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Separable Functors Applied to Graded Rings

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INTRODUCTION

In this paper we introduce the notion of separability for functors. As special cases we mention the restriction (of scalars) functor, φ_*, and the induction functor, φ^*, associated to a ring homomorphism $\varphi: R \to S$; separability of φ_* relates to separability of S over R which is defined in terms of splitting of the canonical map $\psi: S \otimes_R S \to S$ as an S-bimodule map. Separable functors and their properties have various applications, but here we restrict to applications in the theory of graded rings where the morphism φ is usually taken to be the map $R_e \to R$, where e is the neutral element of the group G. However, there are a few very natural functors around that allow interesting applications. After the general properties of separability studied in Section 1 we turn to separability of the restriction functor for graded rings in Section 2. In case R is strongly graded by G then R is separable over R_e if and only if G is finite and the trace function is surjective. The forgetful functor $U: R\text{-gr} \to R\text{-mod}$ (U forgets the gradiation) is always a separable functor.

The body of this paper deals with the properties of the right adjoint F of U in case G is finite of order n. Let $\mathrm{Col}_G(M)$ be the set of $n \times 1$-columns over M and $H: R\text{-mod} \to R\text{-gr}$ the functor obtained by putting $H(M) = \mathrm{Col}_G(M)$ viewing it as a graded R-module in some way, then the functors F and H are equivalent. In Theorem 3.6 we provide a separability criterion for F using smash-product constructions appearing in the duality for coac-

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tions. Separability of F is weaker than separability of R over R_e but in the strongly graded case it amounts to the same thing. The general properties F enjoys are summed up in Theorem 3.10, e.g., F preserves: finitely generated, finitely presented, Noetherian, Artinian, Krull dimension, Gabriel dimension, projective, injective, essentiality, inj.dimension (if $|G| = n$ is invertible in R). If M is a simple R-module then $F(M)$ is semi-simple of finite length (G is finite here!) in R-gr; this provides alternative approaches to results of E. Dade [3] and P. Greszczuk [4]. Finally, we present yet another construction of the functor F using group rings over graded rings; in this way we show that existing techniques in the literature, i.e., smash-products, duality for (co-)actions, group ring constructions, blend together well in the study of the functor F and its separability properties.

From the generality of Section 1 it is clear that several other applications may exist outside the graded context, e.g., separability of representable functors, etc., but we hope to come back to such applications in the future.

1. Separable Functors

Consider categories $\mathscr{C}, \mathscr{D}$ (in most applications these categories are abelian or at least additive).

A covariant functor $F: \mathscr{C} \to \mathscr{D}$ is said to be a separable functor if for all objects M, N in $\mathscr{C}$ there are maps $\varphi^F_{M,N}$, $\varphi^F_{M,N}$: $\mathrm{Hom}_{\mathscr{D}}(F(M), F(N)) \to \mathrm{Hom}_{\mathscr{C}}(M, N)$, satisfying the following conditions:

SF.1. For $\alpha \in \mathrm{Hom}_{\mathscr{C}}(M, N)$ we have $\varphi^F_{M,N}(F(\alpha)) = \alpha$.

SF.2. Given M', N' in $\mathscr{C}$, $\alpha \in \mathrm{Hom}_{\mathscr{C}}(M, M')$, $\beta \in \mathrm{Hom}_{\mathscr{C}}(N, N')$, $f \in \mathrm{Hom}_{\mathscr{D}}(F(M), F(N))$, $g \in \mathrm{Hom}_{\mathscr{D}}(F(M'), F(N'))$ such that the following diagram is commutative:

$$\begin{array}{ccc} F(M) & \xrightarrow{f} & F(N) \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle F(\beta)} \\ F(M') & \xrightarrow{g} & F(N') \end{array}$$

then the following diagram is also commutative:

$$\begin{array}{ccc} M & \xrightarrow{\varphi^F_{M,N}(f)} & N \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow \\ M' & \xrightarrow[\varphi^F_{M',N'}(g)]{} & N' \end{array}$$

Note that condition SF.1 entails that $\varphi^F_{M,M}(1_{F(M)}) = 1_M$. A few elementary properties of separable functors are summed up as:

1.1. Lemma. 1. *An equivalence of categories is separable.*

2. *If $F: \mathscr{C} \to \mathscr{D}$, $G: \mathscr{D} \to \mathscr{E}$ are separable functors then the composed functor GF is separable too.*

3. *Let F, G be as in 2. If GF is separable then F is a separable functor.*

Proof. 1. The map $f \mapsto F(f)$, $\mathrm{Hom}_{\mathscr{C}}(M, N) \to \mathrm{Hom}_{\mathscr{D}}(F(M), F(N))$ is bijective.

2. For M, N is $\mathscr{C}$ and $f \in \mathrm{Hom}_{\mathscr{E}}(GF(M), GF(N))$ we define $\varphi^{GF}_{M,N}(f)$ to be $\varphi^F_{M,N}(\varphi^G_{F(M),F(N)}(f))$.

3. For $f \in \mathrm{Hom}_{\mathscr{D}}(F(M), F(N))$ we define $\varphi^F_{M,N}(f) = \varphi^{GF}_{M,N}(G(f))$. ∎

Note that SF.1 holds for a full and faithful functor $F: \mathscr{C} \to \mathscr{D}$, i.e., when for M, N in $\mathscr{C}$ the map $\mathrm{Hom}_{\mathscr{C}}(M, N) \to \mathrm{Hom}_{\mathscr{D}}(F(M), F(N))$, $f \mapsto F(f)$ is bijective.

The separability of a functor $F: \mathscr{C} \to \mathscr{D}$ sometimes allows one to deduce properties of M in $\mathscr{C}$ from corresponding properties of $F(M)$ in D; the following proposition provides some examples of such properties.

1.2. Proposition. *Let $F: \mathscr{C} \to \mathscr{D}$ be a separable functor and let M, N be objects in $\mathscr{C}$. Then we have:*

1. *If $f \in \mathrm{Hom}_{\mathscr{C}}(M, N)$ is such that $F(f)$ is split then f is split.*

1'. *If $f \in \mathrm{Hom}_{\mathscr{C}}(M, N)$ is such that $F(f)$ is co-split, i.e., there is an $\omega \in \mathrm{Hom}_{\mathscr{D}}(F(N), F(M))$ such that $F(f)\omega = 1_{F(N)}$, then f is co-split.*

For the following properties we assume that $\mathscr{C}$ and $\mathscr{D}$ are abelian categories.

2. *If $F(M)$ is quasi-simple (i.e., every subject splits off) and F conserves monomorphisms then M is quasi-simple in $\mathscr{C}$.*

3. *If F conserves epimorphisms, resp. monomorphisms, and $F(M)$ is projective, resp. injective, then M is projective, resp. injective.*

Proof. 1. There is a map $u: F(N) \to F(M)$ such that $uF(f) = 1_{F(M)}$. Put $g = \varphi^F_{N,M}(u)$. Condition SF.2 yields that $gf = 1_M$ because of the diagram

$$\begin{array}{ccc} F(M) & \xrightarrow[1_{F(M)}]{} & F(M) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle F(1_M)} \\ F(N) & \xrightarrow[u]{} & F(M) \end{array}$$

The proof of 1' is similar. Statements 2, 3 may be derived from 1 and 1' in a straightforward way. ∎

Specializing the categories $\mathscr{C}$ and $\mathscr{D}$ further one may study separable functors in many different situations but we focus here on module categories.

Let $\varphi: R \to S$ be a ring homomorphism. To φ we associate the following functors:

a. $\varphi_*: S-\text{mod} \to R\text{-mod}$, the *restriction* (*of scalars*) functor associating to an S-module M the R-module structure defined on the set M by the ring morphism $\varphi: R \to S$.

b. $\varphi^*: R\text{-mod} \to S\text{-mod}$, the *induction* functor, associating to an R-module M the S-module $S \otimes_R M$ (note that all modules are left modules unless otherwise specified). The functor φ_* is exact but φ^* is only right exact. We say that *S/R is separable* if the map $\psi: S \otimes_R S \to S$, $s \otimes s' \mapsto ss'$, splits as an S–S-bimodule map. Note that this definition is compatible with the definition of separability for commutative ring extensions. We intend to relate separability of S/R and separability of the functors introduced above.

1.3. PROPOSITION. 1. *φ_* is separable if and only if S/R is separable.*

2. *φ^* is separable if and only if φ splits as an R-bimodule map.*

Proof. 1. Suppose that φ_* is separable. Let ${}_SM_S$ and ${}_SN_S$ be S–S-bimodules and $f: M \to N$ an R–S-bimodule morphism. Separability of φ_* entails the existence of a map $\varphi_{M,N}: \text{Hom}_R(\varphi_*(M), \varphi_*(N)) \to \text{Hom}_S(M, N)$. Put $g = \varphi_{M,N}(f)$, $g \in \text{Hom}_S(M, N)$. We claim that g is an S–S-bimodule morphism. For $s \in S$ we define $\alpha_s(x) = xs$ and we obtain the commutative diagram

$$\begin{array}{ccc} {}_RM_S & \xrightarrow{f} & {}_RN_S \\ \downarrow{\scriptstyle \alpha_s} & & \downarrow{\scriptstyle \alpha_s} \\ {}_RM_S & \xrightarrow[f]{} & {}_RN_S \end{array}$$

Since $\alpha_s = \varphi_*(\alpha_s)$, condition SF.2 yields a commutative diagram

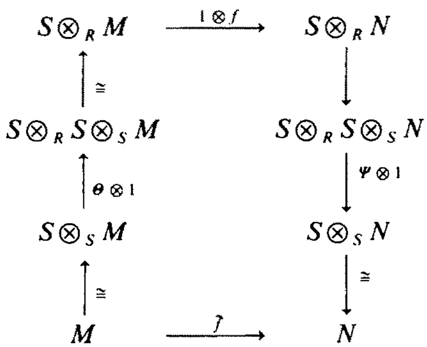
$$\begin{array}{ccc} {}_SM & \xrightarrow{g} & {}_SN \\ \downarrow{\scriptstyle \alpha_s} & & \downarrow{\scriptstyle \alpha_s} \\ {}_SM & \xrightarrow[g]{} & {}_SN \end{array}$$

Hence $g(xs) = g(x)s$ for all $x \in M$, and therefore g is as claimed. To the morphism $\Psi: S \otimes_R S \to S$, $s \otimes s' \mapsto ss'$, we may associate an R–S-bimodule morphism $\Psi': S \to S \otimes_R S$, $s \mapsto 1 \otimes s$, such that $\Psi\Psi' = 1_S$. If we put $\Psi_1 =$

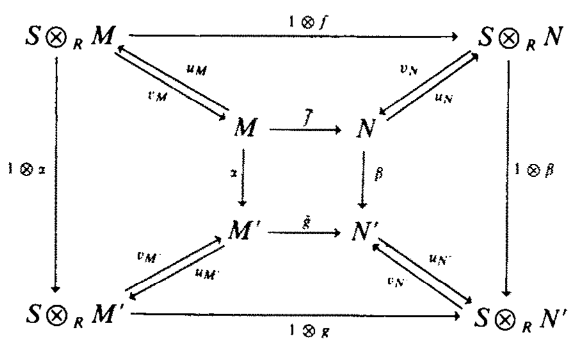
$\varphi_{S,S\otimes_R S}(\Psi')$ then we have $\Psi\Psi_1 = 1_S$ and Ψ_1 is an S–S-bimodule morphism. Hence Ψ is split as an S–S-module morphism.

Conversely, assume that Ψ is split by some S–S-bimodule morphism, Θ say. Let M, N be S-modules and $f \in \mathrm{Hom}_R(\varphi_*(M), \varphi_*(N))$.

Define $\bar{f}$ by the following commutative diagram of S-module maps:



From this diagram it is easily deduced that $\bar{f} = f$ in case f is S-linear. Let u_M: $M \to S\otimes_R M$, v_N: $S\otimes_R N \to N$ be the composition of the vertical maps resp. on the left and on the right in the diagram. Given S-modules M', N' and $\alpha \in \mathrm{Hom}(M, M')$, $\beta \in \mathrm{Hom}_S(N, N')$, $g \in \mathrm{Hom}_R(M', N')$ then one obtains a diagram



In this diagram the diagonal maps are splittings and one easily deduces that the inner square is commutative if the outer square is commutative. Hence φ_* is separable.

2. Suppose first that $S\otimes_R -$ is a separable functor. Then the composition of $1\otimes\varphi$ and γ: $S\otimes_R S \to S\otimes_R R$, $s\otimes s' \mapsto ss'\otimes 1$ is the identity map. Therefore $\tilde{\gamma}$ provides a splitting for φ. Conversely, suppose that φ is

split by an R-linear map Ψ. If $f\colon S\otimes_R M \to S\otimes_R N$ is S-linear then we define $\tilde{f}$ by the commutative diagram

$$\begin{array}{ccc}
S\otimes_R M & \xrightarrow{\;f\;} & S\otimes_R N \\
{\scriptstyle \varphi\otimes}\uparrow & & \downarrow{\scriptstyle \Psi\otimes 1} \\
R\otimes_R M & & R\otimes_R N \\
{\scriptstyle \cong}\uparrow & & \downarrow{\scriptstyle \cong} \\
M & \xrightarrow[\tilde{f}]{} & N
\end{array}$$

A routine verification then establishes (as in 1) that $S\otimes_R -$ is separable. ∎

1.4. COROLLARY. *If $\varphi\colon R\to S$ is an epimorphism in the category of rings then φ_* is a separable functor.*

Proof. Since φ is an epimorphism $S\otimes_R S\to S$, $s\otimes s'\mapsto ss'$, is an isomorphism (e.g., B. Stenstrom, Proposition 1.2, p. 226 [10]). ∎

2. SEPARABILITY OF THE RESTRICTION FUNCTOR FOR GRADED RINGS

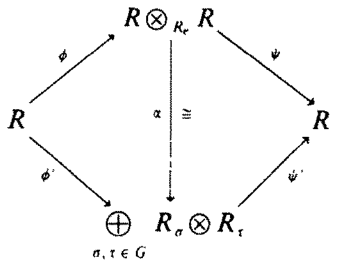
Consider now a ring R graded by a group G, $R=\bigoplus_{\sigma\in G} R_\sigma$. For full detail on graded rings we refer to [10].

A ring R is said to be *strongly graded* if $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma,\tau\in G$. The latter condition makes each R_σ, $\sigma\in G$, into an invertible R_e-bimodule and it is well known that $az=\sigma(z)a$ holds for all $a\in R_\sigma$, z in the centre $Z(R_e)$ of R_e, and $\sigma\in \mathrm{Aut}(Z(R_e))$ associated to the isomorphism class $[R_\sigma]$ of R_σ; cf. [15, 7]. In fact we may give a direct description of this G-action on $Z(R_e)$ (cf. [14]), as follows. Since $R_\sigma R_{\sigma^{-1}} = R_e$ for each $\sigma\in G$, we may fix a decomposition $1=\sum_i a_i b_i$ with $a_i\in R_\sigma$, $b_i\in R_{\sigma^{-1}}$, now put $\sigma(z)=\sum_i a_i z b_i$ for $z\in Z(R_e)$ and one checks directly that $\sigma(z)a=az$ for all $z\in Z(R_e)$ and $a\in R_\sigma$. Let φ_* be the restriction functor associated to the ring morphism $\varphi\colon R_e\to R$. The induction functor φ^* associated to φ is given by $R\otimes_{R_e} -$ and for graded rings this induction is the adjoint of the restriction functor. In Proposition 1.3 we provided criteria for φ_*, φ^* to be separable. Verification of the splitting of φ as an R-bimodule map presents no problem in the graded situation, so we focus on separability of φ_* for a moment. The criterion for separability of R/R_e contained in the following proposition is in fact the same as the one obtained by Y. Miyashita in [7], where finite groups were considered; see also K. H. Ulbrich [15].

2.1. PROPOSITION. *Let R be strongly graded by G, then R/R_e is separable if and only if the trace $t\colon Z(R_e) \to Z(R_e)$, $a \mapsto \sum_{\sigma \in G} \sigma(a)$ is surjective and G is finite (this is yet another version of Maschke's theorem).*

Proof. First suppose that R/R_e is separable, i.e., there is an R-bimodule splitting for the canonical mapping: $\psi\colon R \otimes_{R_e} R \to R$, $s \otimes s' \mapsto ss'$. Say $\phi\colon R \to R \otimes_{R_e} R$ splits the map ψ. Put $\phi(1) = s \in R \otimes_{R_e} R$, then $\psi(s) = 1$. Since ϕ determines an R-bimodule splitting for ψ it is clear that $\lambda_\tau s = s\lambda_\tau$ for all $\lambda_\tau \in R_\tau$, $\tau \in G$. So R/R_e is separable if and only if there is an $s \in R \otimes_{R_e} R$ such that $\psi(s) = 1$ and $\lambda_\tau s = s\lambda_\tau$ for all $\lambda_\tau \in R_\tau$, $\tau \in G$. Because R is strongly graded by G we have isomorphisms: $R \otimes_{R_e} R \cong \bigoplus_{\sigma, \tau \in G} R_\sigma \otimes_{R_e} R_\tau$, $R_\sigma \otimes R_\tau \cong R_{\sigma\tau}$ as R_e-bimodules.

We write ϕ', ψ' for the R-bimodule maps determined by the commutative diagram



We put $\phi'(1) = \alpha(s) = s' = \sum_{\sigma \in S} \sum' a_\sigma \otimes b_{\sigma^{-1}}$ for some finite subset S of G, putting $c_{\sigma,\sigma^{-1}} = \psi'(\sum' a_\sigma \otimes b_{\sigma^{-1}})$. Then $1 = \sum_{\sigma \in S} c_{\sigma,\sigma^{-1}}$ with $c_{\sigma,\sigma^{-1}} \in R_e$. The fact that s' is R_e-centralizing leads to $c_{\sigma,\sigma^{-1}} \in Z(R_e)$. Pick $\lambda_\tau \in R_\tau$, $\tau \in G$, then $\lambda_\tau s' = s'\lambda_\tau$ yields (by comparing homogeneous parts of equal degree) $\sum' \lambda_\tau a_\sigma \otimes b_{\sigma^{-1}} = \sum' a_{\tau\sigma} \otimes b_{\sigma^{-1}\tau^{-1}} \lambda_\tau$. Hence we arrive at $\lambda_\tau c_{\sigma,\sigma^{-1}} = c_{\tau\sigma,\sigma^{-1}\tau^{-1}} \lambda_\tau$, and in view of the definition of the G-action on $Z(R_e)$ we also obtain, for every $\lambda_\tau \in R_\tau$, $(\phi_\tau(c_{\sigma,\sigma^{-1}}) - c_{\tau\sigma,\sigma^{-1}\tau^{-1}})\lambda_\tau = 0$, thus $\phi_\tau(c_{\sigma,\sigma^{-1}}) = c_{\tau\sigma,\sigma^{-1}\tau^{-1}}$ follows. In particular $c_{\tau,\tau^{-1}} = \phi_\tau(c_{e,e})$ and then $1 = \psi'(s') = t(c_{e,e})$ leads to the desired element of trace equal to 1. It also follows that G is finite because if G were infinite then there would exist a $\tau \in G$ such that $\tau\sigma \notin S$ and then $c_{\tau\sigma,\sigma^{-1}\tau^{-1}} = 0$ would lead to $yc_{\sigma,\sigma^{-1}} = 0$ for all $y \in R_\tau$, hence $c_{\sigma,\sigma^{-1}} = 0$, a contradiction (choice of the $c_{\sigma,\sigma^{-1}}$!).

Conversely, if there is an element $u \in Z(R_e)$ having trace one, then we may produce a bimodule splitting of ψ' by sending 1 to $(\phi_\sigma(u))_{\sigma \in G}$; this claim is easy to check. ∎

2.2. *Remarks.* 1. In the situation of the above proposition an explicit description of $\bar{f}$ (as in the proof of Proposition 1.3(1)) for a given $f \in \mathrm{Hom}_{R_e}(M, N)$ is available. Fixing an element $u \in Z(R_e)$ having trace one, and fixing for each $\tau \in G$ a decomposition $1 = \sum_i u_\tau^{(i)} v_{\tau^{-1}}^{(i)}$ with $u_\tau^{(i)} \in R_\tau$, $v_{\tau^{-1}}^{(i)} \in R_{\tau^{-1}}$, then we may write

$$\tilde{f}(m) = \sum_{\tau \in G} \sum_i u_\tau^{(i)} f(u v_{\tau^{-1}}^{(i)} m).$$

The reader may verify (or look up [14]) that this does not depend on the chosen decomposition of 1.

2. In [3], E. Dade also considers the co-induction functor but in the strongly graded case co-induction and induction are isomorphic. Nevertheless, in general, it may be interesting to investigate separability of the co-induction functor.

2.3. COROLLARY. *Let G be a finite group of order n. Then $R[G]$ is separable over R if and only if n is invertible in R.*

A very important functor is the forgetful functor U, U: R-gr $\to$ R-mod, associating to a graded R-module M the underlying ungraded module $\underline{M}$ (usually we write $\underline{M}$ instead of $U(M)$). The following section is devoted to the right adjoint of U. As a transition to the next section we just mention:

2.4. PROPOSITION. *If R is an arbitrary G-graded ring then the functor U is separable.*

Proof. Consider $M, N \in R$-gr and $f \in \mathrm{Hom}_R(\underline{M}, \underline{N})$. If $m \in M$ has decomposition $m = m_{\sigma_1} + \cdots + m_{\sigma_k}$ then we define $\tilde{f}(m) = \sum_i f(m_{\sigma_i})_{\sigma_i}$ and it is easily checked that $\tilde{f} \in \mathrm{Hom}_{R\text{-gr}}(M, N)$ and that the map $\sim: \mathrm{Hom}_R(\underline{M}, \underline{N}) \to \mathrm{Hom}_{R\text{-gr}}(M, N)$ satisfies SF.1 and SF.2. ∎

3. THE RIGHT ADJOINT F OF U

As before let U: R-gr $\to$ R-mod denote the forgetful functor and we will write $\underline{M}$ for $U(M)$. Recall the construction of a right adjoint F of U (cf. [10, p. 4]); if $M \in R$-mod then $F(M)$ is defined to be the additive group $\bigoplus_{\sigma \in G} {}^\sigma M$, where each ${}^\sigma M$ is a copy of M (we write ${}^\sigma M = \{{}^\sigma x, x \in M\}$) and R-module structure is given by $r *^\sigma x = {}^{\rho\sigma}(rx)$ for $r \in R_\rho$. Obviously the gradation of $F(M)$ is given by $F(M)_\sigma = {}^\sigma M$, $\sigma \in G$. If f: $M \to N$ is R-linear then $F(f)$: $F(M) \to F(N)$ is given by $F(f)({}^\sigma x) = {}^\sigma f(x)$ and clearly $F(f)$ is homogeneous of degree $e \in G$, e the neutral element. The functor F is exact and it is a right adjoint for U. Note that $U(F(M))$ need not be a direct sum of copies of M since the component ${}^\sigma M$, $\sigma \in G$, is not an R-submodule of $F(M)$, but it is an R_e-submodule of course. The idea for this construction of

F stems from G. Bergman. Recall that $M(\lambda)$ for $\lambda \in G$ is the shifted R-module graded by $M(\lambda)_\tau = M_{\tau\lambda}$, $\tau \in G$.

3.1. LEMMA. *If $M \in R$-gr then $F(\underline{M}) = \bigoplus_{\lambda \in G} M(\lambda)$.*

Proof. For $x \in M$ we write $x = \sum_{g \in G} x_g$. Define a map $u: F(M) \to \bigoplus_{\lambda \in G} M(\lambda)$, ${}^\sigma x \mapsto (x_\tau)_{\tau \in G}$, where x_τ is considered as an element of $M(\sigma^{-1}\tau)_\sigma = M_\tau$. It is easy to verify that u is an isomorphism in R-gr. ∎

The functor F may also be constructed in at least two other ways. A first alternative construction uses smash-products, and a second construction is based on group rings over graded rings. For the smash-product construction we follow D. Quinn [12].

Assume $|G| = n$, i.e., G is finite. Let $M_G(R)$ denote the $n \times n$-matrices over R where rows and columns are indexed by elements of G. If $\alpha \in M_G(R)$ then we write $\alpha(x, y)$ for the entry in the (x, y)-position of α. For α, β in $M_G(R)$ the matrix product α, β is given by $\alpha\beta(x, y) = \sum_{z \in G} \alpha(x, z)\, \beta(z, y)$. If $x, y \in G$ then we let $e_{x,y}$ be the matrix α with $\alpha(x, y) = 1$ and $\alpha(-, -) = 0$ elsewhere.

Let $p_x = e_{x,x}$, $x \in G$. Define $\eta: R \to M_G(R)$, $r \mapsto \sum_{x, y \in G} r_{xy^{-1}} e_{x,y}$. That η is a ring monomorphism is easily verified. We put $\eta(r) = \tilde{r}$ and $\tilde{R} = \operatorname{Im} \eta$; let $\tilde{R} \# G^*$ be the subring of $M_G(R)$ generated by $\tilde{R}$ and the set of orthogonal idempotents $\{p_x, x \in G\}$. We call $\tilde{R} \# G^*$ the *smash-product* of R by G and it is exactly the construction given by M. Cohen and S. Montgomery in [2]. Clearly $\tilde{R} \# G^*$ is a free (left and right) $\tilde{R}$-module with basis $\{p_x, x \in G\}$ and $(\tilde{r}p_x)(\tilde{s}p_y) = (\tilde{r}\tilde{s}_{xy^{-1}})\, p_y$ for $r, s \in R$, $x, y \in G$; cf. Proposition 1.4 of [2].

Given $g \in G$ we define $\bar{g} = \sum_{x \in G} e_{x,xg}$. Obviously $\bar{g}$ is a unit of $M_G(R)$ and G is isomorphic to $\bar{G} = \{\bar{g}, g \in G\}$. Theorem 1.3 of [12] yields $M_G(R) = (\tilde{R} \# G^*)\bar{G}$ and with Theorem 3.5 (Duality for Coactions) of [2] we have $M_G(R) \cong (\tilde{R} \# G^*) * G$, i.e., $M_G(R)$ is a skew group ring of G over the ring $\tilde{R} \# G^*$. For $M \in R$-mod we let $\operatorname{Col}_G(M)$ be the set of $n \times 1$-columns over M. If $\underline{m} \in \operatorname{Col}_G(M)$ then x_m stands for the element of M appearing in the x-position of $\underline{m}$. Let $\alpha \in M_G(R)$ act on $\underline{m} \in \operatorname{Col}_G(M)$ as follows: ${}^x(\alpha m) = \sum_{y \in G} \alpha(x, y)\, {}^y m$ (cf. D. Quinn [12, p. 160]), so $\operatorname{Col}_G(M)$ is a left $M_G(R)$-module in a natural way and we may view it as an $\tilde{R} \# G^*$-module by restriction of scalars. An $\tilde{R} \# G^*$-module W has a natural structure of a graded R-module given by putting $W_x = p_x W$, $x \in G$, and for $r \in R$, $w \in W$ we have $rw = \eta(r)w$. So we obtain the functor $(-)_{\mathrm{gr}}$: $\tilde{R} \# G$-mod $\to R$-gr which is an equivalence of categories (Theorem 2.2 [2]). In particular, $\operatorname{Col}_G(M)$ has the structure of a graded R-module. As indicated above we also obtain a functor H: R-mod $\to R$-gr, defined by $H(M) = \operatorname{Col}_G(M)$ and considering this as a G-graded R-module.

3.2. LEMMA. *The functors F and H are isomorphic.*

Proof. For $M \in R$-mod define $\varphi(M)\colon F(M) \to H(M)$, $m = ({}^x m)_{x \in G} \mapsto \underline{m}$, where $\underline{m}$ is the column with ${}^x m$ in the position $x \in G$. If $r \in R_\sigma$ and m is homogeneous in $F(M)$, $m = {}^x m$ say, then we claim that $\varphi(M)(r\,{}^x m) = r\varphi(M)({}^x m)$. Indeed, we have $\eta(r) = \sum_{y,z \in G} r_{yz^{-1}} e_{y,z} = r \sum_{y \in G} e_{y,\sigma^{-1}y}$ (since $r \in R_\sigma$). Hence for $t \in G$ we calculate

$$\begin{aligned} {}^t(r\varphi(M)(m)) &= {}^t(\eta(r)\,\varphi(M)(m)) \\ &= \sum_{z \in G} \eta(r)(t, z)\; {}^z(\varphi(M)({}^x m)) \\ &= \eta(r)(t, x)\; {}^x m \qquad \text{(note that } {}^z m = 0 \text{ for } z \neq x\text{).} \end{aligned}$$

However, $\eta(r)(t, x) = r$ when $t = y$ and $x = \sigma^{-1} y$, and $\eta(r)(t, x) = 0$ otherwise. Hence, ${}^t(\eta(r)(\varphi(M)(m))) = rm = r\,{}^x m$, where $t = \sigma x$ and ${}^t(rm) = 0$ when $t \neq \sigma x$. It follows indeed that $\varphi(M)$ is R-linear. It is easy to check the bijectivity of $\varphi(M)$. Now for $x \in G$ we have $\varphi(M)(F(M)_x) = p_x \operatorname{Col}_G(M) = H(M)_x$. Therefore $\varphi(M)$ is a graded isomorphism. On the other hand, it is also clear that $\varphi = (\varphi(M))_{M \in R\text{-mod}}$ is a functional morphism $F \to H$ and so φ is a functorial isomorphism. ∎

3.3. *Remark.* Let $\operatorname{Col}_G(-)\colon R$-mod $\to M_G(R)$-mod be the functor taking M to $\operatorname{Col}_G(M)$ viewed as an $M_G(R)$-module. The foregoing lemma shows that H is the composition of the following functors: $H = (-)_{-\mathrm{gr}} \circ i_* \circ \operatorname{Col}_G(-)$, where the functor $i_*\colon M_G(R)$-mod $\to \tilde{R} \# G^*$-mod is given by the inclusion map $i\colon \tilde{R} \# G^* \to M_G(R)$. The functor $\operatorname{Col}_G(-)$ is an equivalence of categories by a classical result of Morita; cf. Anderson and Fuller [1, p. 265].

3.4. PROPOSITION. *Let R be a strongly graded ring of type G. Then F is a separable functor if and only if R is separable over* R_e.

Proof. In view of Theorem I.3.4 [10], the functor $R \otimes_{R_e} -\colon R_e$-mod $\to$ R-gr given by $N \mapsto R \otimes_{R_e} N$ is an equivalence with inverse functor $(-)_e$: R-mod $\to R_e$-mod. The composition $(-)_e \circ F$ is thus isomorphic to the functor $i_*\colon R$-mod $\to R_e$-mod, where $i\colon R_e \subsetneq R$ is inclusion. Hence F is isomorphic to the functor $(R \otimes_{R_e} -) \circ i_*$. The first being an equivalence of categories, F is separable if and only if i_* is separable, i.e., F is separable if and only if R is separable over R_e. ∎

A more general characterization may be obtained from $\tilde{R} \# G^*$. Let $Z(\tilde{R} \# G^*) = Z$ be the centre of $\tilde{R} \# G^*$.

3.5. LEMMA. $Z = \{\sum_{x \in G} a^{(x)} p_x,\ a^{(x)} \in Z(R_e)\}$, *where* $Z(R_e)$ *is the centre of* R_e, *and* $a^{(x)} \lambda_\sigma = \lambda_\sigma a^{(\sigma^{-1} x)}$ *for all* $\lambda_\sigma \in R_\sigma$, $\sigma \in G$.

Proof. Consider the canonical monomorphism η,

$$\eta\colon R \to \tilde{R} \# G^*, \qquad r \mapsto \sum_{x,y \in G} r_{xy^{-1}} e_{x,y}, \qquad \text{where} \quad r = \sum_g r_g.$$

If we write 1_G for the identity of $M_G(R)$ then for $r \in Z(R_e)$ we obtain $\tilde{r} = \eta(r) = r \sum_{x \in G} e_{x,x} = r \sum_{x \in G} p_x = r \cdot 1_G$.

Put $X = \{\sum_{x \in G} a^{(x)} p_x,\ a^{(x)} \in Z(R_e) \text{ and } a^{(x)}\lambda_\sigma = \lambda_\sigma a^{(\sigma^{-1}x)},\ \lambda_\sigma \in R_\sigma\}$. To establish that $X \subset Z$ it will be sufficient to show for $\lambda_\sigma \in R_\sigma$ that $u(\eta(\lambda_\sigma) p_y) = \eta(\lambda_\sigma) p_y) u$. We now calculate

$$\begin{aligned} u(\eta(\lambda_\sigma) p_y) &= \left(\sum_x a^{(x)} p_x\right) \tilde{\lambda}_\sigma p_y = (a^{(\sigma y)} \lambda_\sigma)^{\sim} p_y \\ &= \eta(a^{(\sigma y)} \lambda_\sigma) p_y = \eta(\lambda_\sigma a^{(y)}) p_y \\ (\eta(\lambda_\sigma) p_y) u &= (\tilde{\lambda}_\sigma p_y) \left(\sum_x a^{(x)} p_x\right) = \sum_x (\tilde{\lambda}_\sigma p_y)(a^x p_x) \\ &= \eta(\lambda_\sigma a^y) p_y \end{aligned}$$

hence $X \subset Z$ follows. Conversely, if $u \in Z$ is given as $u = \sum_{x \in G} \eta(a^{(x)}) p_x$, where $a^{(x)} \in R$ for $x \in G$, then we may derive from $up_y = p_y u$ the equalities $up_y = \eta(a^{(y)}) p_y$ and $p_y u = p_y(\sum_{x \in G} \eta(a^{(x)}) p_x) = \sum_{x \in G} \eta(a^{(x)}_{yx^{-1}}) p_x$. In particular we obtain that $\eta(a^{(y)}) = \eta((a^{(y)})_e)$, hence $a^{(y)} \in R_e$ for any $y \in G$. Therefore $u = \sum_{x \in G} \eta(a^{(x)}) p_x$, where $a^{(x)} \in R_e$ for every $x \in G$. If we consider $\lambda_\sigma \in R_\sigma$ then $u\eta(\lambda_\sigma) = \eta(\lambda_\sigma) u$ yields that $\{a^{(x)}, x \in G\}$ satisfies $a^{(x)} \lambda_\sigma = \lambda_\sigma a^{(\sigma^{-1}x)}$ for $\lambda_\sigma \in R_\sigma$ and consequently $u \in X$. ▮

3.6. Theorem. *The functor F is separable if and only if there is a family $\{a^{(x)}, x \in G\}$ in $Z(R_e)$ such that:*

1. $\sum_x a^{(x)} = 1$.
2. *For all $\lambda_\sigma \in R_\sigma$, $a^{(x)}\lambda_\sigma = \lambda_\sigma a^{(\sigma^{-1}x)}$.*

Proof. By Lemma 3.2 it suffices to establish that H is separable. Taking into account Remark 3.3, the latter comes down to the separability of $i_*\colon M_G(R)\text{-mod} \to \tilde{R} \# G^*$. Hence F is separable if and only if $(\tilde{R} \# G^*) * G$ is separable over $\tilde{R} \# G^*$. Now the result follows easily from Proposition 2.1. ▮

3.7. Corollary. *If $n = |G|$ is invertible in R then F is a separable functor.*

Proof. Put $a^{(x)} = n^{-1}$ for all $x \in G$. ▮

3.8. COROLLARY. *Let R be a G-graded ring and assume that the following properties hold:*

1. *$Z(R_e) \subset Z(R)$, where $Z(R)$ is the centre of R.*

2. *If $aR_\sigma = 0$ for $a \in R_e$, any $\sigma \in G$, then $a = 0$. Then F is separable if and only if n is invertible in R.*

Proof. Theorem 3.6 provides us with a family $\{a^{(x)}, x \in G\}$ in $Z(R_e)$ such that $\sum_x a^{(x)} = 1$, satisfying $a^{(x)}\lambda_\sigma = \lambda_\sigma a^{(\sigma^{-1}x)}$ for $\lambda_\sigma \in R_\sigma$, then $a^{(x)}\lambda_\sigma = a^{(\sigma^{-1}x)}\lambda_\sigma$ and therefore $a^{(x)} = a^{(\sigma^{-1}x)}$. Since $\sigma \in G$ is arbitrary we have $a^{(x)} = a$ for any $x \in G$, thus $na = 1$ and hence n is invertible in R. ▮

3.9. *Remarks.* 1. Note that condition 2 in Corollary 3.8 cannot be dropped. As an example consider $R = \bigoplus_{\sigma \in G} R_\sigma$ with $R_\sigma = 0$ for all $\sigma \neq e$ and $R_e = A$ such that $n = |G|$ is not invertible in A, then F is separable. Indeed for $M \in R$-mod, $F(M) = M^n$. If $u \in \mathrm{Hom}_{R\text{-gr}}(F(M), F(N))$ then u has the form $u = u_1 \times \cdots \times u_n$, where each $u_i: M \to N$ is R-linear. We may define $\varphi_{M,N}$:

$$\mathrm{Hom}_{R\text{-gr}}(F(M), F(N)) \to \mathrm{Hom}_R(M, N) = \mathrm{Hom}_A(M, N), \qquad \varphi_{M,N}(u) = u_1.$$

2. Let k be a perfect field and V a k vectorspace of finite dimension. The trivial extension $R = k \times V = \{(a, x), a \in k, x \in V\}$ has multiplication defined by $(a, x)(b, y) = (ab, bx + ay)$, and the ring R is commutative and $\mathbb{Z}_2$-graded by putting $R_0 = k \times \{0\}$ and $R_1 = \{0\} \times V$. Since R_1 is a nilpotent ideal of R, Corollary b, p. 192, from [11] entails that R is not R_0-separable. On the other hand, if char $k \neq 2$, then the conditions in Corollary 3.6 hold and therefore F is a separable functor. This makes it clear that the separability of F does not entail that R is separable over R_e. However, in case R is strongly graded Proposition 3.4 yields this implication.

We now continue the study of the properties of F when G is finite. For $M \in R$-mod define $\alpha_M: M \to F(M)$ and $\beta_M: F(M) \to M$, by $\alpha_M(m) = ({}^\sigma m)_{\sigma \in G}$, where ${}^\sigma m = m$ for all $\sigma \in G$, $\beta_M(({}^\tau x)_{\tau \in G}) = \sum_{\tau \in G} {}^\tau x$. Clearly, α_M and β_M are R-linear and $(\beta_M \circ \alpha_M)(x) = nx$ for all $x \in M$. The Krull dimension of $M \in R$-mod is denoted by $\mathrm{Kdim}_R M$, similar for $\mathrm{Kdim}_{R_e} M$ when M is viewed as an R_e-module. By $\mathrm{Gdim}_R M$, resp. $\mathrm{Gdim}_{R_e} M$, we denote the Gabriel dimension of M over the ring R, resp. over R_e.

3.10. THEOREM. *The functor F enjoys the following properties:*

1. *If $n = |G|$ is invertible in R then M is isomorphic to a direct summand of $F(M)$.*

2. *If M is finitely generated, resp. finitely presented, then $F(M)$ is a finitely generated R-module, resp. finitely presented R-module.*

3. *If M is a Noetherian R-module then $F(M)$ is a Noetherian R-module .*

4. *If M is an Artinian R-module then $F(M)$ is an Artinian R-module.*

5. *If M has Krull dimension, resp. Gabriel dimension, then $F(M)$ has Krull dimension, resp. Gabriel dimension. Moreover we have*

$$\mathrm{Kdim}_R M = \mathrm{Kdim}_R F(M) \qquad \textit{resp.} \quad \mathrm{Gdim}_R M = \mathrm{Gdim}_R F(M).$$

6. *If M is projective, resp. injective, then $F(M)$ is projective, resp. injective.*

7. *If M' is an essential R-submodule of M and M is n-torsion free then $F(M')$ is an essential R-submodule of $F(M)$.*

8. *If n is invertible in R, and $M \in R$-mod, then* $\mathrm{inj.dim}_R M = \mathrm{inj.dim}_R F(M)$. *Here the injective dimension of M in R-mod is denoted by* $\mathrm{inj.dim}_R(-)$.

Proof. From Proposition 3.4 and Remark 3.3 we retain that F is isomorphic to $(-)_{\mathrm{gr}} \circ i_* \circ \mathrm{Col}_G(-)$, where i_* corresponds to the inclusion $i: \tilde{R} \# G^* \to M_G(R)$. Moreover the functors $\mathrm{Col}_G(-)$ and $(-)_{\mathrm{gr}}$ are equivalences of categories.

1. Since $n^{-1}\beta_M \circ \alpha_M = 1_M$.

2. If M is a finitely generated R-module, $\mathrm{Col}_G(M)$ is a finitely generated $M_G(R)$-module. Since $M_G(R) = (\tilde{R} \# G^*)\bar{G} \simeq (\tilde{R} \# G^*) * G$ and G being finite, it follows that $\mathrm{Col}_G(M)$ is a finitely generated $\tilde{R} \# G^*$-module. Because $(-)_{\mathrm{gr}}$ is an equivalence of categories we obtain that $F(M)$ is finitely generated in R-gr hence also as an R-module. A similar argument may be used in the finitely presented case.

3, 4. If M is Noetherian, resp. Artinian, then the fact that $\mathrm{Col}_G(-)$ is an equivalence of categories entails that $\mathrm{Col}_G(M)$ is Noetherian, resp. Artinian, as an $M_G(R)$-module. Theorem I.8.10 of [10], resp. Theorem I.8.12, yields that $\mathrm{Col}_G(M)$ is Noetherian, resp. Artinian, over the ring $\tilde{R} \# G^*$. Since $(-)_{\mathrm{gr}}$ is an equivalence of categories it follows that $F(M)$ is Noetherian, resp. Artinian, in R-gr. Corollary II.3.3 of [10] then yields that $F(M)$ is Noetherian, resp. Artinian, as an R-module. The same argumentation may be used to established 5, indeed, Theorem I.8.12 and Theorem I.8.14 of [10] yield that $F(M)$ has Krull dimension, resp. Gabriel dimension, in R-gr. Then we may apply Corollary II.5.21 of [10], and statement 5 follows.

6. The case where M is projective is easy. Assume that M is an injective R-module. Then $\mathrm{Col}_G(M)$ is injective as an $M_G(R)$-module. Since

$M_G(R)$ is a free (left and right) module over $\tilde{R} \# G^*$ it follows that $\mathrm{Col}_G(M)$ is injective as an $\tilde{R} \# G^*$-module. Since $(-)_{\mathrm{gr}}$ is an equivalence of categories we obtain that $F(M)$ is injective in R-gr. By Theorem 4.7 of [9] it follows that $F(M)$ is injective in R-mod.

7. Since $\mathrm{Col}_G(-)$ is an equivalence of categories, $\mathrm{Col}_G(M')$ is an essential $M_G(R)$-submodule of $\mathrm{Col}_G(M)$. By the essential version of Maschke's theorem for strongly graded rings, cf. [16], it follows that $\mathrm{Col}_G(M')$ is an essential submodule of $\mathrm{Col}_G(M)$ as $\tilde{R} \# G^*$-modules. Again using the equivalence $(-)_{\mathrm{gr}}$ we obtain that $F(M')$ is an essential subobject of $F(M)$ in the category R-gr. Lemma I.2.8 of [10] yields that $\underline{F(M')}$ is an essential R-submodule of $\underline{F(M)}$.

8. Consider a minimal injective resolution of M in R-mod:

$$0 \to M \to Q_0 \to Q_1 \to Q_2 \to \cdots.$$

Applying 6 and 7 and the fact that F is an exact functor, we obtain a minimal injective resolution of $F(M)$ in R-mod:

$$0 \to F(M) \to F(Q_0) \to F(Q_1) \to F(Q_2) \to \cdots.$$

That $\mathrm{inj.dim}_R\, M = \mathrm{inj.dim}_R\, F(M)$ is now obvious. ∎

3.11. COROLLARY. *Let R be graded by the finite group G. Let $M \in R$-mod, then the following properties hold:*

1. *If M is Noetherian, resp. Artinian, then ${}_{R_e}M$ is Noetherian, resp. Artinian.*

2. *If M has Krull dimension, resp. Gabriel dimension, then ${}_{R_e}M$ has Krull dimension, resp. Gabriel dimension, and* $\mathrm{Kdim}_{R_e} M = \mathrm{Kdim}_R M$, *resp.* $\mathrm{Gdim}_{R_e} M = \mathrm{Gdim}_R M$.

Proof. 1. If M is Noetherian, resp. Artinian, then by Theorem 3.10, $F(M)$ is Noetherian, resp. Artinian, and hence $F(M)_\sigma$ is Noetherian, resp. Artinian, R_e-module for every $\sigma \in G$ (cf. Lemma II.3.2 [10]). Since G is finite, $F(M)$ is Noetherian, resp. Artinian, as an R_e-module. Since M is isomorphic to a submodule of $F(M)$, the statement follows.

2. If M has Krull, resp. Gabriel, dimension then $F(M)$ has Krull dimension, resp. Gabriel dimension, in view of Theorem 3.10. From Corollary II.5.21 of [10] we obtain

$$\mathrm{Kdim}_R\, F(M) = \sup_{\sigma \in G}\{\mathrm{Kdim}_{R_e}\, F(M)_\sigma\} = \mathrm{Kdim}_{R_e}\, F(M).$$

Since $F(M)_\sigma \cong M$ as R_e-modules, $\mathrm{Kdim}_R\, F(M) = \mathrm{Kdim}_{R_e}\, M$. Theorem 3.10

then entails $\mathrm{Kdim}_R M = \mathrm{Kdim}_R F(M)$ and thus $\mathrm{Kdim}_R M = \mathrm{Kdim}_{R_e} M$. A similar argument works for G-dim. ▌

3.12. *Remarks.* 1. The implication ${}_R M$ is Noetherian implies ${}_{R_e}M$ is Noetherian has been proved first by P. Greszczuk in [4].

2. Let M' be an essential R-submodule of M and assume that M is n-torsion free. For any nonzero x in M there are $\sigma \in G$ and $\lambda_\sigma \in R_\sigma$ such that $\lambda_\sigma x \in M'$ and $\lambda_\sigma x \neq 0$. This result is exactly Theorem 1.8 of D. Quinn [12]. Indeed, we consider the commutative diagram

$$\begin{array}{ccc} F(M') & \hookrightarrow & F(M) \\ {\scriptstyle \alpha_{M'}}\uparrow & & \uparrow{\scriptstyle \alpha_M} \\ M' & \hookrightarrow & M \end{array}$$

Since $\alpha_M(x) \neq 0$ and $F(M')$ is essential in $F(M)$ (cf. Theorem 3.10.7) then Lemma I.2.8 of [10] provides us with a $\sigma \in G$ and a $\lambda_\sigma \in R_\sigma$ such that $\lambda_\sigma \alpha_M(x) \in F(M')$ and $\lambda_\sigma \alpha_M(x) \neq 0$. However, we have $\lambda_\sigma \alpha_M(x) = \alpha_{M'}(\lambda_\sigma x)$ and therefore $\lambda_\sigma x \in M'$, $\lambda_\sigma x \neq 0$.

3. If $n = |G|$ is invertible in R then $\mathrm{gldim}\, R = \mathrm{gr.gldim}\, R$. Here $\mathrm{gl.dim}\, R$, resp. $\mathrm{gr.gldim}\, R$, is the homological global dimension of R-mod, resp. of R-gr. Indeed, $\mathrm{gr.gldim}\, R \leqslant \mathrm{gldim}\, R$ is obvious and from Theorem 2.10.8 we retain $\mathrm{inj.dim}_R M = \mathrm{inj.dim}_R F(M) = \mathrm{gr.inj.dim}_R F(M) \leqslant \mathrm{gr.dldim}\, R$. This yields the other inequality $\mathrm{gldim}\, R \leqslant \mathrm{gr.gldim}\, R$. This result is in fact Theorem 4.11 of [9].

3.13. THEOREM. *Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be graded by the finite group G. If $M \in R$-mod is simple then $F(M)$ is a semi-simple object in R-gr of finite length.*

Proof. Since $\mathrm{Col}_G(-)$ is an equivalence, $\mathrm{Col}_G(M)$ is a simple $M_G(R)$-module. By the Clifford theorem for strongly graded rings (cf. Theorem I.3.33 of [10]) $\mathrm{Col}_G(M)$ is semi-simple of finite length as an $\tilde{R} \# G^*$-module. Again, since $(-)_{gr}$ is an equivalence, $F(M)$ is a semi-simple object of R-gr having finite length. ▌

3.14. *Remark.* From Theorem 3.13 we retain that for a simple R-module M there exists a simple object N in R-gr such that M is isomorphic to a submodule of N. This result is then exactly Theorem 12.10 of [3].

3.15. COROLLARY (Greszczuk [4]). *If M is a semi-simple R-module of finite length then ${}_{R_e}M$ is semi-simple of finite length and moreover $l({}_{R_e}M) \leqslant |G|\, l({}_R M)$.*

Proof. Put $n = |G|$. We may assume that M is a simple R-module. Theorem 3.13 shows that $l_{R\text{-gr}}(F(M)) \leqslant n$, where $l_{R\text{-gr}}$ denotes length in the category R-gr. By Lemma I.7.1 of [10] we know that $F(M)_\sigma$ is a semisimple R_e-module and also $l_{R_e}(F(M)_\sigma) \leqslant {}_{R\text{-gr}}(F(M)) \leqslant n$. Since $F(M) = \bigoplus_{\sigma \in G} F(M)_\sigma$ we have $l({}_{R_e}F(M)) \leqslant n^2$. On the other hand, $F(M) \cong M^n$ in R_e-mod, hence $l({}_{R_e}F(M)) = nl({}_{R_e}M)$ and $l({}_{R_e}M) \leqslant n$. ∎

For the sake of completeness let us conclude by constructing F in yet another way, using the construction of group rings over graded rings as introduced by M. Van den Bergh in [13]; cf. also C. Năstăsescu [9].

Let R be graded of type G, G arbitrary now, and let $R[G]$ be the group ring over R graded by $R[G]_\sigma = R\sigma$, $\sigma \in G$. There is a graded subring $S = \sum_{\sigma \in G} R_\sigma \sigma$ in $R[G]$ that is in fact isomorphic to R as a graded ring under the isomorphism j: $R \to S$, $\sum_{\sigma \in G} a_\sigma \mapsto \sum_{\sigma \in G} a_\sigma \sigma$, where $a_\sigma \in R_\sigma$, $\sigma \in R$.

Note that the standard copy $R = Re$ in $R[G]$ is not a graded subring of $R[G]$. If M is an R-module then $M[G]$ is a graded $R[G]$-module in the usual way, note $M[G] = R[G] \otimes_R M$. Obviously ${}_S M[G]$ is a graded S-module and the isomorphism j: $R \to S$ defines on $M[G]$ a structure of a graded R-module by restriction of scalars. In this way we have obtained a functor

$$H'\colon R\text{-mod} \to R\text{-gr}, \qquad H'(M) = i_*({}_S M[G]).$$

3.16. PROPOSITION. *The functors F and H' are isomorphic.*

Proof. For an R-module M define V_M: $F(M) \to H'(M)$ by $V_M({}^\sigma x) = x\sigma$, where ${}^\sigma x \in {}^\sigma M = F(M)_\sigma$, $\sigma \in G$. It is easy to check that V_M is an isomorphism.

Using Proposition 3.16 in combination with the results of [9, 13] we have alternative ways to obtain the properties of the functor F mentioned before. Obviously the notion of separability awaits application to other functors, not necessarily in a graded context. ∎

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