Stability for the $p$-energy of the equator map of the ball into an ellipsoid

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Abstract: For $m \geq 3$ and $b > 0$, let $B^m = \{ x \in \mathbb{R}^m : |x| \leq 1 \}$ and $E^m(b) = \{(w, y) \in \mathbb{R}^m \times \mathbb{R} : |m|^2 + y^2/b^2 = 1 \}$. Set $r = |x|$: the equator map $u^*(x) = (x/r, 0)$ belongs to $H^{1,p}(B^m, E^m(h))$ and is weakly $p$-harmonic if and only if $p < m$. We study the stability of this map for $2 \leq p < m$; in particular, we establish a condition on $p$ and $b$ which is necessary and sufficient for the stability of $u^*$.

Keywords: $p$-harmonic maps, equator map, stability.

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0. Introduction

Let $(M, g)$ and $(N, h)$ be connected Riemannian manifolds of dimension $m$ and $n$ respectively, and suppose that $M$ is compact with boundary. By the theorem of Nash, we can suppose that the target manifold $N$ is embedded in $\mathbb{R}^{n+k}$. For $p \geq 2$, we consider the Sobolev space (see [5,8])

$$H^{1,p}(M, N) = \{ u \in H^{1,p}(M, \mathbb{R}^{n+k}) : u(x) \in N \text{ almost everywhere } \}. \quad (0.1)$$

The $p$-energy of $u$ is defined by

$$E_p(u) = \frac{1}{p} \int_M |Du|^p \, dv_g. \quad (0.2)$$

Let $A$ be the second fundamental form of $N$ in $\mathbb{R}^{n+k}$. We say that $u$ is weakly $p$-harmonic if it is a weak solution of the Euler–Lagrange equation of the functional (0.2), that is

$$- \text{div}_g(|Du|^{p-2} Du) + |Du|^{p-2} A(u)(Du, Du) = 0. \quad (0.3)$$

The case $p = 2$ corresponds to the usual harmonic maps. If $p > 2$, the problem of the regularity of weakly $p$-harmonic maps is difficult in general and is the subject of several papers (for example, see [3,4,8] and references therein). On the other hand, not much is known about the second variation of the energy of $p$-harmonic maps (see [3,6,7]). Let $\bar{v}, \bar{w}$ be two vector fields along $u$, such that $\bar{v}, \bar{w} \in H^{1,p} \cap L^\infty(M, \mathbb{R}^{n+k})$. If $u$ is weakly $p$-harmonic, the second variation of the $p$-energy in the direction of the vector fields $\bar{v}$ and $\bar{w}$ is given by

$$H_p(\bar{v}; \bar{w}) = \int_M |Du|^{(p-2)/2} \{ (\nabla^u \bar{v}, \nabla^u \bar{w}) - \text{Trace}(R^N(Du, \bar{v})Du, \bar{w}) \} \, dv_g + (p - 2) \int_M |Du|^{(p-4)/2} \{ (\nabla^u \bar{v}, Du)\langle \nabla^u \bar{w}, Du \rangle \} \, dv_g. \quad (0.4)$$

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where $\nabla^u$ denotes the covariant differential along $u$ and $R^N$ is the curvature tensor of $N$. We say that the map $u$ is strictly stable if $H_p(\tilde{v}; \tilde{v}) \geq 0$ for all vector fields $\tilde{v}$ along $u$, and equality holds if and only if $\tilde{v} = 0$. The map $u$ is unstable if there exists a vector field $\tilde{v}$ such that $H_p(\tilde{v}; \tilde{v}) < 0$.

The aim of this paper is to study the stability of the equator map of the ball into an ellipsoid; more precisely, for $m \geq 3$ and $b > 0$, let

\[
B^m = \{ x \in \mathbb{R}^m : |x| \leq 1 \},
E^m(b) = \{ (w, y) \in \mathbb{R}^m \times \mathbb{R} : |w|^2 + y^2/b^2 = 1 \}.
\]

The equator map $u^*$ is defined by

\[
u^* : B^m \to E^m(b),

x \to (x/r, 0), \quad \text{where } r = |x|.
\]

In particular $u^*$ belongs to $H^{1,p}(B^m, E^m(b))$ and is weakly $p$-harmonic if and only if $p = m$.

Our result is

**Theorem.** Let $m \geq 3$ and $2 \leq p < m$.

1. If $b^2 < 4(m - 1)/(m - p)^2$, then the equator map $u^*$ is unstable.
2. If $b^2 \geq 4(m - 1)/(m - p)^2$, then the equator map $u^*$ is strictly stable.

**Remark.** Baldes [1] proved the theorem in the case $p = 2$, and Duzaar [3] in the case $2 \leq p < m$, $b = 1$. We note that the method of [3] does not apply if $b \neq 1$. Moreover, it is interesting to note that if $b^2 \geq 4(m - 1)/(m - p)^2$ then $u^*$ is the absolute minimum over the class of equivariant maps which coincide with $u^*$ on $\partial B^m$.

1. Proof of the theorem

If $\tilde{v} \in \dot{H}^{1,p} \cap L^\infty(B^m, \mathbb{R}^{m+1})$ is a vector field along $u^*$, we decompose $\tilde{v} = (v, \eta)$, where $v$ takes its values on $\mathbb{R}^m$ and $\eta$ on $\mathbb{R}$ and $(v(x), x) = 0$ almost everywhere on $B^m$. A short calculation gives

\[
H_p(\tilde{v}; \tilde{v}) = (m - 1)^{(p-2)/2} \{ K(\eta) + H(v) \}
\]

where

\[
K(\eta) = \int_{B^m} r^{2-p} \left[ |D\eta|^2 - \frac{m-1}{b^2 r^2} \eta^2 \right] \, dx \quad (1.2)
\]

and

\[
H(v) = \int_{B^m} r^{2-p} \left[ |Dv|^2 - \frac{m-1}{r^2} |v|^2 + \frac{p-2}{m-1} (\text{div } v)^2 \right] \, dx. \quad (1.3)
\]

To prove part (1) of the theorem, let

\[
v = \frac{(m - p)^2}{4} - \frac{m - 1}{b^2} + \epsilon,
\]
where $\epsilon$ is small enough as to have $\nu < 0$. Next, we take $\tilde{v} = (0; \eta(r))$ where
\[
\eta(r) = \begin{cases} 
(r^{p-m})^{1/2} \sin(\sqrt{-\nu} \log r) & \text{if } r_0 \leq r \leq 1, \\
0 & \text{if } 0 \leq r \leq r_0, \quad r_0 = \exp(-\pi/\sqrt{-\nu}) < 1. 
\end{cases}
\]

A short calculation gives
\[
\frac{d}{dr} [r^{m-p+1} \eta(r)] = r^{m-p+1} \eta(r) \frac{\epsilon - m - 1}{b^2}.
\]

Next, an integration by parts leads us to the equality
\[
K(\eta) = -\frac{\epsilon}{b^2} \text{vol}(S^{m-1}) \int_{r_0}^{1} \left[ \frac{\eta^2(r)}{r^2} r^{m-p+1} \right] dr < 0,
\]
so ending the proof of (1).

To prove part (2), we will need two lemmas which are extensions of the results of [3] and [1] respectively.

**Lemma 1.** If $b^2 \geq 4(m-1)/(m-p)^2$ and $\eta \neq 0$, $\eta \in \tilde{H}^{1,p} \cap L^\infty(B^m, \mathbb{R})$, then $K(\eta) > 0$.

**Proof of Lemma 1.** We introduce polar coordinates $r, \theta$ on $B^m$. After an integration by parts, we get
\[
\int_{B^m} r^{-p} \eta^2(x) \, dx = -\frac{2}{m-p} \int_{B^m} r^{1-p} \eta(x) \frac{\partial \eta}{\partial r}(x) \, dx.
\]

Next, we use the following Young's inequality
\[
a_1 b_1 \leq \frac{a_1^2}{2} + \frac{b_1^2}{2}
\]
where $a_1 = \frac{\partial \eta}{\partial r} r^{-1/p} -1$ and $b_1 = \sqrt{1/2} \eta r^{-p/2}$

and we get
\[
\int_{B^m} r^{-p} \eta^2(x) \, dx \leq \frac{4}{(m-p)^2} \int_{B^m} r^{2-p} \left( \frac{\partial \eta}{\partial r} \right)^2 (x) \, dx.
\]

If we replace (1.5) into (1.2), we conclude that
\[
K(\eta) \geq \left[ 1 - \frac{4(m-1)}{b^2(m-p)^2} \right] \int_{B^m} r^{2-p} \left( \frac{\partial \eta}{\partial r} \right)^2 (x) \, dx \geq 0
\]
and $K(\eta) = 0$ if and only if equality holds in (1.4), a fact which implies that $\eta \equiv 0$.

**Lemma 2.** Set $\tilde{B}^m = \overline{B}^m - \{0\}$. For all $v \in \tilde{H}^{1,p} \cap C^\infty(\tilde{B}^m, \mathbb{R}^m)$ such that $\langle v(x), x \rangle = 0$ almost everywhere, there exists a sequence $v_k \in \tilde{H}^{1,p} \cap C^\infty(\tilde{B}^m, \mathbb{R}^m)$ such that $\langle v_k(x), x \rangle = 0$ on $\tilde{B}^m$ and $H(v_k) \to H(v)$ when $k \to +\infty$.
Proof of Lemma 2. We choose a sequence \( w_k \in C_c^\infty (B^m, \mathbb{R}^m) \) such that \( w_k \to v \) in \( H^{1,p} \). We set \( u_k = w_k - \langle w_k, x \rangle x/r^2; \) so \( u_k \in C_c^\infty (\tilde{B}^m, \mathbb{R}^m) \) and \( \langle u_k, x \rangle = 0 \) on \( \tilde{B}^m \). Moreover, we prove that \( u_k \in H^{1,p} \). We set \( u_k = v - w_k \); since \( \langle v, x \rangle = 0 \), we have \( v - u_k = u_k - \langle u_k, x \rangle x/r^2 \).

Now, we note that

\[
H(u_k) = H_1(u_k) - (m - 1)H_2(u_k) + \frac{p - 2}{m - 1}H_3(u_k)
\]

where

\[
H_1(u_k) = \int_{B^m} r^{2-p}|Du_k|^2 \, dx,
\]

\[
H_2(u_k) = \int_{B^m} r^{2-p} \frac{|u_k|^2}{r^2} \, dx,
\]

\[
H_3(u_k) = \int_{B^m} (\text{div}(u_k))^2 r^{2-p} \, dx.
\]

To end the proof, it is sufficient to prove that \( H_i(u_k) \to H_i(v) \) for \( i = 1, 2, 3 \). A short calculation shows that there exists \( C > 0 \) such that

\[
\left| \frac{d}{dx^i}[v - u_k] \right|^2 \leq C \left\{ \left| \frac{du_k}{dx^i} \right|^2 + \frac{|u_k|^2}{r^2} \right\}
\]

(1.6)

Next, there exist two constants \( C_1, C_2 > 0 \) such that

\[
H_i(v - u_k) \leq C_1 H_1(u_k) + C_2 H_2(u_k).
\]

(1.7)

Using Holder’s inequality, it follows that there exists a constant \( C_3 > 0 \) such that

\[
H_1(u_k) \leq C_3 |u_k|^2_{1,p}
\]

(1.8)

Moreover, an argument similar to Lemma 1, applied to the components of \( u_k \), shows that

\[
H_2(u_k) \leq \frac{4}{(m - p)^2} H_1(u_k)
\]

(1.9)

Now, using (1.8) and (1.9) in (1.7), we deduce that \( H_1(u_k) \to H_1(v) \). Next, by replacing \( u_k \) by \( v_k - v \) in (1.9), we get \( H_2(u_k) \to H_2(v) \).

Finally, since \( (\text{div}(v))^2 \leq m |Dv|^2 \), we have the following inequality

\[
|(\text{div}(v))^2 - (\text{div}(v_k))^2| \leq m |Dv - u_k||D(v + u_k)|.
\]

(1.10)

Now, thanks to (1.10) and to Schwartz’s inequality, there exists a constant \( C_4 > 0 \) such that

\[
\int_{B^m} |(\text{div}(v))^2 - (\text{div}(v_k))^2| r^{2-p} \, dx \leq C_4 [H_1(u_k - v)]^{1/2} [H_1(u_k + v)]^{1/2}.
\]

(1.11)

For \( k \) arbitrarily large, there exist two constants \( C_5, C_6 > 0 \) (independent of \( k \)) such that

\[
H_1(u_k + v) \leq C_5 [H_1(v_k) + H_1(v)] \leq C_6. \]

Then, by passing to the limit when \( k \to +\infty \), we get \( H_3(u_k) \to H_3(v) \), so ending the proof of Lemma 2. \( \Box \)
Let $S^{m-1}(r) = \{ x \in \mathbb{R}^m : |x| = r \}$ and $S^{m-1}(1) = S^{m-1}$. We denote by $D_\theta$ and $\text{div}_\theta$, the gradient and the divergence on $S^{m-1}$ respectively. We have

$$|Dv|^2 = \left| \frac{\partial v}{\partial r} \right|^2 + \frac{1}{r^2} |D_\theta v|^2$$

and

$$\text{div}(v) = \frac{1}{r} \text{div}_\theta(v)$$

because $(v(x), x) = 0$ almost everywhere on $B^m$. We identify $v(x)$ on $S^{m-1}(r)$ and $v(rx)$ on $S^{m-1}$.

For all $v \in \dot{H}^{1,p} \cap C^\infty(B^m, \mathbb{R}^m)$ with $(v(x), x) = 0$, the quadratic form $H(v)$ in (1.3) becomes

$$H(v) = \int_{S^{m-1}} \left( |D_\theta v|^2 - (m - 1)|v|^2 + \frac{p - 2}{m - 1} (\text{div}_\theta(v))^2 \right) d\theta + \int_0^1 \frac{r^{m-p+1}}{r^2} T(v) dr$$

where

$$T(v) = \int_{S^{m-1}} \left\{ |D_\theta v|^2 - (m - 1)|v|^2 + \frac{p - 2}{m - 1} (\text{div}_\theta(v))^2 \right\} d\theta$$

and $J_p^{id}$ is the Jacobi operator of the identity map of $S^{m-1}$ for the $p$-energy (see [7]). El Soufi--Jeune [7] studied the spectrum of the operator $J_p^{id}$: if $m > p + 1$ the only negative eigenvalue is $\lambda = p - m + 1$. So, if $m > p + 1$ we get the inequality

$$T(v_k) \geq \int_{S^{m-1}} (p - m + 1)|v_k|^2 d\theta$$

for all $v_k \in \dot{H}^{1,p} \cap C^\infty(B^m, \mathbb{R}^m)$ with $(v_k(x), x) = 0$ on $B^m$.

Using (1.14) in (1.13), we have

$$H(v_k) \geq \int_{B^m} r^{2-p} \left\{ \left| \frac{\partial v_k}{\partial r} \right|^2 + (p - m + 1) \frac{|v_k|^2}{r^2} \right\} dx.$$
Then, if $m \neq p + 2$ we have $H(v) > 0$. And, if $m - p = 2$ we have $H(v) \geq 0$ by (1.18). Moreover, passing to the limit in (1.15) and (1.16), $H(v) = 0$ implies that

$$
\int_{B^n} r^{2-p} \frac{|v|^2}{r^2} \, dx \geq \int_{B^n} r^{2-p} \left| \frac{\partial v}{\partial r} \right|^2 \, dx \quad \text{and} \quad \int_{B^n} r^{2-p} \frac{|v|^2}{r^2} \, dx \leq \int_{B^n} r^{2-p} \left| \frac{\partial v}{\partial r} \right|^2 \, dx.
$$

By an application of Young’s inequality, equality holds if $v \equiv 0$. We conclude that if $m - p > 1$ then $H(v) > 0$ for all $v \in H^{1,p} \cap L^\infty(B^n, \mathbb{R}^m)$ such that $\langle v(x), x \rangle = 0$ almost everywhere $v \neq 0$. Now, if $m - 1 \leq p < m$ the identity map of $S^{m-1}$ is stable (see[7]) so $T(v_k) \geq 0$. Then, we get the inequality

$$
H(v_k) \geq \int_{B^n} r^{2-p} \left| \frac{\partial v_k}{\partial r} \right|^2 \, dx. \tag{1.19}
$$

Next, letting $k \to +\infty$ in (1.19) yields

$$
H(v) \geq \int_{B^n} r^{2-p} \left| \frac{\partial v}{\partial r} \right|^2 \, dx \geq 0
$$

and $H(v) = 0$ if and only if $v \equiv 0$. So, the map $u^*$ is strictly stable. This completes the proof of the theorem.

References