - \left( 2 \int_0^1 u(t) \sin \pi t \, dt \right) \sin \pi x \]

\[
+ \left( 2 \int_0^1 u(t) \sin \pi t \, dt \right) \sin \pi x,
\]

for \( x \in [0, 1] \). We define the canonical projection operators \( P: Y \to Y_1; Q: Y \to Y_2 \) by

\[
P(u) = u(x) - \left( 2 \int_0^1 u(t) \sin \pi t \, dt \right) \sin \pi x,
\]

\[
Q(u) = \left( 2 \int_0^1 u(t) \sin \pi t \, dt \right) \sin \pi x,
\]

for \( u \in Y \). Clearly, \( Q = I - P \), where \( I \) denotes the identity mapping on \( Y \), and the projections \( P \) and \( Q \) are continuous. Now let \( X_2 = X \cap Y_2 \). Clearly \( X_2 \) is a closed subspace of \( X \). Let \( X_1 \) be the closed subspace of \( X \) such that \( X = X_1 \oplus X_2 \). We note that \( P(X) \subseteq X_1 \), \( Q(X) \subseteq X_2 \) and the projections \( P|X: X \to X_1 \), \( Q|X: X \to X_2 \) are continuous. Similarly, we obtain \( H = H_1 \oplus H_2 \) and the continuous projections \( P|H: H \to H_1 \), \( Q|H: H \to H_2 \). In the following, \( X, Y, H, P, Q \), etc., will refer to Banach spaces, Hilbert space
and the projections as defined above and we shall not distinguish between $P, P\mid X, P\mid H$ (resp. $Q, Q\mid X, Q\mid H$) and depend on the context for proper meaning.

Also for $u \in X, v \in Y$ let $(u, v) = \int_0^1 u(x) v(x) \, dx$ denote the duality pairing between $X$ and $Y$. We note that for $u \in X, v \in Y$ so that $u = Pu + Qu, \quad v = Pv + Qv$, we have

$$(u, v) = (Pu, Pv) + (Qu, Qv). \quad (2.4)$$

Define a linear operator $L: D(L) \subset X \rightarrow Y$ by setting

$$D(L) = \{u \in X \mid u', u'', u''' \in AC[0, 1],
\quad u(0) = u(1) = u''(0) = u''(1) = 0\} \quad (2.5)$$

and for $u \in D(L)$,

$$Lu = \frac{d^4u}{dx^4} - \pi^4 u. \quad (2.5),$$

(Here $AC[0, 1]$ denotes the space of real valued absolutely continuous functions on $[0, 1]$.) Now, for $u \in D(L)$, we see using Fourier series and Parseval's identity that

$$(Lu, u) = \int_0^1 \frac{d^4u}{dx^4} u \, dx - \pi^4 \int_0^1 u'^2 \, dx = \int_0^1 u''^2 - \pi^4 \int_0^1 u^2 \geq 0. \quad (2.6)$$

**Lemma 2.1.** For $h \in Y_1$, i.e., $h \in L^1[0, 1]$ with $\int_0^1 h(t) \sin \pi t \, dt = 0$, the linear boundary value problem

$$\frac{d^4u}{dx^4} - \pi^4 u = h(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0 \quad (2.7)$$

has a unique solution $u(x)$ with $\int_0^1 u(x) \sin \pi x \, dx = 0$.

**Proof.** For $x \in [0, 1]$ define

$$v_1(x) = e^{i\pi x} \int_0^x h(t) e^{-i\pi t} \, dt,$$

$$v_2(x) = e^{-i\pi x} \int_0^x v_1(t) e^{i\pi t} \, dt,$$

$$v_3(x) = e^{i\pi x} \int_0^x v_2(t) e^{-i\pi t} \, dt,$$

$$v(x) = e^{-i\pi x} \int_0^x v_3(t) e^{i\pi t} \, dt.$$
Then \( u(x) = c_1 e^{\pi x} + c_2 e^{-\pi x} + c_3 e^{inx} + c_4 e^{-inx} + v(x) \) is such that \( Rl(u(x)) \) is a general solution of the equation \( (d^4u/dx^4) - \pi^4u = h(x) \).

Next we compute \( c_1, c_2, c_3, c_4 \) using the boundary conditions \( u(0) = u(1) = u''(0) = u''(1) = 0 \) and the condition \( \int_0^1 u(x) \sin \pi x \, dx = 0 \). Now \( u(0) = u''(0) = 0 \) give

\[
c_1 + c_2 = c_3 + c_4 = 0, \tag{2.8}
\]

and \( u(1) = u''(1) = 0 \) give

\[
c_1 e^{\pi} + c_2 e^{-\pi} - c_3 - c_4 + v(1) = 0, \tag{2.9}
\]

in view of the assumption \( \int_0^1 h(t) \sin \pi t \, dt = 0 \). It is now easy to compute \( c_1, c_2, c_3, c_4 \) uniquely from (2.8), (2.9) and the equation \( \int_0^1 u(x) \sin \pi x \, dx = 0 \) to get \( Rl(u(x)) \) as the unique solution for (2.7).

Let, now, for \( e \in Y_1 \), i.e., \( e \in L^1[0, 1] \) with \( \int_0^1 e(x) \sin \pi x \, dx = 0 \), \( u = Ke \) denote the unique solution of the problem

\[
\frac{d^4u}{dx^4} - \pi^4u = e(x), \quad 0 < x < 1,
\]

\[
u(0) = u(1) = u''(0) = u''(1) = 0,
\]

such that \( \int_0^1 u(x) \sin \pi x \, dx = 0 \). It is immediate that the linear mapping \( K: Y_1 \to X_1 \) is bounded and is such that for

\[
u \in Y, \, KPU \in D(L), \, LKP(u) - P(u), \text{ and } (KP(u), P(u)) \geq 0. \tag{2.10}
\]

Also we see, using Fourier series and Parseval's identity, for \( u \in H_1 \) (i.e., \( u \in L^2[0, 1] \) with \( \int_0^1 u(x) \sin \pi x \, dx = 0 \)) that

\[
(Ku, u) \leq \frac{1}{15\pi^4} \|u\|^2_H, \tag{2.11}
\]

with equality if and only if \( u = C \sin 2\pi x \) for some \( C \in \mathbb{R} \).

**Definition 2.2.** \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfies Caratheodory's conditions for \( L^1[0, 1] \) (resp. \( L^2[0, 1] \)) if \( g(x, \cdot) \) is continuous for a.e. \( x \in [0, 1] \), \( g(\cdot, u) \) is measurable for every \( u \in \mathbb{R} \), and for each \( r \in \mathbb{R} \) there is a function \( \alpha_r(x) \in L^1[0, 1] \) (resp. \( L^2[0, 1] \)) such that \( |g(x, u)| \leq \alpha_r(x) \) for a.e. \( x \) in \([0, 1] \) and every \( u \) in \( \mathbb{R} \) with \( |u| \leq r \).
Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) be given and \( N: X \to Y \) be the nonlinear mapping defined by
\[
(Nu)(x) = g(x, u(x)), \quad x \in [0, 1],
\]
(2.12)
for \( u \in X \). Also it is easy to see, using Arzela–Ascoli Theorem, that \( KPN: X \to X_1 \) is a well defined compact mapping and \( QN: X \to X_2 \) is a bounded mapping.

For \( e(x) \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \), the boundary value problem
\[
\frac{d^4u}{dx^4} - \pi^4u + g(x, u) = e(x), \quad x \in [0, 1],
\]
\[
u(0) = u(1) = u''(0) = u''(1) = 0,
\]
(2.13)
now reduces to the functional equation
\[
Lu + Nu = e
\]
in \( X \) with a given \( e \in Y_1 \).

**Theorem 2.3.** Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfy Caratheodory’s conditions for \( L^1[0, 1] \) and
\[
g(x, u)u \geq 0 \quad \text{for} \quad x \in [0, 1], u \in \mathbb{R}.
\]
(2.15)
Then, for each \( e \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \), the boundary value problem
\[
\frac{d^4u}{dx^4} - \pi^4u + g(x, u) - e(x), \quad x \in [0, 1],
\]
\[
u(0) = u(1) = u''(0) = u''(1) = 0,
\]
(2.16)
has at least one solution in \( X = C[0, 1] \).

**Proof.** In the following \( X, Y, X_1, X_2, Y_1, Y_2, L, K, P, Q, N \) will be as defined above from the beginning of this section to just before the statement of this theorem.

As noted above, the boundary value problem (2.16) reduces to the functional equation
\[
Lu + Nu = e,
\]
(2.17)
in $X$ with $e \in Y_1$. Now to solve the functional equation (2.17) it suffices to solve the system of equations

$$Pu + KPNu = e_1,$$

$$QNu = 0,$$

(2.18)

$u \in X$, $e_1 = Ke$. Indeed, if $u \in X$ is a solution of (2.18) then $u \in D(L)$ and

$$LPu + LKPNu = Lu + PNu = Le_1 = e,$$

$$QNu = 0,$$

which gives on adding that $Lu + Nu = e$.

Now, (2.18) is clearly equivalent to the single equation

$$Pu + QNu + KPNu = e_1,$$

(2.19)

which has the form of a compact perturbation of the Fredholm operator $P$ of index zero. We can therefore apply the version given in [2, Theorem 1, Corollary 1] or [3, Theorem IV.4] or [4] of the Leray–Schauder continuation theorem which ensures the existence of a solution for (2.19) if the set of solutions of the family of equations

$$Pu + (1 - \lambda)Qu + \lambda QNu + \lambda KPNu = \lambda e_1,$$

(2.20)

$\lambda \in (0, 1)$, is a priori bounded independently of $\lambda$. Notice that (2.20) is then equivalent to the system of equations

$$Pu + \lambda KPNu = \lambda e_1,$$

$$QNu = 0.$$ (2.21)

If $u_\lambda \in X$ is a solution of (2.21) for $\lambda \in (0, 1)$ then $u_\lambda \in D(L)$ and

$$(Pu_\lambda, PNu_\lambda) + \lambda(KPNu_\lambda, PNu_\lambda) = \lambda(e_1, PNu_\lambda),$$

$$(1 - \lambda)(Qu_\lambda, QNu_\lambda) + \lambda(QNu_\lambda, QNu_\lambda) = 0.$$ (2.22)

Consequently, we have using (2.10) that

$$(Pu_\lambda, PNu_\lambda) \leq \lambda(e_1, PNu_\lambda),$$

$$(Qu_\lambda, QNu_\lambda) \leq 0.$$ (2.22)

Next, it is easy to prove from our assumptions on $g$ that for every $k \in \mathbb{R}$, $k \geq 0$, there is a constant $C(k) \geq 0$ such that

$$(Nu, u) \geq k \|Nu\|_Y - C(k), \quad u \in Y.$$ (2.23)
Using, now, (2.23) and (2.22) we see that for each $k \in \mathbb{R}$, $k \geq 0$, there is a constant $C(k) \geq 0$ such that
\[
k \| Nu_j \|_Y - C(k) \leq (Nu_j, u_j) \leq \lambda(e_1, PNu_j)
\]
\[
\leq \| e_1 \|_X \| PNu_j \|_Y
\]
\[
\leq C_0 \| e_1 \|_X \| Nu_j \|_Y,
\]
where $C_0 \geq 0$ is such that $\| Pu \|_Y \leq C_0 \| u \|_Y$.

Thus,
\[
(k - C_0 \| e_1 \|_X) \| Nu_j \|_Y \leq C(k). \tag{2.24}
\]

Also we obtain from the first equation in (2.21) that
\[
\| Pu_j \|_X \leq \| KP\lambda Nu_j \|_X + \| e_1 \|_X \leq \| K \| \| P\lambda Nu_j \|_Y + \| e_1 \|_X
\]
\[
\leq C_0 \| K \| \| Nu_j \|_Y + \| e_1 \|_X. \tag{2.25}
\]

Taking $k > C_0 \| e_1 \|_X$ we see from (2.24) and (2.25) that there is a constant $C \geq 0$, independent of $\lambda$ in $(0, 1)$ such that
\[
\| Nu_j \|_Y \leq C, \quad \| Pu_j \|_X \leq C, \quad \lambda \in (0, 1). \tag{2.26}
\]

It only remains to prove that there is a constant $C_1$, independent of $\lambda \in (0, 1)$ such that $\| Qu_j \|_X \leq C_1$, $\lambda \in (0, 1)$. Let us suppose, on the other hand, that the set
\[
\{ \| Qu_j \|_X : \lambda \in (0, 1) \}
\]
is unbounded. \tag{2.27}

We now have from the first equation in (2.21) that
\[
LPu_j + \lambda LKP\lambda Nu_j = \lambda Le_1,
\]
i.e.,
\[
J. Pu_j + \lambda PNu_j = \lambda e,
\]
so that
\[
\| LPu_j \|_Y \leq \lambda \| Pu_j \|_Y + \lambda \| e \|_Y
\]
\[
\leq \| Pu_j \|_Y + \| e \|_Y
\]
\[
\leq C_0 \| Nu_j \|_Y + \| e \|_Y
\]
\[
\leq C_0 C + \| e \|_Y \equiv C_1,
\]
for $\lambda \in (0, 1)$. 
Since, now,

\[ LPu_\lambda = \frac{d^4}{dx^4} (Pu_\lambda) - \pi^4 Pu_\lambda, \]

and \( \|Pu_\lambda\|_X \leq C, \|PNu_\lambda\|_Y \leq C_0 \|Nu_\lambda\|_Y \leq C_0 C \) we see easily that there is a constant \( C_2 \) independent of \( \lambda \in (0, 1) \) such that

\[ \left\| \frac{d^4}{dx^4} (Pu) \right\|_Y \leq C_2, \quad \lambda \in (0, 1). \quad (2.28) \]

Also, since \( u_{\lambda,1} \in D(L) \) so that \( u_\lambda(0) = u_\lambda(1) = u_\lambda'(0) = u_\lambda'(1) = 0 \), it follows easily from (2.28) that

\[ \|u_{\lambda,1}'\|_X \leq C_2, \quad \lambda \in (0, 1). \]

We next use the well-known estimate (see, e.g., estimate (16) of [5])

\[ \left| \frac{v(x)}{\sin \pi x} \right| \leq \frac{1}{2} \max_{s \in [0,1]} |v'(s)|, \quad x \in [0,1], \quad (2.29) \]

for \( v \in X, v(0) = v(1) = 0 \), to get

\[ |(Pu_\lambda)(x)| \leq \frac{1}{2} C_2 \sin \pi x \quad \text{for} \quad x \in [0,1], \lambda \in (0, 1). \quad (2.30) \]

Now by (2.27) we see that there is a sequence \( \{\lambda_n\}, \lambda_n \in (0, 1) \), such that

\[ \|Qu_{\lambda_n}\|_X = \left| 2 \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \right| \to \infty, \]

as \( n \to \infty \). We may now assume that

\[ \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \to \infty, \]

as \( n \to \infty \), so that there is an \( n_0 \) such that

\[ \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \geq \frac{1}{4} C_2 \quad \text{for} \quad n \geq n_0. \quad (2.31) \]

So, for \( n \geq n_0, x \in [0,1] \) we have using (2.30), (2.31) that

\[ u_{\lambda_n}(x) = Qu_{\lambda_n}(x) + Pu_{\lambda_n}(x) \]

\[ = \left( 2 \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \right) \sin \pi x + Pu_{\lambda_n}(x) \]

\[ \geq 2 \cdot \frac{1}{4} C_2 \sin x - \frac{1}{2} C_2 \sin \pi x = 0. \]
Since, now, \( g(x, u)u \geq 0 \) for \( x \in [0, 1] \), \( u \in \mathbb{R} \) we have \( g(x, u_n(x)) \geq 0 \) for \( n \geq n_0 \), \( x \in [0, 1] \) and accordingly,

\[
(QU_{\lambda_n}, QU_{\lambda_n}) \geq 0 \quad \text{for} \quad n \geq n_0. \tag{2.32}
\]

Also, it follows from the second equation in (2.21) and (2.32) that

\[
(1 - \lambda_n)(QU_{\lambda_n}, QU_{\lambda_n}) = 2(1 - \lambda_n) \left( \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \right)^2 \leq 0,
\]

for \( n \geq n_0 \), a contradiction. Similarly, assuming \( \int_0^1 u_{\lambda_n}(t) \sin \pi t \, dt \to -\infty \) leads to a contradiction.

Thus the set \( \left\{ \|QU_{\lambda_n}\| : \lambda \in (0, 1) \right\} \) is bounded by a constant independent of \( \lambda \in (0, 1) \). We have, accordingly, proved that the set of solutions of the system of equations (2.21) is bounded by a constant independent of \( \lambda \in (0, 1) \) and the proof of the theorem is complete. \( \square \)

**Theorem 2.4.** Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfy Caratheodory's conditions for \( L^1[0, 1] \) and suppose that for each \( x \) in \( [0, 1] \), \( g(x, \cdot) \) be non-decreasing.

Then for each \( e \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \), the boundary value problem

\[
\frac{d^4u}{dx^4} - \pi^4u + g(x, u) = e(x), \quad x \text{ in } [0, 1], \tag{2.33}
\]

\[
u(0) = u(1) = u''(0) = u''(1) = 0,
\]

has at least one solution is \( X = C[0, 1] \) if and only if there exists an \( \alpha \in \mathbb{R} \) such that

\[
\tilde{g}(\alpha) \equiv \int_0^1 g(x, \alpha \sin \pi x) \sin \pi x \, dx = 0. \tag{2.34}
\]

**Proof.** Necessity of (2.34). If \( u \) is a solution of (2.33) then it follows by multiplying the equation in (2.33) by \( \sin \pi x \) and integrating the resulting equation over \([0, 1]\) that

\[
\int_0^1 g(x, u(x)) \sin \pi x \, dx = 0. \tag{2.35}
\]

Letting, now, \( R = \max_{x \in [0, 1]} |u'(x)| \) and using estimate (2.29) for \( u \), (2.35) and the non-decreasing nature of \( g(x, \cdot) \) for each \( x \) in \([0, 1]\) we have

\[
\tilde{g}(-\frac{1}{2} R) \leq 0 \leq \tilde{g}(\frac{1}{2} R).
\]
The existence of $\alpha \in \mathbb{R}$ such that $g(\alpha) = 0$ is now immediate from intermediate value theorem. We remark that this proof of necessity of (2.34) is essentially the same as the proof of Lemma 2 of [5] and is included here for completeness.

Sufficiency of (2.34). Suppose that there exists an $\alpha \in \mathbb{R}$ such that

$$\int_0^1 g(x, \alpha \sin \pi x) \sin \pi x \, dx = 0.$$ 

Setting, now,

$$g_1(x, u) = g(x, u) - g(x, \alpha \sin \pi x),$$

$$e_1(x) = e(x) - g(x, \alpha \sin \pi x),$$

$$v = u - \alpha \sin \pi x,$$

$$g_2(x, v) = g_1(x, v + \alpha \sin \pi x),$$

we see that (2.33) is equivalent to

$$\frac{d^4 v}{dx^4} - \pi^4 v + g_2(x, v) = e_1(x), \quad 0 < x < 1,$$

$$v(0) = v(1) = v''(0) = v''(1) = 0.$$ 

(2.36)

Clearly, $g_2: [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions for $L^1[0, 1]$ and

$$g_2(x, v)v = g_1(x, v + \alpha \sin \pi x)v$$

$$= \left[ g(x, v + \alpha \sin \pi x) - g(x, \alpha \sin \pi x) \right] v \geq 0$$

for $x \in [0, 1], v \in \mathbb{R}$ and the existence of a solution for (2.36) is immediate from Theorem 2.3 above. 

**Theorem 2.5** Let $g: [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions for $L^2[0, 1]$ and

(i) $g(x, u)u \geq 0$ for $x \in [0, 1], u \in \mathbb{R},$

(ii) there is a constant $\beta \geq 0$ such that

$$\lim_{|u| \to \infty} \frac{g(x, u)}{u} = \beta < 15\pi^4,$$

uniformly for $x \in [0, 1]$. 

Then, for each $e \in Y = L^1[0,1]$ with $\int_0^1 e(t) \sin \pi t \, dt = 0$ the boundary value problem

$$\frac{d^4 u}{dx^4} + \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

has at least one solution $u$ in $X = C[0,1]$.

Proof. Let us set $\tilde{L} = -L$ and $\tilde{K} = -K$, where $L : D(L) \subset X \to Y$ is the linear operator defined by (2.5), (2.5)$_1$ and $K : Y \to X_1$ is the bounded linear mapping as in (2.10), (2.11). Accordingly, we have for

$$u \in Y, \quad \tilde{K}P(u) \in D(\tilde{L}), \quad \tilde{L}\tilde{K}P(u) = P(u), \quad (2.38)$$

and for $u \in H_1$,

$$(\tilde{K}u, u) \geq -\frac{1}{15\pi^4} \|u\|_{H^4}^2.$$  

(2.39)

Now, as in the proof of Theorem 2.3 the boundary value problem (2.37) reduces to

$$\tilde{L}u + Nu = e,$$

in $X$ with $e \in Y$, and it suffices to show that the set of solutions of the system of equations

$$Pu + \lambda \tilde{K}PNu = e_1,$$

$$(1 - \lambda) Qu + \lambda QNu = 0,$$

(2.40)

where $e_1 = \tilde{K}e$, is a priori bounded in $X$ independently of $\lambda \in (0,1)$.

Let $u_{\lambda}$ be a solution of (2.40) for $\lambda \in (0,1)$. Then we obtain as in Theorem 2.3

$$(Pu_{\lambda}, PNu_{\lambda}) + \lambda(\tilde{K}PNu_{\lambda}, PNu_{\lambda}) = \lambda(e_1, PNu_{\lambda}),$$

$$(1 - \lambda)(Qu_{\lambda}, QNu_{\lambda}) + \lambda(Qu_{\lambda}, QNu_{\lambda}) = 0,$$

and, hence, we get using (2.39) that

$$(Pu_{\lambda}, PNu_{\lambda}) - \frac{1}{15\pi^4} \|PNu_{\lambda}\|_H^2 \leq \lambda(e_1, PNu_{\lambda}),$$

$$(Qu_{\lambda}, QNu_{\lambda}) \leq 0.$$  

(2.41)
Since, now, \( \limsup_{|u| \to \infty} \left( g(x, u) / u \right) = \beta < 15\pi^4 \) uniformly in \( x \in [0, 1] \), we see, choosing \( \varepsilon > 0 \) such that \( \beta + \varepsilon < 15\pi^4 \), that there is a constant \( C(\varepsilon) > 0 \) such that

\[
(Nu, u) \geq \frac{1}{\beta + \varepsilon} \| Nu \|_H^2 - C(\varepsilon) \tag{2.42}
\]

for \( u \in H \). We next have from (2.41), (2.42) that

\[
\frac{1}{\beta + \varepsilon} \| Nu_\lambda \|_H^2 - C(\varepsilon) \leq (Nu_\lambda, u_\lambda) \\
\leq \frac{1}{15\pi^4} \| PNu_\lambda \|_H^2 + \| e_1 \|_X \| PNu_\lambda \|_Y \\
\leq \frac{1}{15\pi^4} \| Nu_\lambda \|_H^2 + C_0 \| e_1 \|_X \cdot \| Nu_\lambda \|_Y.
\]

Consequently,

\[
\left( \frac{1}{\beta + \varepsilon} - \frac{1}{15\pi^4} \right) \| Nu_\lambda \|_H^2 \leq C_0 \| e_1 \|_X \cdot \| Nu_\lambda \|_H + C(\varepsilon),
\]

so that there is a constant \( C > 0 \), independent of \( \lambda \in (0, 1) \) such that

\[
\| Nu_\lambda \|_H \leq C.
\]

Also, we obtain from the first equation in (2.40) that

\[
\| Pu_\lambda \|_X \leq \| \tilde{K} P Nu_\lambda \|_X + \| e_1 \|_X \\
\leq \| \tilde{K} \| \| P Nu_\lambda \|_Y + \| e_1 \|_X \\
\leq C_0 \| \tilde{K} \| \| Nu_\lambda \|_H + \| e_1 \|_X \\
\leq C_0 \| \tilde{K} \| C + \| e_1 \|_X \equiv C_1.
\]

The boundedness of \( \{ \| Q u_\lambda \|_X : \lambda \in (0, 1) \} \) now follows as in the proof of Theorem 2.3 above.

Thus, we have shown that the set of solutions of (2.40) is, a priori, bounded in \( X \), independently of \( \lambda \in (0, 1) \) and the proof of the theorem is complete. \( \blacksquare \)

**Theorem 2.6.** Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) be continuous on \( [0, 1] \times \mathbb{R} \) and

(i) for each \( x \) in \([0, 1]\), \( g(x, u) \) be non-decreasing in \( u \);
(ii) there is a constant $\beta \geq 0$ such that

$$\limsup_{|u| \to \infty} \frac{g(x, u)}{u} = \beta < 15\pi^4,$$

uniformly for $x$ in $[0, 1]$.

Then, for each $e \in Y = L^1[0, 1]$ with $\int_0^1 e(t) \sin \pi t \, dt = 0$ the boundary value problem

$$-\frac{d^4u}{dx^4} + \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

has at least one solution $u$ in $X = C[0, 1]$ if and only if there exists an $\alpha \in \mathbb{R}$ such that

$$\tilde{g}(x) \equiv \int_0^1 g(x, \alpha \sin \pi x) \sin \pi x \, dx = 0.$$

Theorem 2.6 follows from Theorem 2.5 in a similar way as Theorem 2.4 followed from Theorem 2.3. Accordingly, the proof of Theorem 2.6 is left to the reader.

3. **Uniqueness Theorems**

It was remarked by Usmani [6] that the boundary value problem

$$\frac{d^4u}{dx^4} - \pi^4 u = 0, \quad 0 < x < 1,$$

$$u(0) - u(1) = u''(0) = u''(1), \quad \text{(3.1)}$$

does not have a unique solution since $u(x) = C \sin \pi x$, with $C$ arbitrary, is a solution for (3.1). Now, the kernel of the linear operator $L: D(L) \subset X \to Y$ defined by (2.5), (2.5), is given by

$$\ker L = \{ \alpha \sin \pi x | \alpha \in \mathbb{R} \},$$

and $\ker L \neq \{0\}$. This is what causes the non-uniqueness of a solution for the boundary value problem (3.1). It turns out that any two solutions of the boundary value problem (1.6) differ by $\alpha \sin \pi x$, for some $\alpha \in \mathbb{R}$, if $g(x, u)$ is non-decreasing in $u$ for every $x$ in $[0, 1]$. 
THEOREM 3.1. Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfy Caratheodory's conditions for \( L^1[0, 1] \) such that

(i) \( g(x, u) \) is non-decreasing in \( u \) for every \( x \) in \([0, 1]\) and,

(ii) \( g(x, 0) = 0 \) for a.e. \( x \) in \([0, 1]\).

Then for each \( e \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \), any two solutions of the boundary value problem

\[
\frac{d^4 u}{dx^4} - \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,
\]

\[
u(0) = u(1) = u''(0) = u''(1) = 0
\]

\( (3.2) \)

differ by \( \alpha \sin \pi x \), for some \( \alpha \in \mathbb{R} \).

**Proof.** Since \( g(x, 0) = 0 \) for \( x \) in \([0, 1]\), we have \( g(x, u)u \geq 0 \) and existence of a solution for \( (3.2) \) follows from Theorem 2.3. Using, now, the notation of Section 2 we can write \( (3.2) \) as a functional equation

\[
Lu + Nu = e,
\]

in \( X \). Let, now, \( u_1, u_2 \) be any two solutions of \( (3.2) \). We then have using \((2.6)\) that

\[
0 = (Lu_1 - Lu_2, u_1 - u_2) + (Nu_1 - Nu_2, u_1 - u_2)
\]

\[
\geq (Nu_1 - Nu_2, u_1 - u_2)
\]

\[
\geq \int_0^1 (g(x, u_1(x)) - g(x, u_2(x))(u_1(x) - u_2(x)) \, dx \geq 0,
\]

since \( g(x, u) \) is non-decreasing in \( u \) for a.e. \( x \) in \([0, 1]\). Accordingly, it follows that

\[
g(x, u_1(x)) = g(x, u_2(x)) \quad \text{for a.e. } x \text{ in } [0, 1],
\]

that is,

\[
Nu_1 = Nu_2,
\]

and hence

\[
Lu_1 = Lu_2.
\]

So \( u_1 - u_2 \in \ker L \) and there must exist an \( \alpha \in \mathbb{R} \) such that \( u_1(x) - u_2(x) = \alpha \sin \pi x \) for \( x \) in \([0, 1]\).

Hence the theorem. \( \blacksquare \)
THEOREM 3.2. Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) be as in Theorem 3.1. Suppose additionally that \( g(x, u) \) is strictly increasing in \( u \) for a.e. \( x \) in \([0, 1]\). Then the boundary value problem (3.2) has exactly one solution for each given \( e \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \).

Proof. The existence of a solution for (3.2) is obvious since \( g \) satisfies conditions of Theorem 3.1. Also we have

\[
g(x, u_1(x)) - g(x, u_2(x)) \quad \text{for a.e. } x \in [0, 1],
\]
as in the proof of Theorem 3.1. Since, now \( g \) is strictly increasing we have \( u_1(x) = u_2(x) \) for a.e. \( x \) in \([0, 1]\) and hence \( u_1 \equiv u_2 \) since \( u_1, u_2 \) are continuous on \([0, 1]\).

Hence the theorem. \( \blacksquare \)

THEOREM 3.3. Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) be continuous on \([0, 1] \times \mathbb{R}\) and suppose that

(i) \( g(x, u) \) is non-decreasing in \( u \) for a.e. \( x \) in \([0, 1]\) and

(ii) there exists an \( a \in \mathbb{R} \) such that \( \int_0^1 g(x, u \sin \pi x) \sin \pi x \, dx = 0 \).

Then for each \( e \in Y = L^1[0, 1] \) with \( \int_0^1 e(t) \sin \pi t \, dt = 0 \)

(a) any two solutions of the boundary value problem (3.2) differ by \( c \sin \pi x \), for some \( c \in \mathbb{R} \);

(b) the boundary value problem (3.2) has exactly one solution if \( g(x, u) \) is strictly increasing in \( u \) for a.e. \( x \) in \([0, 1]\).

Proof. Let us define \( e_1 \in Y = L^1[0, 1] \) and \( g_2: [0, 1] \times \mathbb{R} \to \mathbb{R} \) as in the proof of Theorem 2.4. Then the boundary value (3.2) is equivalent to the boundary value problem

\[
\frac{d^4 u}{dx^4} - \pi^4 u + g_2(x, u) = e_1(u), \quad 0 < x < 1,
\]

\[
u(0) - u(1) = u''(0) = u''(1) = 0.
\]

Also \( g_2(x, 0) \equiv 0 \) for \( x \) in \([0, 1]\), \( g_2(x, u) \) is non-decreasing in \( u \) for a.e. \( x \) in \([0, 1]\). Further, \( g_2(x, u) \) is strictly increasing in \( u \) for a.e. \( x \) in \([0, 1]\) if \( g(x, u) \) is strictly increasing in \( u \) for a.e. \( x \) in \([0, 1]\).

Theorem 3.3 is now immediate from Theorems 3.1, 3.2. \( \blacksquare \)

THEOREM 3.4. Let \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfy Caratheodory's conditions for \( L^2[0, 1] \). Suppose \( 0 \leq \beta < 15\pi^4 \) be such that

\[
(g(x, u_1) - g(x, u_2))(u_1 - u_2) \geq \frac{1}{\beta} (g(x, u_1) - g(x, u_2))^2
\]

(3.3)
for a.e. $x$ in $[0, 1]$ and all $u_1$, $u_2$ in $\mathbb{R}$. Also let $g(x, 0) \equiv 0$ for $x$ in $[0, 1]$. Then for each $e \in Y = L^4[0, 1]$ with $\int_0^1 e(t) \sin \pi t \, dt = 0$

(i) any two solutions of the boundary value problem

$$\frac{d^4 u}{dx^4} + \pi^4 u + g(x, u) = e(x), \quad 0 < x < 1,$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$  \hfill(3.4)

differ by $c \sin \pi x$, for some $c \in \mathbb{R}$;

(ii) the boundary value problem (3.4) has exactly one solution if $g(x, u)$ is additionally strictly increasing in $u$ for a.e. $x$ in $[0, 1]$.

**Proof.** Since $g(x, 0) \equiv 0$ for $x$ in $[0, 1]$ it follows easily from (3.3) that $g(x, u)u \geq 0$ and $(g(x, u)/u) \leq \beta$ for $x$ in $[0, 1]$, $u$ in $\mathbb{R}$. The existence of a solution for (3.4) then follows from Theorem 2.5. Using the notation used in the proof of Theorem 2.5 we see that (3.4) is equivalent to the functional equation

$$Lu + Nu = e.$$

Let now $u_1$, $u_2$ be two solutions of (3.4). Then we have,

$$0 = (\tilde{L}u_1 - \tilde{L}u_2, u_1 - u_2) + (Nu_1 - Nu_2, u_1 - u_2)$$

$$\geq - \frac{1}{15\pi^4} \|\tilde{L}u_1 - \tilde{L}u_2\|_H^2 + \frac{1}{\beta} \|Nu_1 - Nu_2\|_H^2$$

$$= \left(\frac{1}{\beta} - \frac{1}{15\pi^4}\right) \|\tilde{L}u_1 - \tilde{L}u_2\|_H^2$$

in view of (2.39) and (3.3). Since, now, $\beta < 15\pi^4$, we get $\tilde{L}u_1 - \tilde{L}u_2 = 0$, i.e., $u_1 - u_2 \in \ker \tilde{L} = \ker L$ and so there is a $c \in \mathbb{R}$ such that $u_1 - u_2 = c \sin \pi x$. This proves (i).

If we, additionally, assume that $g(x, u)$ is strictly increasing in $u$ for a.e. $x$ in $[0, 1]$, we see from $\tilde{L}u_1 = \tilde{L}u_2$ that $Nu_1 = Nu_2$, i.e., $g(x, u_1(x)) = g(x, u_2(x))$, for, i.e., $x$ in $[0, 1]$. Hence $u_1(x) = u_2(x)$ for a.e. $x$ in $[0, 1]$. This proves (ii) and the proof of the theorem is complete. \hfill \square

**Remark 3.5.** The conclusion of Theorem 3.4 remains valid if we replace the assumption $g(x, 0) \equiv 0$ for $x$ in $[0, 1]$ by the assumption that there exists an $\alpha$ in $\mathbb{R}$ such that

$$\int_0^1 g(x, \alpha \sin \pi x) \sin \pi x \, dx = 0.$$
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REFERENCES