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Rankin–Cohen brackets and formal quantization

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Abstract

In this paper, we use the theory of deformation quantization to understand Connes' and Moscovici's results [A. Connes, H. Moscovici, Rankin–Cohen brackets and the Hopf algebra of transverse geometry, Mosc. Math. J. 4 (1) (2004) 111–130, 311]. We use Fedosov's method of deformation quantization of symplectic manifolds to reconstruct Zagier's deformation [D. Zagier, Modular forms and differential operators, in: K.G. Ramanathan Memorial Issue, Proc. Indian Acad. Sci. Math. Sci. 104 (1) (1994) 57–75] of modular forms, and relate this deformation to the Weyl–Moyal product. We also show that the projective structure introduced by Connes and Moscovici is equivalent to the existence of certain geometric data in the case of foliation groupoids. Using the methods developed by the second author [X. Tang, Deformation quantization of pseudo (symplectic) Poisson groupoids, Geom. Funct. Anal. 16 (3) (2006) 731–766], we reconstruct a universal deformation formula of the Hopf algebra $\mathcal{H}_1$ associated to codimension one foliations. In the end, we prove that the first Rankin–Cohen bracket RC_1 defines a noncommutative Poisson structure for an arbitrary $\mathcal{H}_1$ action.

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Keywords: Modular forms; Rankin–Cohen brackets; Hopf algebra; Deformation quantization

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1. Introduction

In the study of transversal index theory, Connes and Moscovici introduced a Hopf algebra, $\mathcal{H}_1$, which governs the local symmetry in calculating the index of a transversal elliptic operator. Interestingly, Connes and Moscovici [4] discovered an action of $\mathcal{H}_1$ on the modular Hecke algebras.

Inspired by this action, Connes and Moscovici found many similarities between the theory of codimension one foliations and the theory of modular forms. For example, they showed that the Hopf cyclic version of the Godbillon–Vey cocycle gives rise to a 1-cocycle on $PSL(2,\mathbb{Q})$ with values in an Eisenstein series of weight 2, and that the Schwarzian 1-cocycle corresponds to an inner derivation implemented by a level 1 Eisenstein series of weight 4. In particular, inspired by Zagier's [11] Rankin–Cohen deformation on modular forms, Connes and Moscovici [5] constructed a universal deformation formula for an action of $\mathcal{H}_1$ with a projective structure. In this paper, we aim to reconstruct this deformation formula using noncommutative Poisson geometry as developed by the second author [9,10].

The origin of the Rankin–Cohen deformation is a work of Rankin. Rankin in 1956 described all polynomials in the derivatives of modular forms with values again in modular forms. Based on Rankin's work, in 1977, Cohen defined a sequence of bilinear operations on modular forms indexed by nonnegative integer n, which assigns to two modular forms, f of weight k and g of weight l, a modular form of weight $k+l+2n$. Their results showed that for any given integer $n \geqslant 0$, there is essentially (up to a constant) only one bilinear operator mapping[1] $\mathcal{M}_p \otimes \mathcal{M}_q$ to $\mathcal{M}_{p+q+2n}$ $\forall p,q \in \mathbb{Z}_{\geqslant 0}$. They are later called Rankin–Cohen brackets and usually denoted by RC_n. These operators were further studied and played an important role in the theory of modular forms. Zagier [11] observed that the sum of Rankin–Cohen brackets defines an associative product on the algebra $\mathcal{M} := \sum_{l\geqslant 0} \mathcal{M}_l$. Zagier's complete proof of the associativity of this product, which involves infinitely many equalities, was rather combinatoric. Cohen, Manin, and Zagier [3] explained this deformation using the theory of automorphic pseudo differential operators. The calculation still involves many interesting and complicated combinatoric identities. In this paper, we will first reconstruct Zagier's Rankin–Cohen deformation using the methods of deformation quantization of symplectic manifolds developed by Fedosov [6]. In particular, we will show that this deformation is isomorphic to the standard Moyal product. The calculation involved in our construction is easier and more transparent than those [3] and [11].

To reconstruct Connes–Moscovici's Rankin–Cohen deformation for $\mathcal{H}_1$ action, we need to first understand the projective structure introduced by Connes and Moscovici [5]. The notion of a projective structure of $\mathcal{H}_1$ is a generalization of the projective structure on an elliptic curve (see [3]). Our idea to understand this structure is to look at the defining action of $\mathcal{H}_1$ on a groupoid algebra associated to a codimension one foliation. In this case, we discovered that the existence of a projective structure is equivalent to the existence of a certain type of invariant symplectic connection. This geometric explanation provides a natural connection to the results in Tang [9], where he studied the deformation quantization of a groupoid algebra. The existence of an invariant symplectic connection is a sufficient condition for the existence of a deformation quantization of a groupoid algebra. Therefore, in the case of a codimension one foliation, Tang's construction [9] implies that with a projective structure, one can construct a deformation quantization (a star product) of the corresponding foliation groupoid algebra. Furthermore, our

[1] $\mathcal{M}_p$ is the space of modular forms of weight p.

calculation in Section 5 exhibits that when the symplectic connection is flat, the star product on the groupoid algebra can be expressed by an element RC in $\mathcal{H}_1 \otimes \mathcal{H}_1[[\hbar]]$. To obtain a universal deformation for an $\mathcal{H}_1$ action with a projective structure as Connes and Moscovici [5], we construct a fully injective $\mathcal{H}_1$ action on the union of groupoid algebras of those foliation groupoids with a fixed type of invariant symplectic connections. Therefore, we are able to reconstruct the universal deformation formula on $\mathcal{H}_1$ by pulling back the star products on the groupoid algebras.

All the above deformations, including [3,5,11], are all formal deformation, which means that the deformation parameter t is a formal variable. It is more interesting to ask whether one can make a deformation strict in the sense of Rieffel. This will be studied in the next paper [1].

2. Prerequisites

In this section, we review the materials needed for this paper.

2.1. Codimension one foliations and the Hopf algebra

For a constant rank foliation on M, we choose a complete flat orientable transversal X. We look at the oriented frame bundle FX of X with the lifted holonomy foliation groupoid action, which defines an étale groupoid $\mathcal{G} \rightrightarrows FX$. Connes and Moscovici found a Hopf algebra $\mathcal{H}_k$ acting on the smooth groupoid algebra $C_c^\infty(\mathcal{G})$, where k is the codimension of the foliation. We exhibit this Hopf algebra in the case of $k = 1$.

In the case of a codimension one foliation, the complete transversal X is a flat 1-dim manifold, and FX is isomorphic to $X \times \mathbb{R}^+$ by fixing a flat connection on $FX \to X$. We introduce coordinates x on the X component and y on the $\mathbb{R}^+$ component. Let Γ be a pseudogroup associated to the foliation acting on X. The lifted action of Γ on FX is

$$(x, y) \mapsto \big(\phi(x), \phi'(x)y\big), \quad \forall \phi \in \Gamma.$$

We look at the groupoid $FX \rtimes \Gamma \rightrightarrows FX$. It is an étale groupoid with a natural symplectic form $\omega = \frac{dx \wedge dy}{y^2}$.

On FX, we consider vector fields $X = y\partial_x$ and $Y = y\partial_y$. It is easy to check that Y is invariant under the Γ action, but X is not, and has the following commutation relation,

$$U_\phi X U_\phi^{-1} = X - y\frac{\phi^{-1''}(x)}{\phi^{-1'}(x)}Y.$$

We introduce the following operators on $\mathcal{A}$.

$$\begin{aligned}
X(fU_\phi) &= X(f)U_\phi, \\
Y(fU_\phi) &= Y(f)U_\phi, \\
\delta_1(fU_\phi) &= \mu_{\phi^{-1}} f U_\phi, \\
\delta_n(fU_\phi) &= X^{n-1}(\mu_{\phi^{-1}}) f U_\phi,
\end{aligned} \tag{1}$$

where $\mu_{\phi^{-1}}(x, y) = y\frac{\phi^{-1''}(x)}{\phi^{-1'}(x)}$.

The commutation relation among the above operators are

$$[Y, X] = X, \qquad [X, \delta_n] = \delta_{n+1},$$
$$[Y, \delta_n] = n\delta_n, \qquad [\delta_n, \delta_m] = 0.$$

The operators $X, Y, \delta_n, n \in \mathbb{N}$, form an infinite dimensional Lie algebra H_1, and the Hopf algebra $\mathcal{H}_1$ is defined to be the universal enveloping algebra of H_1.

We define the following operations on $\mathcal{H}_1$:

1. Product $\cdot : \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{H}_1$ by the product on $\mathcal{H}_1$ as the universal enveloping algebra of H_1.
2. Coproduct $\Delta : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1$ by

$$\Delta Y = Y \otimes 1 + 1 \otimes Y,$$
$$\Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$
$$\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,$$
$$\Delta \delta_n = [\Delta X, \Delta \delta_{n-1}].$$

3. Counit $\epsilon : \mathcal{H}_1 \to \mathbb{C}$ by taking the value of the identity component.
4. Antipode $S : \mathcal{H}_1 \to \mathcal{H}_1$ by

$$S(X) = -X + \delta_1 Y, \qquad S(Y) = -Y, \qquad S(\delta_1) = -\delta_1.$$

It is straightforward to check that $(\mathcal{H}_1, \cdot, \Delta, S, \epsilon, \mathrm{id})$ defines a Hopf algebra.

2.2. *Deformation quantization á la Fedosov*

Fedosov's construction of deformation quantizations of a symplectic manifold can be formulated as follows.

Let (M, ω) be a $2n$ dimensional symplectic manifold. At each fiber T_xM of the tangent bundle, which is a symplectic vector space, we define a Weyl algebra W_x to be an associative algebra over $\mathbb{C}$ with a unit, whose elements are of the form

$$a(y, \hbar) = \sum_{k,\, |\alpha| \geqslant 0} \hbar^k a_{k,\alpha} y^\alpha,$$

where $\hbar$ is a formal parameter and $y = (y^1, \ldots, y^{2n}) \in T_xM$ is a tangent vector, $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a multi-index, $y^\alpha = (y^1)^{\alpha_1} \cdots (y^{2n})^{\alpha_{2n}}$.

The product of elements $a, b \in W_x$ is defined as follows:

$$a \circ b = \exp\left(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(y, \hbar)b(z, \hbar)\Big|_{z=y}$$
$$= \sum_{k=0}^{\infty}\left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!}\omega^{i_1 j_1} \cdots \omega^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}}.$$

We consider the Weyl algebra bundle W over (M,ω) for which the fiber at the point x is W_x, and denote $C^\infty(W)$ to be the algebra of smooth sections of W with pointwise multiplication $\circ$. To introduce the Fedosov connection, we look at the algebra $C^\infty(W\otimes\Lambda)=\bigoplus_{q=0}^{2n}\Gamma^\infty(W\otimes\Lambda^q)$, where Λ^q is the set of smooth q-forms.

We introduce several operations on $C^\infty(W\otimes\Lambda)$.

1. Commutator, i.e. $[a,b]=a\circ b-(-1)^{\deg(a)\deg(b)}b\circ a$.
2. $\delta,\delta^*: C^\infty(W\otimes\Lambda)\to C^\infty(W\otimes\Lambda)$, i.e.

$$\delta a=dx^k\wedge\frac{\partial a}{\partial y^k},\qquad \delta^*a=y^k i\left(\frac{\partial}{\partial x^k}\right)a.$$

A *Fedosov connection* on the Weyl algebra bundle W is a connection D such that for any section $a\in C^\infty(W\otimes\Lambda)$,

$$D^2a=\frac{i}{\hbar}[\Omega,a]=0.$$

Fedosov in [6] showed that given a torsion free symplectic connection ∇ on M with Christoffel symbol Γ_{ijk}, one can construct an abelian connection on W of the following form,

$$D=-\delta+\partial+\frac{i}{\hbar}[r,\cdot],$$

where $\partial a:=da+\frac{i}{\hbar}[\Gamma,a]$, with $\Gamma=\frac{1}{2}\Gamma_{ijk}y^iy^j\,dx^k$, and r is a local 1-form with values in W.

We look at the subalgebra $W_D\subset C^\infty(W)$ consisting of flat sections of D. The main theorem that we will use is the following:

Theorem 2.1. *For any $a_0\in C^\infty(M)[[\hbar]]$, there exists a unique section $a\in W_D$, which is denoted by $\sigma^{-1}(a_0)$, such that $\sigma(a)=a_0$, where $\sigma(a)$ means the projection onto the center*: $\sigma(a)=a(x,0,h)$.

This implies that there is a one-to-one correspondence between W_D and $C^\infty(M)[[\hbar]]$. Accordingly we can define on $C^\infty(M)[[\hbar]]$ an associative star product

$$a\star b=\sigma\left(\sigma^{-1}(a)\circ\sigma^{-1}(b)\right).\tag{2}$$

2.3. Deformation quantization of groupoids

The second named author [9] considered deformation quantization of the groupoid algebra of a pseudo étale groupoid and proved that one can construct star products on such groupoids. As a special case, we have that for an étale groupoid with an invariant symplectic structure and an invariant symplectic connection on the base, the groupoid algebra can be formally deformation quantized. In this subsection, we recall the basic concepts and constructions from Tang [9].

Definition 1. (Block, Getzler and Xu) A Poisson structure on an associative algebra A is an element $[\Pi]$ of the Hochschild cohomology group $H^2(A,A)$ such that the cohomology class of the Gerstenhaber bracket $[\Pi,\Pi]$ vanishes.

Definition 2. Let $(A, [\Pi])$ be a noncommutative Poisson algebra, and $A[[\hbar]]$ the space of formal power series with coefficients in A. A *formal deformation quantization* of $(A, [\Pi])$ (or in other words *star product*) is an associative product

$$\star : A[[\hbar]] \times A[[\hbar]] \to A[[\hbar]], \quad (a_1, a_2) \mapsto a_1 \star a_2 = \sum_{k=0}^{\infty} \hbar^k c_k(a_1, a_2)$$

satisfying the following properties:

1. Each one of the maps $c_k : A[[\hbar]] \otimes A[[\hbar]] \to A[[\hbar]]$ is $\mathbb{C}[[\hbar]]$-bilinear;
2. One has $c_0(a_1, a_2) = a_1 \cdot a_2$ for all $a_1, a_2 \in A$;
3. The relation

$$a_1 \star a_2 - c_0(a_1, a_2) - \frac{i}{2}\hbar \Pi(a_1, a_2) \in \hbar^2 A[[\hbar]]$$

   holds true for some representative $\Pi \in Z^2(A, A)$ of the Poisson structure and all $a_1, a_2 \in A$.

For an étale groupoid $\mathcal{G}$ with an invariant symplectic form ω and a invariant symplectic connection ∇ on the base, we define a Hochschild 2-cochain on $C^\infty(\mathcal{G})$ by

$$\Pi(a_1, a_2)(g) = \sum_{g_1 g_2 = g} \pi(g)\big(da_1(g_1), da_2(g_2)\big), \quad g \in \mathcal{G},\ a_1, a_2 \in C^\infty(\mathcal{G}), \tag{3}$$

where $da_1(g_1)$ and $da_2(g_2)$ have been pulled back to g along the maps t and s, and π is the Poisson structure associated to the symplectic form ω. This definition is legitimate because t and s are local diffeomorphisms. It was proved [9] that this Hochschild 2-cochain gives rise to a Poisson structure on $C^\infty(\mathcal{G})$ if there is an invariant symplectic connection.

Tang [9] showed that the above noncommutative Poisson structure Π on the groupoid algebra admits a formal deformation quantization. Such a deformation can be constructed as follows: first using Fedosov's construction [6], given an invariant symplectic connection, we construct an invariant star product on the algebra of smooth functions on the unit space $\mathcal{G}^{(0)}$. The deformation of the groupoid algebra $C^\infty(\mathcal{G})$ is a crossed product algebra of the above deformation on the base $C^\infty(\mathcal{G}^{(0)})$ and the associated pseudogroup $\mathcal{G}$ action.

2.4. Rankin–Cohen deformation

It is well known that if $f(z)$ is a modular form, $\frac{1}{2\pi i}\frac{d}{dz} f$ is not a modular form any more. Following [4], we introduce a differential operator X as

$$X \stackrel{\mathrm{def}}{=} \frac{1}{2\pi i}\frac{d}{dz} - \frac{1}{12\pi i}\frac{d}{dz}(\log \Delta) \cdot Y,$$

where $\Delta(z) = (2\pi)^{12}\eta^{24}(z) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1 - q^n)^{24}$, $q = e^{2\pi z}$ and $Y(f) = \frac{k}{2} f$, $\forall f \in \mathcal{M}_k$, the space of modular forms of weight k.

It is straightforward to check that X and Y acts on $\mathcal{M} = \bigoplus_k \mathcal{M}_k$ satisfying $[Y, X] = X$. Under these two operators, the Rankin–Cohen bracket RC_n can be written as follows, for $f \in \mathcal{M}_k$, $g \in \mathcal{M}_l$

$$RC_n(f, g) = \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+l-1}{r} f^{(r)} g^{(s)},$$

where $f^{(r)}$ (or $g^{(s)}$) is the rth (or sth) derivative of f (or g), and $(\alpha)_k \stackrel{\text{def}}{=} \alpha(\alpha+1)\cdots(\alpha+k-1)$.

In [11], Zagier observed that $\sum_n RC_n$ defines an associative product on $\mathcal{M}$. This product actually defines a universal deformation formula of the Lie algebra h_1, consisting of X, Y with $[Y, X] = X$, since h_1 acts on $\mathcal{M}$ injectively. It is worth mentioning that h_1 is the Lie algebra of the "$ax + b$" group.

Inspired by the Rankin–Cohen brackets, Connes and Moscovici [5] introduced a family of Rankin–Cohen type elements in $(\mathcal{H}_1 \otimes \mathcal{H}_1)[[\hbar]]$ as follows.

Definition 2.2. [5] Let $\mathcal{H}_1$ act on an algebra A. This action is called projective if $\delta_2' \stackrel{\text{def}}{=} \delta_1^2 - \frac{1}{2}\delta_2$ is inner implemented by an element $\Omega \in A$, so that

$$\delta_2'(a) = [\Omega, a], \quad \forall a \in A,$$

and

$$\delta_k(\Omega) = 0, \quad \forall k \in \mathbb{N}.$$

Assume that the action of $\mathcal{H}_1$ action an algebra A is projective. Define

$$\begin{aligned} RC &= \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} \frac{A_k}{k!}(2Y+k)_{n-k} \otimes \frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k, \\ A_{m+1} &= S(X)A_m - m\Omega^0\left(Y - \frac{m-1}{2}\right)A_{m-1}, \\ B_{m+1} &= XB_m - m\Omega\left(Y - \frac{m-1}{2}\right)B_{m-1}, \end{aligned} \tag{4}$$

where Ω^0 is the right multiplication of Ω.

Connes and Moscovici [5] proved that RC defines a universal deformation formula of a projective $\mathcal{H}_1$ action.

3. Universal deformation of h_1

If we set all δ_n to be 0, the Lie algebra H_1 is reduced to h_1, the Lie algebra of the "$ax + b$" group, and $\mathcal{H}_1$ becomes $\mathcal{U}(h_1)$, the universal enveloping algebra of h_1. In this case, RC defined by (4) is simplified to the following universal deformation formula of h_1,

$$RC_n(a, b) \stackrel{\text{def}}{=} \sum_{k=0}^{n} \left[\frac{(-1)^k}{k!} X^{n-k}(2Y+k)_{n-k}(a) \frac{1}{k!} X^{n-k}(2Y+n-k)_k(b) \right], \tag{5}$$

where $X, Y \in h_1$ are such that $[Y, X] = X$, $(\alpha)_k \stackrel{\text{def}}{=} \alpha(\alpha+1)\cdots(\alpha+k-1)$, and $a, b \in A$.

We spend this section studying this universal deformation.

3.1. *Giaquinto–Zhang's deformation of* h_1

A nice deformation formula for h_1 has already been given by Giaquinto and Zhang [7, Theorem 2.20]: Given two elements X, Y with $[Y, X] = X$, the following expression defines a universal deformation formula (UDF) of the Hopf algebra associated to h_1,

$$F = \sum_{n=0}^{\infty} \frac{t^n}{n!} F_n = 1 \times 1 + tX \wedge Y + \frac{t^2}{2!}\left(X^2 \otimes Y_2 - 2XY_1 \otimes XY_1 + Y_2 \otimes X^2\right) + \cdots,$$

where F_n is defined to be $F_n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} X^{n-r} Y_r \otimes X^r Y_{n-r}$.

Proposition 3.1. *The above defined F can be realized by the standard Moyal product.*

Proof. We consider the space $\mathbb{R} \times \mathbb{R}_+$ on which X and Y act as $Y = -y\frac{\partial}{\partial y}$, and $X = \frac{1}{y}\frac{\partial}{\partial x}$. It is obvious that the action of X and Y on $\mathbb{R} \times \mathbb{R}_+$ is injective.

With the following identity,

$$Y_r = Y(Y+1)\cdots(Y+r-1) = (-y)^r \frac{\partial^r}{\partial y^r},$$

it is straightforward to check that the above defined F in this representation is equal to the Moyal product. □

3.2. *Rankin–Cohen deformation of* h_1

We should point out that the above universal deformation formula of h_1 is not equal to the one induced from *RC* in Eq. (5). However, we will show that it is equivalent to the Giaquinto–Zhang's deformation.

We set $(V, \omega) := (\mathbb{R}^2 = \{(p, q)\}, dp \wedge dq)$ and denote by $\mathfrak{h} = \mathfrak{h}(V, \omega) := V \times \mathbb{R}$ the associated Heisenberg algebra. Setting $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{R}) = \mathtt{span}_{\mathbb{R}}\{H, E, F\}$ ($[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$), we form the natural semi-direct product $\tilde{\mathfrak{g}} := \mathfrak{g} \times \mathfrak{h}$. The (infinitesimal) affine linear action $\tilde{\gamma} \to \Gamma(T(V))$ is then strongly Hamiltonian. We let $\lambda : \tilde{\mathfrak{g}} \to C^\infty(V)$ denote the corresponding moment map. Explicitly, denoting fundamental vector fields by $A^\star_x := \frac{d}{dt}|_{t=0} \exp(-tA) \cdot x$, $A \in \tilde{\mathfrak{g}}$, one has

$$H^\star = -p\partial_p + q\partial_q; \quad E^\star = -q\partial_p; \quad F^\star = -p\partial_q; \quad P^\star = -\partial_p; \quad Q^\star = -\partial_q;$$
$$\lambda_H = pq; \quad \lambda_E = \frac{1}{2}q^2; \quad \lambda_F = -\frac{1}{2}p^2; \quad \lambda_P = q; \quad \lambda_Q = -p.$$

We have that $[A^\star, B^\star] = [A, B]^\star$ and $\lambda_{[A,B]} = \{\lambda_A, \lambda_B\}$ where $\{u, v\} = \partial_p u \partial_q v - \partial_p v \partial_q u$, and $A, B \in \tilde{\mathfrak{g}}$.

Let $S := AN = \exp(\mathtt{span}\{H, E\})$ denote the Iwasawa component in $SL(2, \mathbb{R})$, which is the "$ax + b$" group. We consider the open orbit $\mathcal{O} \stackrel{\text{def}}{=} S \cdot (0, 1)$ in V, which is equal to the set $[q > 0]$.

Since S acts simply transitively on $\mathcal{O}$, we have the identification $\phi : S \to \mathcal{O} : g \mapsto g \cdot (1,0)$. We still denote by $\lambda : \tilde{\mathfrak{g}} \to C^\infty(S)$ the transported restricted moment map, that is:

$$\lambda_A := \phi^\star(\lambda_A|_{\mathcal{O}}) \quad (A \in \tilde{\mathfrak{g}}). \tag{6}$$

Lemma 3.2. *Denoting by* $\tilde{X}_g := \frac{d}{dt}|_{t=0} g \exp(tX)$ *the left-invariant vector field associated to* $X \in h_1 = \mathrm{Lie}(S)$*, one has*:

(i) $\tilde{H} . \lambda_{X+v} = (-2)\lambda_X + (-1)\lambda_v$ *for* $X \in \mathfrak{g}$ *and* $v \in V$;
(ii) $\tilde{E}^r . \lambda_X = 0$ *for* $r \geqslant 3$*, for all* $X \in \mathfrak{g}$;
(iii) $\tilde{E}^r . \lambda_v = 0$ *for* $r \geqslant 2$*, for all* $v \in V$.

Proof. A convenient parametrization of the group manifold S is given by:

$$\mathbb{R}^2 \to S : (a, \ell) \mapsto \exp(aH)\exp(\ell E).$$

In these coordinates, the group law reads $(a, \ell) \cdot (a', \ell') = (a + a', e^{-2a'}\ell + \ell')$. We deduce the expressions for the left-invariant vector fields:

$$\tilde{H} = \partial_a - 2\ell\partial_\ell; \qquad \tilde{E} = \partial_\ell.$$

The corresponding chart on the orbit $\mathcal{O} \simeq S$ is given by

$$p = e^a \ell; \qquad q = e^{-a}.$$

Note that this is a global Darboux chart on $\mathcal{O}$ as for $da \wedge d\ell = \pm\phi^\star\omega|_{\mathcal{O}}$. The corresponding (uncomplete) moment map reads as

$$\lambda_H = \ell; \qquad \lambda_E = \frac{1}{2}e^{-2a}; \qquad \lambda_F = -\frac{1}{2}\ell^2 e^{2a}; \qquad \lambda_P = e^{-a}; \qquad \lambda_Q = -e^a\ell.$$

A straightforward computation then yields the lemma. □

From (5), for any left $\mathcal{U}(h_1)$ action on an algebra A, the Rankin–Cohen brackets on $\mathcal{U}(h_1)$ is defined by,

$$RC_n(a, b) := \sum_{k=0}^{n} \left[\frac{(-1)^k}{k!} X^k (2Y + k)_{n-k}(a) \frac{1}{k!} X^{n-k} (2Y + n - k)_k(b) \right],$$

where $X, Y \in h_1$ are such that $[Y, X] = X$, $(\alpha)_k \stackrel{\text{def}}{=} \alpha(\alpha + 1) \cdots (\alpha + k - 1)$, and $a, b \in A$.

Since h_1 acts as left invariant vector fields on S, $\mathcal{U}(h_1)$ acts as left invariant differential operators on $C^\infty(S)$, and RC_n, an element of $\mathcal{U}(h_1) \otimes \mathcal{U}(h_1)$, acts as a left invariant bidifferential operator on $C^\infty(S)$. Since $[H, E] = 2E$, we set

$$\tilde{H} = 2Y \quad \text{and} \quad \tilde{E} = X.$$

Lemma 3.3. *For all A in $\tilde{\mathfrak{g}}$, we have*

$$[\lambda_A, u]_n \overset{\text{def}}{=} RC_n(\lambda_A, u) - RC_n(u, \lambda_A) = 0 \quad \textit{for } n \neq 1. \tag{7}$$

Proof. For $X \in \mathfrak{g}$ and $v \in V$, Lemma 3.2 implies that $X^k(2Y + r)_s.\lambda_{X+v} = (-2+r)_s X^k \lambda_X + (-1+r)_s X^k \lambda_v = 0$ if $k > 2$. Therefore, in the expression (5) of $RC_n(\lambda_{X+v}, u)$ only the first three terms corresponding to $k = 0, 1, 2$ contribute. In each of them the following (left-hand side) factor occurs:

- for $k = 0$: $\quad (-2)_n \lambda_X + (-1)_n \lambda_v;$ (8)
- for $k = 1$: $\quad \tilde{E}.\big[(-1)_{n-1}\lambda_X + (0)_{n-1}\lambda_v\big];$ (9)
- for $k = 2$: $\quad \tilde{E}^2.\big[(0)_{n-2}\lambda_X + (1)_{n-2}\lambda_v\big].$ (10)

1. The first expression (8) vanishes identically for $n \geqslant 3$. Indeed, $(-2)_n = (-2)(-2+1)(-2+2)\cdots(-2+n-1)$ is zero as soon as $n-1 \geqslant 2$; and similarly for $(-1)_n$.
2. In the same way, the second expression (9) vanishes for $n-2 \geqslant 1$, i.e. $n \geqslant 3$.
3. At last, the third expression (10) is equal to $(n-2)!\tilde{E}^2(\lambda_v)$ which is identically zero by Lemma 3.2, item (iii). We conclude by observing that RC_0 and RC_2 are symmetric. □

By Lemma 3.3, the Rankin–Cohen deformation (4) defines a $\tilde{\mathfrak{g}}$ invariant star product on (V, ω). In Corollary 2, Section 2.7 of [8], Gutt showed that there is a unique $\tilde{\mathfrak{g}}$-invariant star product on (V, ω), which is the standard Moyal product. We conclude that the Rankin–Cohen deformation on $C^\infty(S)$ is identical to the Moyal product.

Proposition 3.4. *The reduced Rankin–Cohen deformation realized on $\mathcal{O} \subset V$ coincides with the restriction to $\mathcal{O}$ of the standard Moyal product on (V, Ω).*

To generalize the construction in Proposition 3.4, we explain its relation to Fedosov's construction of deformation quantization of symplectic manifolds.

The natural action of $S \simeq$ "$ax + b$" on $\mathbb{R}$,

$$\exp(aH + nE) \cdot x_1 := e^{2a} x_1 + n e^a,$$

lifts to $T^\star(\mathbb{R}) = \mathbb{R}^2$ as

$$\exp(aH + nE) \cdot (x_1, x_2) := \big(e^{2a} x_1 + n e^a, e^{-2a} x_2\big).$$

The S-orbit $\tilde{\mathcal{O}}$ of point $\tilde{o} := (0, 1) = dx_1|_0 \in T^\star(\mathbb{R}^2)$ is then naturally isomorphic as S-homogeneous space to $\mathcal{O} \subset V$; namely one has the identification:

$$\varphi : \mathcal{O} \to \tilde{\mathcal{O}} : g \cdot e_2 \mapsto g \cdot \tilde{o}.$$

In (p, q)-coordinates on $\mathcal{O}$, this reads:

$$\varphi(p, q) = \left(\frac{p}{2q}, q^2\right).$$

Identifying $\tilde{\mathcal{O}}$ with S (via $\varphi \circ \phi$), we obtain the expressions for the left invariant vector fields:

$$\tilde{H} = -2x_2\partial_{x_2}; \qquad \tilde{E} = \frac{1}{x_2}\partial_{x_1}.$$

In particular, we set

$$\tilde{H} = 2Y \quad \text{and} \quad \tilde{E} = X.$$

By letting $\nabla^{\mathcal{O}}$ denote the restriction to $\mathcal{O}$ of the standard symmetric flat connection on V ($\nabla^{\mathcal{O}}_{\partial_p}\partial_p = \nabla^{\mathcal{O}}_{\partial_q}\partial_p = \nabla^{\mathcal{O}}_{\partial_q}\partial_q = 0$), and setting

$$\nabla^{\tilde{\mathcal{O}}} := \varphi\big(\nabla^{\mathcal{O}}\big),$$

we obtain a symplectic connection on $\tilde{\mathcal{O}}$,

$$\nabla^{\tilde{\mathcal{O}}}_{\partial_{x_1}}\partial_{x_1} = 0; \qquad \nabla^{\tilde{\mathcal{O}}}_{\partial_{x_1}}\partial_{x_2} = \frac{1}{2x_2}\partial_{x_1}; \qquad \nabla^{\tilde{\mathcal{O}}}_{\partial_{x_2}}\partial_{x_2} = -\frac{1}{2x_2}\partial_{x_2}. \tag{11}$$

We identify $\tilde{\mathcal{O}}$ with $\mathbb{R} \times \mathbb{R}^+$, and use $\nabla^{\tilde{\mathcal{O}}}$ to construct deformation quantization of ($\mathbb{R} \times \mathbb{R}^+$, $\omega \overset{\text{def}}{=} dx \wedge dy$) as described in Section 2.2.

Corollary 3.5. *The reduced Rankin–Cohen deformation on $\tilde{\mathcal{O}}$ is identical to Fedosov's construction of the star product on $(\tilde{\mathcal{O}}, \omega)$ using the connection $\nabla^{\tilde{\mathcal{O}}}$ with the characteristic form equal to $\frac{1}{i\hbar}\omega$.*

4. Projective structures

To reconstruct Connes–Moscovici's Rankin–Cohen deformation, we need to understand the geometric meaning of their Definition 2.2, a projective structure.

4.1. The flat case

We look at the connection $\nabla^{\tilde{\mathcal{O}}}$ considered in Section 3, (11).

Proposition 4.1. *The connection $\nabla^{\tilde{\mathcal{O}}}$ (11) is invariant under the local diffeomorphism $\phi : x_1 \mapsto \tilde{x}_1 \overset{\text{def}}{=} \phi(x_1)$, $x_2 \mapsto \tilde{x}_2 \overset{\text{def}}{=} \frac{x_2}{\phi'(x_1)}$ if and only if $\delta_2'(\phi) = 0$. Here $\mathcal{H}_1$ acts on ϕ as in Section* 2.1.

Notation. We use ∇ to replace $\nabla^{\tilde{\mathcal{O}}}$ in the rest of the paper.

Proof. We have the following transformation rules of vector fields.

$$\frac{\partial}{\partial \tilde{x}_1} = \frac{1}{\phi'(x_1)}\frac{\partial}{\partial x_1} + \frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial x_2},$$
$$\frac{\partial}{\partial \tilde{x}_2} = \phi'\frac{\partial}{\partial x_2}.$$

The invariance of ∇ implies that we should have

$$\begin{aligned}
\nabla_{\phi_*(\frac{\partial}{\partial x_1})}\phi_*\left(\frac{\partial}{\partial x_1}\right) &= \nabla_{\phi'(x_1)\frac{\partial}{\partial\tilde{x}_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}}\left(\phi'(x_1)\frac{\partial}{\partial\tilde{x}_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}\right)\\
&= \phi'^2\nabla_{\frac{\partial}{\partial\tilde{x}_1}}\frac{\partial}{\partial\tilde{x}_1}+\phi'\frac{\partial}{\partial\tilde{x}_1}(\phi')\frac{\partial}{\partial\tilde{x}_1}-\frac{\phi''}{\phi'}x_2\nabla_{\frac{\partial}{\partial\tilde{x}_1}}\frac{\partial}{\partial\tilde{x}_2}-\phi'\frac{\partial}{\partial\tilde{x}_1}\left(\frac{\phi''}{\phi'^2}x_2\right)\frac{\partial}{\partial\tilde{x}_2}\\
&\quad-\frac{\phi''}{\phi'}x_2\nabla_{\frac{\partial}{\partial\tilde{x}_2}}\frac{\partial}{\partial\tilde{x}_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}(\phi')\frac{\partial}{\partial\tilde{x}_1}+\left(\frac{\phi''}{\phi'^2}x_2\right)^2\nabla_{\frac{\partial}{\partial\tilde{x}_2}}\frac{\partial}{\partial\tilde{x}_2}\\
&\quad+\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}\left(\frac{\phi''}{\phi'^2}x_2\right)\frac{\partial}{\partial\tilde{x}_2}\\
&= \phi'\frac{1}{\phi'}(\phi'')\frac{\partial}{\partial\tilde{x}_1}-\phi'\frac{\phi''}{\phi'^2}x_2\frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_1}\\
&\quad-\phi'\left[\frac{1}{\phi'}\frac{\phi'''\phi'^2-2\phi''^2\phi'}{(\phi'^2)^2}x_2+\left(\frac{\phi''}{\phi'^2}\right)^2x_2\right]\frac{\partial}{\partial\tilde{x}_2}\\
&\quad-\phi'\frac{\phi''}{\phi'^2}x_2\frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_1}+0+\left(\frac{\phi''}{\phi'^2}x_2\right)^2\frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_2}+\frac{\phi''}{\phi'^2}x_2\phi'\frac{\phi''}{\phi'^2}\frac{\partial}{\partial\tilde{x}_2}\\
&= -\frac{\phi'''\phi'-\frac{3}{2}\phi''^2}{\phi'^3}x_2\frac{\partial}{\partial\tilde{x}_2},
\end{aligned}$$

$$\begin{aligned}
\nabla_{\phi_*(\frac{\partial}{\partial x_1})}\phi_*\left(\frac{\partial}{\partial x_2}\right) &= \nabla_{\phi'(x_1)\frac{\partial}{\partial\tilde{x}_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}}\left(\frac{1}{\phi'}\frac{\partial}{\partial\tilde{x}_2}\right)\\
&= \phi'\frac{1}{\phi'}\nabla_{\frac{\partial}{\partial\tilde{x}_1}}\frac{\partial}{\partial\tilde{x}_2}+\phi'\frac{\partial}{\partial\tilde{x}_1}\left(\frac{1}{\phi'}\right)\frac{\partial}{\partial\tilde{x}_2}-\frac{\phi''}{\phi'^2}x_2\frac{1}{\phi'}\nabla_{\frac{\partial}{\partial\tilde{x}_2}}\frac{\partial}{\partial\tilde{x}_2}\\
&\quad-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde{x}_2}\left(\frac{1}{\phi'}\right)\frac{\partial}{\partial\tilde{x}_2}\\
&= \frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_1}+\phi'\frac{1}{\phi'}\left(-\frac{\phi''}{\phi'^2}\right)\frac{\partial}{\partial\tilde{x}_2}-\frac{\phi''}{\phi'^2}x_2\frac{1}{\phi'}\left(-\frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_2}\right)-0\\
&= \frac{1}{2\tilde{x}_2}\frac{\partial}{\partial\tilde{x}_1}-\frac{1}{2}\frac{\phi''}{\phi'^2}\frac{\partial}{\partial\tilde{x}_2}=\phi_*\left(\frac{1}{2x_2}\frac{\partial}{\partial x_1}\right),
\end{aligned}$$

$$\begin{aligned}
\nabla_{\phi_*(\frac{\partial}{\partial x_2})}\phi_*\left(\frac{\partial}{\partial x_2}\right) &= \nabla_{\frac{1}{\phi'}\frac{\partial}{\partial\tilde{x}_2}}\left(\frac{1}{\phi'}\frac{\partial}{\partial\tilde{x}_2}\right)=\frac{1}{\phi'^2}\nabla_{\frac{\partial}{\partial\tilde{x}_2}}\left(\frac{\partial}{\partial\tilde{x}_2}\right)+\frac{1}{\phi'}\frac{\partial}{\partial\tilde{x}_2}\left(\frac{1}{\phi'}\right)\frac{\partial}{\partial\tilde{x}_2}\\
&= \frac{1}{\phi'^2}\left(-\frac{1}{2\tilde{x}_2}\right)\frac{\partial}{\partial\tilde{x}_2}+0=\phi_*\left(-\frac{1}{2x_2}\frac{\partial}{\partial x_2}\right)\bigg|_{(\tilde{x}_1,\tilde{x}_2)}.
\end{aligned}$$

We see easily that the invariance of the connection under ϕ is equivalent to $\phi'''\phi'-\frac{3}{2}\phi''^2=0$, i.e. $\delta_2'(\phi)=0$. $\square$

4.2. The general case

For the general case of nontrivial δ_2', we look at the following connection.

$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1}=\mu(x_1,x_2)\frac{\partial}{\partial x_2},\qquad \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2}=\frac{1}{2x_2}\frac{\partial}{\partial x_1},$$
$$\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_1}=\frac{1}{2x_2}\frac{\partial}{\partial x_1},\qquad \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}=-\frac{1}{2x_2}\frac{\partial}{\partial x_2}. \tag{12}$$

Here μ is a suitable function.

Theorem 4.2. *Let Γ be a pseudogroup generated by local diffeomorphisms on $\mathbb{R}$ acting on $\mathbb{R}\times\mathbb{R}^+$ by $\phi: x_1\mapsto\phi(x_1)$, $x_2\mapsto\frac{x_2}{\phi'(x_1)}$, $\forall\phi\in\Gamma$. Assume that the dimension of the fixed point set of each element $\phi\in\Gamma$ is strictly less than 2. The connection ∇ in* (12) *is invariant under Γ if and only if the $\mathcal{H}_1$ action on the corresponding groupoid algebra $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ is projective.*

Proof. Given a local diffeomorphism ϕ, we have the following quantity different from the proof of Proposition 4.1. All the others are same.

$$\begin{aligned}
\nabla_{\phi_*(\frac{\partial}{\partial x_1})}\phi_*\left(\frac{\partial}{\partial x_1}\right)&=\nabla_{\phi'(x_1)\frac{\partial}{\partial\tilde x_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde x_2}}\left(\phi'(x_1)\frac{\partial}{\partial\tilde x_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde x_2}\right)\\
&=\phi'^2\nabla_{\frac{\partial}{\partial\tilde x_1}}\frac{\partial}{\partial\tilde x_1}+\phi'\frac{\partial}{\partial\tilde x_1}(\phi')\frac{\partial}{\partial\tilde x_1}-\frac{\phi''}{\phi'}x_2\nabla_{\frac{\partial}{\partial\tilde x_1}}\frac{\partial}{\partial\tilde x_2}\\
&\quad-\phi'\frac{\partial}{\partial\tilde x_1}\left(\frac{\phi''}{\phi'^2}x_2\right)\frac{\partial}{\partial\tilde x_2}-\frac{\phi''}{\phi'}x_2\nabla_{\frac{\partial}{\partial\tilde x_2}}\frac{\partial}{\partial\tilde x_1}-\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde x_2}(\phi')\frac{\partial}{\partial\tilde x_1}\\
&\quad+\left(\frac{\phi''}{\phi'^2}x_2\right)^2\nabla_{\frac{\partial}{\partial\tilde x_2}}\frac{\partial}{\partial\tilde x_2}+\frac{\phi''}{\phi'^2}x_2\frac{\partial}{\partial\tilde x_2}\left(\frac{\phi''}{\phi'^2}x_2\right)\frac{\partial}{\partial\tilde x_2}\\
&=\phi'^2\mu(\tilde x_1,\tilde x_2)\frac{\partial}{\partial\tilde x_2}+\phi'\frac{1}{\phi'}(\phi'')\frac{\partial}{\partial\tilde x_1}-\phi'\frac{\phi''}{\phi'^2}x_2\frac{1}{2\tilde x_2}\frac{\partial}{\partial\tilde x_1}\\
&\quad-\phi'\left[\frac{1}{\phi'}\frac{\phi'''\phi'^2-2\phi''^2\phi'}{(\phi'^2)^2}x_2+\left(\frac{\phi''}{\phi'^2}\right)^2x_2\right]\frac{\partial}{\partial\tilde x_2}-\phi'\frac{\phi''}{\phi'^2}x_2\frac{1}{2\tilde x_2}\frac{\partial}{\partial\tilde x_1}\\
&\quad+\left(\frac{\phi''}{\phi'^2}x_2\right)^2\frac{1}{2\tilde x_2}\frac{\partial}{\partial\tilde x_2}+\frac{\phi''}{\phi'^2}x_2\phi'\frac{\phi''}{\phi'^2}\frac{\partial}{\partial\tilde x_2}\\
&=\left[\phi'^2\mu(\tilde x_1,\tilde x_2)-\frac{\phi'''\phi'-\frac{3}{2}\phi''^2}{\phi'^3}x_2\right]\frac{\partial}{\partial\tilde x_2}.
\end{aligned}$$

By the invariance of ∇, we have

$$\left[\phi'^2\mu(\tilde x_1,\tilde x_2)-\frac{\phi'''\phi'-\frac{3}{2}\phi''^2}{\phi'^3}x_2\right]\frac{\partial}{\partial\tilde x_2}=\phi_*\left(\mu(x_1)\frac{\partial}{\partial x_2}\right)=\mu(x_1,x_2)\frac{1}{\phi'}\frac{\partial}{\partial\tilde x_2},$$

and

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^3} x_2 = \phi'^2 \mu\left(\phi(x_1), \frac{x_2}{\phi'}\right) - \frac{1}{\phi'}\mu(x_1, x_2). \tag{13}$$

By Eq. (13), we have

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^2} x_2^2 = \phi'^4 \tilde{x}_2 \mu\left(\phi(x_1), \frac{x_2}{\phi'}\right) - x_2\mu(x_1, x_2). \tag{14}$$

1. $\Rightarrow$. Let ϕ be an element in Γ.
We introduce $\nu = \frac{\mu(x_1,x_2)}{x_2}$, and Eq. (14) is equivalent to

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^2} = \phi'^2 \nu\left(\phi(x_1), \frac{x_2}{\phi'}\right) - \nu(x_1, x_2).$$

Define $\omega(x_1, x_2) = \nu(x_1, \frac{1}{x_2})$, and we have

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^2} = \phi'^2 \nu\left(\phi(x_1), \frac{x_2}{\phi'}\right) - \nu(x_1, x_2) = \phi'^2 \omega\left(\phi(x_1), \frac{\phi'}{x_2}\right) - \omega\left(x_1, \frac{1}{x_2}\right).$$

Introduce $y = \frac{1}{x_2}$, the above equation gives

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^2} = \phi'^2 \omega\big(\phi(x_1), \phi' y\big) - \omega(x, y). \tag{15}$$

Finally, letting $\Omega(x, y) = y^2\omega(x, y)$, $x_1 = x$, we see that Eq. (15) implies

$$\frac{\phi'''\phi' - \frac{3}{2}\phi''^2}{\phi'^2} y^2 = \phi'^2 y^2 \omega\big(\phi(x_1), \phi' y\big) - \omega(x, y) y^2 = \big(\phi^{-1}\big)^*(\Omega)(x, y) - \Omega(x, y).$$

The left-hand side of the above equation is equal to the expression of $\delta_2'(\phi^{-1})$. The above equality shows that δ_2' is inner when we consider the $\mathcal{H}_1$ action on the foliation groupoid $FX \rtimes \mathcal{G}$ as in Section 2.1.

2. $\Leftarrow$. Suppose that the $\mathcal{H}_1$ action on $\Gamma \ltimes C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$ is projective.

We first show that if the $\mathcal{H}_1$ action is projection on $\Gamma \ltimes C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, the support of Ω has to be on the unit space. We write $\Omega = \sum_{\alpha\in\Gamma} \Omega_\alpha U_\alpha$ and $\delta_2'(U_\phi)U_\phi = [\Omega, U_\phi]$, and have the following observations.

(a) From $\delta_i(\Omega) = 0$, $\forall i > 0$, we know that $\delta_i(U_\alpha)\Omega_\alpha = 0$, $\forall \alpha$.
(b) From $\delta_i(f) = 0$ for any $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$, we have that $[\Omega, f] = \sum_{\alpha\in\Gamma}(\alpha^*(f) - f)\Omega_\alpha U_\alpha$. Therefore $(\alpha^*(f) - f)\Omega_\alpha = 0$, for all $\alpha \in \Gamma$.

For a given $\alpha \in \Gamma$ not equal to identity, we have that $\delta_i(U_\alpha)\Omega_\alpha = 0$, $\forall i > 0$, and $(\alpha^*(f) - f)\Omega_\alpha = 0$. If there is $x_0 \in \mathbb{R} \times \mathbb{R}^+$ such that $\Omega_\alpha(x_0) \neq 0$, then at x_0, there is a neighborhood N of x_0 on which $\delta_i(U_\alpha) = 0$. In particular $\delta_1(U_\alpha) = \log((\alpha^{-1})')' = 0$. Solving this differential equation, we know that α on N must act like $\alpha:(x_1,x_2)\mapsto(ax_1+b,ax_2)$. By the fact that $(\alpha^*(f)-f)\Omega_\alpha(x_0)=0$ on N, for any smooth function, we know that $\alpha(x_0)=x_0$. The same argument show that all $x\in N$ has to be fixed by α, since $\Omega_\alpha(x)\neq 0$. But this contradicts our assumption that the fixed point set of α is at most 1 dimensional. This shows that $\Omega_\alpha=0$.

From the above argument, we know that Ω has to be supported on the unit space. At this time, the projective condition is equivalent to

$$\delta_2'(\phi^{-1})=y^2\frac{\phi'''\phi'-\frac{3}{2}\phi''^2}{\phi'^2}U_\phi=\bigl(\Omega-\phi^*(\Omega)\bigr)U_\phi.$$

From (15) and the transformation there, we know that the existence of Ω implies the existence of an invariant connection like (12). □

Remark 4.3. Here, for calculation convenience, we have identified the Frame bundle $F\mathbb{R}$ with the cotangent bundle $T^*\mathbb{R}$ by $\tau:(x,y)\mapsto(x,\frac{1}{y})$. The connection ∇ is defined on $T^*\mathbb{R}$. By τ, it is also defined on $F\mathbb{R}$.

In Theorem 4.2, the assumption that the fixed point set of any element in Γ is at most one dimensional is only used in the sufficient part of the proof. Generally, Ω is supported on the fixed point set $B^{(0)}$ of Γ, i.e. $\{(\gamma,x)\mid\gamma\in\Gamma,\ \gamma(x)=x\}$. Γ acts on $B^{(0)}$, by conjugation action. The similar result of Theorem 4.2 is extended to this general situation without any extra effort.

Theorem 4.2′. *Let Γ be a pseudogroup generated by local diffeomorphisms on $\mathbb{R}$ and $B^{(0)}=\{(\gamma,x)\in\Gamma\times\mathbb{R}\times\mathbb{R}^+\mid\gamma\cdot x=x\}$ be the fixed point set. The projective action (ρ,Ω) of $\mathcal{H}_1$ on $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ is one to one correspondent to a Γ invariant connection ∇ on $\mathbb{R}\times\mathbb{R}^+$ of form* (12) *and a smooth function f on $\Gamma\times\mathbb{R}\times\mathbb{R}^+$, which is supported on $B^{(0)}-\{(\mathrm{id},x)\mid x\in\mathbb{R}\times\mathbb{R}^+\}$ and invariant under Γ conjugation action.*

5. Universal deformation formula for $\mathcal{H}_1$

In this section, we will use a Fedosov type construction to reconstruct the universal deformation formula of $\mathcal{H}_1$ originally constructed by Connes and Moscovici [5].

5.1. Zagier's deformation

In this subsection, we discuss the influence of the above new connection (12) on the star product (2).

Corollary 5.1. *The connection ∇* (12) *is flat if and only if $\mu(x_1,x_2)=x_2\nu(x_1)$, where $\nu(x_1)$ is an arbitrary smooth function on $\mathbb{R}$.*

Proof. The curvature of ∇ can be directly calculated to be equal to

$$R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1}\right)=\left(\frac{\mu}{x_2}-\frac{\partial\mu}{\partial x_2}\right)\frac{\partial}{\partial x_2},$$
$$R\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_2}\right)=0.$$

Therefore, $R=0$ if and only if $\frac{\mu}{x_2}-\frac{\partial\mu}{\partial x_2}=0$. The solution of this first order differential equation is that $\mu=x_2\nu(x_1)$, where $\nu(x_1)$ is an arbitrary smooth function on $\mathbb{R}$. $\square$

In this section, we restrict ourselves to the case that the connection (12) is flat, which means that $\mu(x_1,x_2)=x_2\nu(x_1)$. We consider the deformation quantization of $(\mathbb{R}\times\mathbb{R}^+, dx_1\wedge dx_2)$ using this connection.

The Christoffel symbols of the connection $\nabla^{\tilde{\mathcal{O}}}$ are calculated as follows,

$$\Gamma_{11}^1=\Gamma_{12}^2=\Gamma_{21}^2=\Gamma_{22}^1=0,\qquad \Gamma_{11}^2=\mu,\qquad \Gamma_{12}^1=\Gamma_{21}^1=\frac{1}{2x_2},\qquad \Gamma_{22}^2=-\frac{1}{2x_2}.$$

Taking the formula (5.1.8) in [6] with the same notations, we have

$$\begin{aligned}
&\Gamma_{111}=\omega_{11}\Gamma_{11}^1+\omega_{12}\Gamma_{11}^2=\omega_{12}\mu, &&\Gamma_{211}=\omega_{21}\Gamma_{11}^1+\omega_{22}\Gamma_{11}^2=0,\\
&\Gamma_{112}=\omega_{11}\Gamma_{12}^1+\omega_{12}\Gamma_{12}^2=0, &&\Gamma_{121}=\omega_{11}\Gamma_{21}^1+\omega_{12}\Gamma_{21}^2=0,\\
&\Gamma_{212}=\omega_{21}\Gamma_{12}^1+\omega_{22}\Gamma_{12}^2=\frac{1}{2x_2}\omega_{21}, &&\Gamma_{221}=\omega_{21}\Gamma_{21}^1+\omega_{22}\Gamma_{21}^2=\frac{1}{2x_2}\omega_{21},\\
&\Gamma_{122}=\omega_{11}\Gamma_{22}^1+\omega_{12}\Gamma_{22}^2=-\frac{1}{2x_2}\omega_{12}, &&\Gamma_{222}=\omega_{21}\Gamma_{22}^1+\omega_{22}\Gamma_{22}^2=0.
\end{aligned}$$

We have the following expression for Γ, $\Gamma\circ a$, $a\circ\Gamma$, and $[\Gamma,a]$.

$$\Gamma=\frac{1}{2}\omega_{21}\left\{\left[-\mu\left(u^1\right)^2+\frac{1}{2}\left(2u^2\right)^2\right]dx_1+\frac{1}{2}2u^1u^2\,dx_2\right\},$$

and

$$\begin{aligned}
\Gamma\circ a=\Gamma a+\left(\frac{-ih}{2}\right)\frac{1}{1!}\bigg[&\omega^{12}\left(\frac{\omega_{21}}{2}\left(-\mu 2u^1\,dx_1+\frac{1}{2x_2}2u^2\,dx_2\right)\right)\sum a_{m,n}\left(u^1\right)^m n\left(u^2\right)^{n-1}\\
&+\omega^{21}\frac{\omega_{21}}{2}\frac{1}{2x_2}\left(2u^2\,dx_1+2u^1\,dx_2\right)\sum a_{m,n}m\left(u^1\right)^{m-1}\left(u^2\right)^n\bigg],
\end{aligned}$$

and

$$\begin{aligned}
\frac{i}{h}[\Gamma,a]=\sum&\left(\frac{1}{2}(-\mu)2a_{m,n}\left(u^1\right)^m n\left(u^2\right)^{n-1}-\frac{1}{4x_2}2a_{m,n}m\left(u^1\right)^{m-1}\left(u^2\right)^{n+1}\right)dx_1\\
&+\frac{1}{4x_2}\left(2a_{m,n}\left(u^1\right)^m n\left(u^2\right)^n-2a_{m,n}m\left(u^1\right)^m\left(u^2\right)^n\right)dx_2.
\end{aligned}$$

It is a direct check that when $\mu=x_1\nu(x_2)$, ∇^2 and D^2 are both 0. By Theorem 2.1, for each $f\in C^\infty(\mathbb{R}\times\mathbb{R}_+)[[\hbar]]$, there is a unique solution of the equation $Da=0$ with $a_{0,0}=f$. In the following, we calculate the explicit expression of a.

The expression of Da is calculated as follows.

$$\begin{aligned}
Da &= \partial a - \delta a = -\delta a + da + \frac{i}{h}[\Gamma, a] \\
&= -\sum a_{m,n} m\left(u^1\right)^{m-1}\left(u^2\right)^n\, dx_1 - \sum a_{m,n}\left(u^1\right)^m n\left(u^2\right)^{n-1}\, dx_2 \\
&\quad + \sum \frac{\partial a_{m,n}}{\partial x_1}\left(u^1\right)^m\left(u^2\right)^n\, dx_1 + \sum \frac{\partial a_{m,n}}{\partial x_2}\left(u^1\right)^m\left(u^2\right)^n\, dx_2 \\
&\quad + \left[-\mu \sum a_{m,n} n\left(u^1\right)^{m+1}\left(u^2\right)^{n-1} - \sum \frac{a_{m,n}}{2x_2} m\left(u^1\right)^{m-1}\left(u^2\right)^{n+1}\right] dx_1 \\
&\quad + \sum \frac{a_{m,n}}{2x_2}(n-m)\left(u^1\right)^m\left(u^2\right)^n\, dx_2.
\end{aligned}$$

The equation $Da = 0$ gives the following system of differential equations:

$$-a_{m+1,n}(m+1) + \frac{\partial a_{m,n}}{\partial x_1} - (n+1)\mu a_{m-1,n+1} - \frac{a_{m+1,n-1}}{2x_2}(m+1) = 0,$$

and

$$-a_{m,n+1}(n+1) + \frac{\partial a_{m,n}}{\partial x_2} + \frac{a_{m,n}}{2x_2}(n-m) = 0.$$

Given $a_{0,0} = f$, we solve the system of equations by induction.

$$a_{m,0} = \frac{1}{m}\left(\frac{\partial a_{m-1,0}}{\partial x_1} - \mu a_{m-2,1}\right) = \frac{1}{m}\left(\frac{\partial a_{m-1,0}}{\partial x_1} - \mu\left(\frac{\partial}{\partial x_2} - \frac{m-2}{2x_2}\right) a_{m-2,0}\right),$$
$$a_{m,n} = \frac{1}{n!}\left(\frac{\partial}{\partial x_2} - \frac{m}{2x_2}\right)\cdots\left(\frac{\partial}{\partial x_2} + \frac{n-m-1}{2x_2}\right) a_{m,0}.$$

If we set

$$X = \frac{1}{x_2}\frac{\partial}{\partial x_1},$$
$$Y = -x_2\frac{\partial}{\partial x_2},$$

it is a direct check that

$$A_{m+1} = -X A_m - m\frac{\mu}{x_2^3}\left(Y - \frac{m-1}{2}\right) A_{m-1},$$
$$B_{m+1} = X B_m - m\frac{\mu}{x_2^3}\left(Y - \frac{m-1}{2}\right) B_{m-1},$$
$$a_{m,n} = \frac{x_2^{m-n}}{n!}\frac{A_m}{m!}(2Y+m)\cdots(2Y+m+n-1)a,$$
$$b_{n,m} = \frac{x_2^{n-m}}{m!}\frac{B_n}{n!}(2Y+n)\cdots(2Y+m+n-1)b.$$

The above expression of A_m, B_m is exactly identical to the recurrence relation as described in (2.9) of [3] of Connes and Moscovici with $S(X) = -X$, and $\Omega = \frac{\mu}{x_2^3} = \frac{\nu}{x_2^2}$. The star product constructed in this way defines the Zagier's deformation [11] for h_1 constructed from Rankin–Cohen brackets on modular forms with a fourth degree element.

Remark 5.2. For computation reasons, we have chosen that a special form of connections defined by Eq. (12), which is flat. Because of the flatness, the calculation is quite simple and transparent. When the connection is not flat, Fedosov's construction still works, but the calculation is much more complicated. However, the star product should be able to be expressed by the same formula.

Remark 5.3. As explained in Remark 4.3, the connection and the star product discussed in this subsection are both on the cotangent bundle $T^*\mathbb{R}$. However, all these constructions can be pulled back to the frame bundle by τ (see Remark 4.3) without any difficulty.

5.2. *Full injectivity*

We have shown in the last subsection that the deformation quantization of the standard symplectic structure on the upper half plane using the connection (12) with $\mu(x_1, x_2) = x_2\nu(x_1)$ gives rise to Zagier's deformation formula on modular forms. To generalize this deformation to a universal deformation formula of a projective $\mathcal{H}_1$ action, we adapt the method used by Connes and Moscovici [5, Section 3] to our situation. We briefly recall their construction in the following, and refer to [5] for the detail.

Firstly, we introduce a free abelian algebra P with a set of generators indexed by $\mathbb{Z}_{\geqslant 0}$, $Z_0, Z_1, \dots, Z_n, \dots$. On P, we define a $\mathcal{H}_1$ action as follows,

$$Y(Z_j) \stackrel{\text{def}}{=} (j+2)Z_j, \quad X(Z_j) \stackrel{\text{def}}{=} Z_{j+1}, \quad \delta_k(p) = 0, \quad \forall p \in P,\ j \geqslant 0.$$

Secondly, we consider the crossed product algebra $\tilde{\mathcal{H}}_1 \stackrel{\text{def}}{=} P \rtimes \mathcal{H}_1 \ltimes P$, which is equal to $P \otimes \mathcal{H}_1 \otimes P$ as a vector space. Denote this algebra by $\tilde{\mathcal{H}}_1$. Connes and Moscovici defines on $\tilde{\mathcal{H}}_1$ an Hopf algebra structure over P, with $\alpha, \beta : P \to \tilde{\mathcal{H}}_1$ defined by

$$\alpha(p) = p \rtimes 1 \ltimes 1, \quad \beta(q) = 1 \rtimes 1 \ltimes q, \quad \forall p, q \in P.$$

Thirdly, to deal with the projective structure, we define $\tilde{\delta}_2' \stackrel{\text{def}}{=} \delta_2 - \frac{1}{2}\delta_2 - \alpha(Z_0) + \beta(Z_0)$, $\tilde{\mathcal{H}}_s$ as the quotient of $\tilde{\mathcal{H}}_1$ by the ideal generated by $\tilde{\delta}_2'$. $\tilde{\mathcal{H}}_s$ is still a Hopf algebra over P because $\Delta(\tilde{\delta}_2') = \tilde{\delta}_2' \otimes 1 + 1 \otimes \tilde{\delta}_2'$.

Fixing a function $\mu(x_1, x_2)$, we consider a pseudogroup Γ action on $\mathbb{R}$ whose lifting onto $T^*\mathbb{R}$ preserves the connection ∇ (12) defined by μ. By Theorem 4.2, the $\mathcal{H}_1$ action on the corresponding groupoid algebra $\mathcal{A}_{\mu,\Gamma} \stackrel{\text{def}}{=} \Gamma \ltimes C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$ is projective with Ω defined in the proof.

We define $\rho_{\mu,\Gamma} : P \to \mathcal{A}_{\mu,\Gamma}$ by $\rho(Z_k) = X^k(\Omega)$ and make $\mathcal{A}_{\mu,\Gamma}$ into a module algebra over $\tilde{\mathcal{H}}_1|P$ by

$$\chi_{\mu,\Gamma}(p \rtimes h \rtimes q)(U_\gamma f) \stackrel{\text{def}}{=} \rho_{\mu,\Gamma}(p) h(U_\gamma f) \rho_{\mu,\Gamma}(q).$$

One easily checks that $\mathcal{A}_{\mu,\Gamma}$ becomes a module algebra over $\tilde{\mathcal{H}}_s|P$ because when the $\mathcal{H}_1$ action is projective, $\tilde{\delta}_2'$ acts as 0.

We define action $\chi^n_{\mu,\Gamma}$,

$$\chi^{(n)}_{\mu,\Gamma} : \underbrace{\tilde{\mathcal{H}}_s \otimes_P \cdots \otimes \tilde{\mathcal{H}}_s}_{n} \to \mathcal{L}(\underbrace{\mathcal{A}_{\mu,\Gamma} \otimes \cdots \mathcal{A}_{\mu,\Gamma}}_{n}, \mathcal{A}_{\mu,\Gamma})$$

by means of acting on each components, where $\mathcal{L}$ means the set of linear maps.

We fix $\mu = x_1\nu(x_1)$, and have the following proposition analogous to [5, Proposition 12].

Proposition 5.4. *For each $n \in \mathbb{N}$, $\bigcap_{\nu(x_1),\Gamma} \operatorname{Ker} \chi^{(n)}_{x_2\nu(x_1),\Gamma} = 0$.*

Proof. There is no difference between the proofs for different n. Therefore, for simplicity, we only prove the proposition for $n = 1$.

Following the proof of [5, Proposition 12], an arbitrary element of $\tilde{\mathcal{H}}_s$ can be written uniquely as a finite sum of the form

$$H = \sum_{j,k,l,m} \alpha(p_{jklm})\beta(q_{jklm})\delta_1^j X^k Y^l,$$

where $p, q \in P$.

Let $\chi_{x_2\nu(x_1),\Gamma}(H) = 0$, for arbitrary $\nu(x_1)$ and pseudogroup Γ preserving the connection defined by $x_2\nu(x_1)$. From the proof of Theorem 4.2, we know that in this case, $\Omega = x_2^2\nu(x_1)$.

If $U_\gamma f \in \mathcal{A}_{x_2\nu(x_2),\Gamma}$, then

$$\sum_{j,k,l,m} \rho_{x_1\nu(x_2),\Gamma}(p_{jklm})\gamma^*\big(\rho_{x_1\nu(x_2)}(q_{jklm})\big)\delta_1(\gamma)^j X^k Y^l(f) = 0.$$

We notice that f can be arbitrary smooth function on $\mathbb{R} \times \mathbb{R}^+$, and $X^k Y^l = x_2^{m+l}\frac{d^k}{dx_1^m}\frac{d^l}{dx_2^l}$. This implies that

$$\sum_{j,m} \rho_{x_1\nu(x_2),\Gamma}(p_{jklm})\gamma^*\big(\rho_{x_1\nu(x_2)}(q_{jklm})\big)\delta_1(\gamma)^j = 0,$$

for any l, m.

To prove the proposition, we consider the following family of algebras, $\mathcal{A}_{x_2\nu(x_2),\Gamma}$.

Fix a diffeomorphism ϕ_{O_1,O_2} from an open set $O_1 \subset \mathbb{R}$ to the other open set $O_2 \subset \mathbb{R}$, with O_1 disjoint from O_2. The disjointness between O_1 and O_2 makes the set $\Gamma_\phi \stackrel{\text{def}}{=} \{\mathrm{id}|_{\mathbb{R}},\ \mathrm{id}|_{O_1},\ \mathrm{id}|_{O_2},\ \phi,\ \phi^{-1}\}$ into a pseudogroup. Starting with any connection ∇_1 of the form (12) with $\mu = x_2\nu(x_1)$ on O_1, we first push forward this connection to O_2 by ϕ, and then extend the connection defined on O_1 and O_2 to a global connection $\tilde{\nabla}$ on $\mathbb{R} \times \mathbb{R}^+$. The extension of the connection is well defined because O_1 is disjoint from O_2, and is Γ_ϕ invariant by its definition. According to our construction, we have that $\tilde{\mathcal{H}}_s$ act on the corresponding groupoid algebra $\mathcal{A}_{\phi_{O_1,O_2},\tilde{\nabla}}$.

Now at any $x \in \mathbb{R}$, we fix O_1 containing x, and let O_2, ϕ, ∇_1 vary. It is not hard to see that if H vanishes on this family of algebra $\mathcal{A}_{\phi_{O_1,O_2},\tilde{\nabla}}$, we must have that H vanishes at x, because H

has only finite number of terms but this family of algebras has infinitely many freedoms. Hence H has to be equal to 0. $\square$

5.3. Universal deformation $\mathcal{H}_1$ with a projective structure

We consider the groupoid algebra $\mathcal{A}_{x_2\nu(x_1),\Gamma}$. Because the connection defined by $x_1\nu(x_1)$ in (12) is Γ invariant, the results in Section 2.3 implies that the symplectic form $\frac{dx\wedge dy}{y^2}$ on $\mathbb{R}\times\mathbb{R}_+$, which is invariant under any Γ, defines a noncommutative Poisson structure on $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$. Furthermore, we extend this Poisson structure to a deformation of $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$. This deformation can be realized by the crossed product of the star product constructed in Section 5.1 with Γ.

In Section 5.1, the $\star$ product is expressed as follows: for $f,g\in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$,

$$f\star g=\sum_{n=0}^{\infty}\hbar^n\sum_{k=0}^{n}\frac{A_k}{k!}(2Y+k)_{n-k}(a)\frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k(b),$$

$$A_{m+1}=XA_m-mx_2\mu\left(Y-\frac{m-1}{2}\right)A_{m-1}=XA_m-m\Omega\left(Y-\frac{m-1}{2}\right)A_{m-1},$$

$$B_{m+1}=XB_m-mx_2\mu\left(Y-\frac{m-1}{2}\right)B_{m-1}=XB_m-m\Omega\left(Y-\frac{m-1}{2}\right)B_{m-1}.$$

The crossed product of $\star$ with Γ is written as $U_\gamma f_\gamma * U_\beta g_\beta \overset{\text{def}}{=} U_{\gamma\beta}\beta^*(f_\gamma)\star g_\beta$ defines a deformation quantization of $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$.

According to the formulas of $\star$ and the Γ crossed product, the deformed product $*$ on $\Gamma\ltimes C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$ can be expressed by $\tilde{\mathcal{H}}_s$ as follows,

$$RC=\sum_{n=0}^{\infty}\hbar^n\sum_{k=0}^{n}\frac{A_k}{k!}(2Y+k)_{n-k}\otimes\frac{B_{n-k}}{(n-k)!}(2Y+n-k)_k,$$

$$A_{m+1}=S(X)A_m-m\Omega^0\left(Y-\frac{m-1}{2}\right)A_{m-1},$$

$$B_{m+1}=XB_m-m\Omega\left(Y-\frac{m-1}{2}\right)B_{m-1},$$

where Ω^0 is the right multiplication of Ω.

By Proposition 5.4, we conclude RC can be pulled back to $\tilde{\mathcal{H}}_s$ and defines an associative universal deformation for any projective $\mathcal{H}_1$ actions.

6. Deformation without projective structures—Noncommutative Poisson structure

In the above deformation (4), we have assumed the action to be projective. One can ask whether one can go beyond this. Recently, a construction of Bressler, Gorokhovsky, Nest, and Tsygan [2] strongly suggests that this general RC deformation may still exist.

In this section, we look at the first order approximation of the general deformation. We prove that RC_1 generally defines a noncommutative Poisson structure without any assumptions.

Proposition 6.1. *For an $\mathcal{H}_1$ action on an A, $RC_1 = -X \otimes 2Y + 2Y \otimes X + \delta_1 Y \otimes 2Y$ defines a noncommutative Poisson structure on A.*

Proof. The proof of this proposition is a calculation. We need to find an element B in $\mathcal{H}_1 \otimes \mathcal{H}_1$, such that for any $a, b, c \in A$,

$$aB(b,c) - B(ab,c) + B(a,bc) - B(a,b)c = RC_1\big(RC_1(a,b),c\big) - RC_1\big(a, RC_1(b,c)\big).$$

In order to find such a B, we first look at the special case where the Hopf algebra action is projective. In this case, the associativity of the Connes–Moscovici's universal deformation formula of $\mathcal{H}_1$ implies that RC_2 is a right choice of B.

For a general $\mathcal{H}_1$ action, we first look at the following term

$$\begin{aligned} B' &= S(X)^2 \otimes Y(2Y+1) + S(X)(2Y+1) \otimes X(2Y+1) + Y(2Y+1) \otimes X^2 \\ &\quad + 2\delta_2' Y \otimes Y^2 + \delta_2' Y \otimes Y. \end{aligned}$$

We calculate the difference between the Hochschild coboundary of B' and $[RC_1, RC_1]$.

$$\begin{aligned}
&\big(b(B') - [RC_1, RC_1]\big)(a,b,c) \\
&= 4\delta_2' YaYbYc + 4Ya\delta_2' YbYc + 2Y^2 a\delta_2' bYc - 2\delta_2' aYbY^2c + Ya\delta_2' bYc - \delta_2' aYbYc \\
&= -2\big[a\delta_2' Y^2 bYc - \delta_2' Y^2(ab)Yc + \delta_2' Y^2 aY(bc) - \delta_2' Y^2 a(Yb)c\big] - 2\delta_2' aY^2 bYc \\
&\quad - 2\delta_2' aYbY^2c + Ya\delta_2' bYc - \delta_2' aYbYc \\
&= -2\big[a\delta_2' Y^2 bYc - \delta_2' Y^2(ab)Yc + \delta_2' Y^2 aY(bc) - \delta_2' Y^2 a(Yb)c\big] \\
&\quad - \frac{2}{3}\big[a\delta_2' bY^3c - \delta_2'(ab)Y^3c + \delta_2' aY^3(bc) - \delta_2' a\big(Y^3b\big)c\big] + Ya\delta_2' bYc - \delta_2' aYbYc \\
&= -2\big[a\delta_2' Y^2 bYc - \delta_2' Y^2(ab)Yc + \delta_2' Y^2 aY(bc) - \delta_2' Y^2 a(Yb)c\big] \\
&\quad - \frac{2}{3}\big[a\delta_2' bY^3c - \delta_2'(ab)Y^3c + \delta_2' aY^3(bc) - \delta_2' a\big(Y^3b\big)c\big] \\
&\quad - \big[a\delta_2' YbYc - \delta_2' Y(ab)Yc + \delta_2' YaY(bc) - \delta_2' Ya(Yb)c\big] \\
&\quad - 2\delta_2' aYbYc \\
&= -2\big[a\delta_2' Y^2 bYc - \delta_2' Y^2(ab)Yc + \delta_2' Y^2 aY(bc) - \delta_2' Y^2 a(Yb)c\big] \\
&\quad - \frac{2}{3}\big[a\delta_2' bY^3c - \delta_2'(ab)Y^3c + \delta_2' aY^3(bc) - \delta_2' a\big(Y^3b\big)c\big] \\
&\quad - \big[a\delta_2' YbYc - \delta_2' Y(ab)Yc + \delta_2' YaY(bc) - \delta_2' Ya(Yb)c\big] \\
&\quad - \big[a\delta_2' bY^2c - \delta_2'(ab)Y^2c + \delta_2' aY^2(bc) - \delta_2' aY^2bc\big],
\end{aligned}$$

where $b(B')$ is the Hochschild coboundary of B' and $\delta_2' = \delta_2 - \frac{1}{2}\delta_1^2$.

It is straightforward to check the following identities.

$$b\big(\delta_2' Y^2 \otimes Y\big)(a,b,c) = a\delta_2' Y^2 bYc - \delta_2' Y^2 (ab)Yc + \delta_2' Y^2 aY(bc) - \delta_2' Y^2 a(Yb)c,$$
$$b\big(\delta_2' \otimes Y^3\big)(a,b,c) = a\delta_2' bY^3 c - \delta_2'(ab)Y^3 c + \delta_2' aY^3(bc) - \delta_2' a\big(Y^3 b\big)c,$$
$$b\big(\delta_2' Y \otimes Y\big)(a,b,c) = a\delta_2' YbYc - \delta_2' Y(ab)Yc + \delta_2' YaY(bc) - \delta_2' Ya(Yb)c,$$
$$b\big(\delta_2' \otimes Y^2\big)(a,b,c) = a\delta_2' bY^2 c - \delta_2'(ab)Y^2 c + \delta_2' aY^2(bc) - \delta_2' aY^2 bc.$$

Therefore, the calculation suggests the introduction of $B'' = +2\delta_2' Y^2 \otimes Y + \frac{2}{3}\delta_2' \otimes Y^3 + \delta_2' Y \otimes Y + \delta_2' \otimes Y^2$ and $B = B' + B''$. And we have $b(B) = b(B' + B'') = [RC_1, RC_1]$. $\square$

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