Wolfe-Type Duality Involving (B, η)-Invex Functions
For a Minmax Programming Problem

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A sufficient optimality theorem is proved for a certain minmax programming problem under the assumptions of proper (B, η)-invexity conditions on the functions involved in the objective and in the constraints. Next a dual is presented for such a problem and duality theorems relating the primal and the dual are proved. The dual for a minimax (generalized) fractional programming problem is presented as special case of the main problem considered in the paper.

1. INTRODUCTION

Fractional programming involving the optimization of single ratio duality has been studied extensively (e.g., see [3, 23]). During the last 10 years or so some results have been obtained for vector valued fractional programming [4, 11–13, 15], and minmax fractional programming involving several ratios in the objective function [5–7, 14, 16, 17, 19, 25] (such a problem is named generalized fractional programming problem [17]). Various approaches are developed by different researchers for arriving at a variety of dual problems for such problems. These duals, however, are related to each other in one or the other sense. Bector [1, 2] introduced the concept of strong pseudo-convex function and used it to establish the nature of quotients, products, rational powers, and compositions of convex-like functions [1, 2]. Bector and Singh in [9] later renamed strong pseudo-convex functions as B-vex functions by and discussed various properties of such functions. Further properties of such functions are discussed in [22]. Hanson [18] introduced the concept of invex functions and showed that an
appropriately defined optimization problem containing invex functions satisfy Karush–Kuhn–Tucker optimality conditions [21].

Recently, Bector et al. [8, 10] unified the concept of B-vex functions and invex functions, naming such functions as B-invex functions. Independently, Jeyakumar and Mond [20] introduced the idea of V-invex functions which are similar to B-invex functions [8, 10]. Both B-invex functions [8, 10] and V-invex [20] functions unify the duality of vector valued fractional programs [11–13, 15]. Since B-invex functions [8, 10] and V-invex functions [20] unify the concepts of B-vex [9] and invex functions [18]; therefore, in the present paper we name them as properly (B, η)-invex functions. A useful consequence of proper (B, η)-invexity is that duality for pseudolinear multiobjective problems and certain nonlinear multiobjective fractional programming problems does not require a separate treatment and all results on optimality conditions and duality can be derived by using the general concept of proper (B, η)-invexity. The purpose of this paper is to use the notion of proper (B, η)-invexity to establish a sufficient optimality theorem and duality results for a class of minmax programming problems, and thereby show that the duality results for a certain class of generalized fractional programming can be derived as a special case.

2. PRELIMINARIES AND MAIN PROBLEM

In this section we provide some definitions, the main problem considered in the paper, and some results that we shall use in the sequel. The following definitions are due to Hanson [6].

**Definition 2.1.** A differentiable function $f_i : S \to R$ is said to be η-invex (η-incave) if there exist functions $\eta : S \times S \to R^n$ such that for each $x, u \in S$,

$$[f_i(x) - f_i(u)] \geq (\leq) \eta(x, u) \nabla f_i(u) \quad \text{for } i = 1, 2, \ldots, p.$$

**Definition 2.2.** A differentiable function $f_i : S \to R$ is said to be strictly η-invex (strictly η-incave) if there exist functions $\eta : S \times S \to R^n$ such that for each $x, u \in S$,

$$[f_i(x) - f_i(u)] > (\prec) \eta(x, u) \nabla f_i(u) \quad \text{for } i = 1, 2, \ldots, p.$$

We now introduce the following definitions [8, 9, 21].

**Definition 2.3.** A differentiable function $f_i : S \to R$ is said to be properly (b, η)-invex if there exist functions $\eta : S \times S \to R^n$ and $b_i : S \times S \to R^+ \setminus \{0\}$ such that for each $x, u \in S$,

$$b_i(x, u)[f_i(x) - f_i(u)] \geq \eta(x, u) \nabla f_i(u) \quad \text{for } i = 1, 2, \ldots, p.$$
For \( b_i(x, u) = 1 \) the above definition reduces to the definition of \( \eta \)-in\-\-vexity [18].

**Definition 2.4.** A differentiable function \( f_i: S \rightarrow R \) is said to be properly strictly \((b_i, \eta)\)-invex if there exist functions \( \eta: S \times S \rightarrow R^n \) and \( b_i: S \times S \rightarrow R^+ \setminus \{0\} \) such that for each \( x, u \in S \),

\[
b_i(x, u) [f_i(x) - f_i(u)] > \eta(x, u) \nabla f_i(u) \quad \text{for } i = 1, 2, \ldots, p
\]

For \( b_i(x, u) = 1 \) the above definition reduces to the definition of strict \( \eta \)-invexity [18]. The following theorem, which we shall use in the sequel, is easy to prove; therefore, we state it without proof.

**Theorem 2.1.** Let \( f \) and each of \( h_{ij}, j = 1, 2, \ldots, m \) be properly \((b, \eta)\)-invex on \( R^n \). If \( \lambda_i \geq 0 \) and \( y_{ij} \geq 0, j = 1, 2, \ldots, m \), then \( \lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij} \) is properly \((b, \eta)\)-invex on \( R^n \). If \( f_i \) be properly strictly \( b_i \)-invex and \( \lambda_i > 0 \), and/or at least one of \( h_{ij} \) for which the corresponding \( y_{ij} > 0 \), be properly strictly \((b, \eta)\)-invex, then \( \lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij} \) is properly strictly \((b, \eta)\)-invex on \( R^n \).

**Primal Problem.** In the present paper we consider the following generalized minmax programming problem as the primal problem:

\[
\begin{align*}
q^* = \min_{x \in S} \max_{1 \leq i \leq p} [f_i(x)],
\end{align*}
\]

where

(A1) \( S = \{x \in X_0 \subseteq R^n; h_{ij}(x) \leq 0, i = 1, 2, \ldots, p; j = 1, 2, \ldots, m\} \) is nonempty and compact;

(A2) \( f_i, i = 1, 2, \ldots, p, \) and \( h_{ij}(x) \leq 0, i = 1, 2, \ldots, p; j = 1, 2, \ldots, m, \) are differentiable on \( R^n \);

(A3) (i) Each \( f_i, i = 1, 2, \ldots, p, \) and \( h_{ij}(x) \leq 0, i = 1, 2, \ldots, p; j = 1, 2, \ldots, m, \) is a properly \((b_i, \eta)\)-invex function on \( R^n \); or

(ii) \( \lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij} \) is properly \((b_i, \eta)\)-invex on \( R^n \) for \( i = 1, 2, \ldots, p, \) and \( \lambda_i \geq 0, y_{ij} \geq 0 \) \( \forall i = 1, 2, \ldots, p; j = 1, 2, \ldots, m \).

We now consider the following programming problem (E) which is equivalent to (P) in the sense of the Lemmas 2.2 and 2.3 given below.

\[
\begin{align*}
\min q \\
\text{subject to} \\
f_i(x) \leq q \quad &i = 1, 2, \ldots, p, \quad (2.2) \\
h_{ij}(x) \leq 0 \quad &i = 1, 2, \ldots, p; j = 1, 2, \ldots, m, \quad (2.3) \\
x \in X_0.
\end{align*}
\]
Remark 2.1. We may note here that the problem \( E \) has \( p \times m \) number of constraints.

**Lemma 2.1.** Let \( x \) be \((P)\)-feasible. Then there exists \( q \in R \) such that \((x, q)\) is \((E)\)-feasible, and if \((x, q)\) be \((E)\)-feasible then \( x \) is \((P)\)-feasible.

**Lemma 2.2.** Let \( x^* \) be \((P)\)-optimal. Then there exists \( q^* \in R \) such that \((x^*, q^*)\) is \((E)\)-optimal, and if \((x^*, q^*)\) be \((E)\)-optimal then \( x^* \) is \((P)\)-optimal with \( q^* \) as the optimal value of the \((P)\)-objective.

If \( X_0 \) is a convex set and \( f_i \) (\( i = 1, 2, \ldots, p \)) and \( h_{ij} \) (\( i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, m \)) are convex functions then \( E \) is a convex programming problem for which optimality conditions and Wolfe-type [11, 2, 3] duality results are easily derived. However, if we take \( f_i \) to be a properly \((b_i, \eta)\)-invex function on \( R^n \), as under (A3), then \( f_i(x) - q \) in (2.2) is not a properly \((b_i, \eta)\)-invex function on \( R^{n+1} \). Such a phenomenon necessitates a separate study of minmax fractional programming problems. In the present paper, it is seen that the notion of \((b_i, \eta)\)-invexity facilitates the study of minmax fractional programming in a unified manner and provides a Wolfe-type dual [11], which can be taken as a generalization of the fractional programming dual of Bector and Chandra [5] and Bector et al. [6].

### 3. Optimality Conditions

**Theorem 3.1** (Necessary optimality conditions). Let \( x^* \) be \((P)\)-optimal. Let an appropriate constraint qualification [21] for \((P)\). Then there exists \( q^* \in R, \lambda^* \in R^p, \) and a matrix \( Y^* \in R^p \times R^m \), such that \((x^*, q^*, \lambda^*, Y^*)\) satisfies

\[
\sum_{i=1}^{p} \nabla \left[ \lambda_i f_i(x) + \sum_{j=1}^{m} y_{ij} h_{ij}(x) \right] = 0
\]

\[
\lambda_i [f_i(x) - q] = 0, \quad i = 1, 2, \ldots, p
\]

\[
y_{ij} h_{ij}(x) = 0 \quad \forall i = 1, 2, \ldots, p; \ j = 1, 2, \ldots, m,
\]

\[
f_i(x) \leq q, \quad i = 1, 2, \ldots, p
\]

\[
h_{ij}(x) \leq 0, \quad j = 1, 2, \ldots, m,
\]

\[
\sum_{i=1}^{p} \lambda_i = 1
\]

\[
q \in R, \lambda \in R^p, Y \in R^{p \times m}, \quad \lambda \geq 0, \quad Y \geq 0.
\]
Proof. Since $x^*$ is $(P)$-optimal, therefore, by Lemma 2.2, there exists a $q^*$ such that $(x^*, q^*)$ is $(E)$-optimal. The theorem now follows by applying the Kuhn–Tucker theorem [21] to $(E)$ at $(x^*, q^*)$.

**Theorem 3.2 (Sufficient optimality conditions).** If $(x^*, q^*, \lambda^*, Y^*)$ satisfy (3.1)–(3.7), then $x^*$ is $(P)$-optimal.

Proof. First we prove that $(x^*, q^*)$ is $(E)$-optimal. Since $(x^*, q^*, \lambda^*, Y^*)$ satisfies (3.1), therefore, we have

$$
\eta(x, x^*) \sum_{i=1}^{p} \nabla \left[ \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right] = 0 \quad (3.8)
$$

for all $(EP)$-feasible solutions $x$.

From Theorem 2.1 and (A3), $\lambda_i f_i + \sum_{j=1}^{m} y_{ij} h_{ij}$ is properly $(b_i, \eta)$-invex on $\mathbb{R}^n$. Therefore,

$$
b_i(x, x^*) \left[ (\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - \left( \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right) \right] \geq \eta(x, x^*) \nabla \left[ \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right]. \quad (3.9)
$$

In (3.9) taking summation over $i$, we obtain

$$
\sum_{i=1}^{p} b_i(x, x^*) \left[ (\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - \left( \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right) \right] \geq \eta(x, x^*) \sum_{i=1}^{p} \nabla \left[ \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right]. \quad (3.10)
$$

(3.8) and (3.10) yield

$$
\sum_{i=1}^{p} b_i(x, x^*) \left[ (\lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x)) - \left( \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right) \right] \geq 0. \quad (3.11)
$$
Now multiplying both sides of (2.2) by \( \lambda_i^* \geq 0 \) we obtain
\[
\lambda_i^* f_i(x) \leq \lambda_i^* q, \quad i = 1, 2, \ldots, p.
\] (3.12)

Multiplying both sides of (2.3) by \( y_{ij}^* \) and summing over \( j = 1, 2, \ldots, m \) we obtain
\[
\sum_{j=1}^{m} y_{ij}^* h_{ij}(x) \leq 0, \quad i = 1, 2, \ldots, p;
\] (3.13)

adding (3.12) and (3.13) we get
\[
\left[ \lambda_i^* f_i(x) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x) \right] \leq \lambda_i^* q.
\] (3.14)

Since \((x^*, q^*, \lambda^*, Y^*)\) satisfies (3.2) and (3.3) we have
\[
\left[ \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right] = \lambda_i^* q^*.
\] (3.15)

Using (3.14) and (3.15) in (3.11) we obtain
\[
(q - q^*) \left[ \sum_{i=1}^{p} \left[ b_i(x, x^*) \right] \lambda_i^* \right] \geq 0.
\] (3.16)

Since \( \sum_{i=1}^{p} \lambda_i^* = 1 \), \( b_i(x, x^*) > 0 \) \( \forall i = 1, 2, \ldots, p \), and \( \sum_{i=1}^{p} \lambda_i^* = 1 \), \( \lambda_i^* \geq 0 \) \( \forall i = 1, 2, \ldots, p \), therefore, \( \sum_{i=1}^{p} \left[ b_i(x, x^*) \lambda_i^* \right] > 0 \). Hence, from (3.16) we have \( q \geq q^* \) for \((x^*, q^*)\) and all feasible points \((x, q)\) of (E). Thus \((x^*, q^*)\) is (E)-optimal. Hence, by Lemma 2.2, \( x^* \) is (P)-optimal with \( q^* \) as the optimal value of (P)-objective.

**Remarks 3.1.** From (3.2), (3.3), and (3.6) it may be observed that \( q^* \), the optimal value of (P)-objective, is equal to \( \sum_{i=1}^{p} \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \).

4. DUAL PROBLEMS AND DUALITY THEOREMS

We now introduce the following dual (D) to (E). Since, by virtue of Lemmas 2.1 and 2.2, (E) is equivalent to (P), therefore, all the duality results that relate (D) and (E) relate (D) and (P) also. Hence, (D) may be
considered to be a dual to (P):

\[(D) \quad \max \ u \]

subject to

\[
\sum_{i=1}^{p} \sum_{j=1}^{m} \left( \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right) = 0 \quad (4.1)
\]

\[
\left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] \geq \lambda_i v, \quad i = 1, 2, \ldots, p \quad (4.2)
\]

\[
\sum_{i=1}^{p} \lambda_i = 1 \quad (4.3)
\]

\[
u \in X_0, \lambda \in \mathbb{R}^p, Y \in \mathbb{R}^{p \times m}, \lambda \geq 0, Y \geq 0. \quad (4.4)
\]

From now on we shall denote the set of (E)-feasible solutions by \(W\) and the set of (D)-feasible solutions by \(T\).

**Theorem 4.1 (Weak duality).** For any \((x, q) \in W\) and any \((u, v, \lambda, Y) \in T\), \(q \geq v\).

**Proof.** For any \((x, q) \in W\) and any \((u, v, \lambda, Y) \in T\) we have from (4.1),

\[
\eta(x, x^+) \sum_{i=1}^{p} \sum_{j=1}^{m} \left( \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right) = 0. \quad (4.5)
\]

Using Theorem 2.1 (as in Theorem 3.2) and (4.5) we obtain for \((x, q) \in W\) and any \((u, v, \lambda, Y) \in T\),

\[
\sum_{i=1}^{p} b_i(x, u) \left[ \left( \lambda f_i(x) + \sum_{j=1}^{m} y_{ij}(x) \right) - \left( \lambda f_i(u) + \sum_{j=1}^{m} y_{ij}(u) \right) \right] \geq 0. \quad (4.6)
\]

Now (as in Theorem 3.2), using the constraints (2.2) and (2.3) and \(E\) and
the constraints (4.2) and (4.4) in (4.6) we obtain for \((x, q) \in W\) and any \((u, v, \lambda, Y) \in T\)

\[
(q - v) \left[ \sum_{i=1}^{p} (b(x, u)\lambda_i) \right] \geq 0.
\]  

(4.7)

But \(\sum_{i=1}^{p} b(x, u)\lambda_i > 0\) in (4.7). Therefore, \(q \geq v\) for \((x, q) \in W\) and any \((u, v, \lambda, Y) \in T\).

**Corollary 4.1.** For \((x^*, q^*) \in W\) and \((u^*, v^*, \lambda^*, Y^*) \in T\) let \(q^* = v^*\). Then \((x^*, q^*)\) is \((E)\)-optimal and \((u^*, v^*, \lambda^*, Y^*)\) is \((D)\)-optimal.

**Theorem 4.2** (Direct duality). Let \((x^*, q^*) \in W\), at which a constraint qualification [21] holds, be \((E)\)-optimal. Then there exist \(\lambda^* \in \mathbb{R}^p\) and \(Y^* \in \mathbb{R}^{p \times m}\) such that \((x^*, q^*, \lambda^*, Y^*) \in T\), at \((x^*, q^*, \lambda^*, Y^*)\) the \((D)\)-objective value is equal to the \((E)\)-objective value, and \((x^*, q^*, \lambda^*, Y^*)\) is \((D)\)-optimal.

**Proof.** Since \((x^*, q^*)\) is \((E)\)-optimal, therefore, there exist \(\lambda^* \in \mathbb{R}^p\) and \(Y^* \in \mathbb{R}^{p \times m}\) such that at \((x^*, q^*, \lambda^*, Y^*)\) conditions (3.1)–(3.7) are satisfied. (3.1)–(3.3) and (3.6), (3.7) yield that \((x^*, q^*, \lambda^*, Y^*)\) is \((D)\)-feasible. Also, we see that the \((D)\)-objective value is equal to \(q^*\) which is the same as the \((E)\)-objective. Using Corollary 4.1, we get that \((x^*, q^*, \lambda^*, Y^*)\) is \((D)\)-optimal.

**Theorem 4.3** (Strict converse duality). Let \((x^*, q^*) \in W\), at which a constraint qualification [21] holds, be \((E)\)-optimal and let \((u^*, v^*, \lambda^*, Y^*) \in T\) be \((D)\)-optimal. For \(i = 1, 2, \ldots, p\) and for all \((E)\)-feasible if at least one of \(f_i\), for which the corresponding \(\lambda_i > 0\), be properly strictly \((b_i, \eta)\)-invex, and/or at least one of \(h_{ij}\) for which the corresponding \(y_{ij} > 0\), be properly strictly \((b_i, \eta)\)-invex, then \((x^*, q^*) = (u^*, v^*)\).

**Proof.** We assume \((x^*, q^*) \neq (u^*, v^*)\) and exhibit a contradiction. Since \((x^*, q^*)\) is \((E)\)-optimal, there exists \(\lambda^0 \in \mathbb{R}^p\) and \(Y^0 \in \mathbb{R}^{p \times m}\) such that \((x^*, q^*, \lambda^0, Y^0)\) is in \(T\) and is \((D)\)-optimal. Also \((u^*, v^*, \lambda^0, Y^0) \in T\) is \((D)\)-optimal, therefore,

\[
q^* = v^*.
\]  

(4.8)

Since \((u^*, v^*, \lambda^0, Y^0) \in T\) is \((D)\)-optimal, it satisfies dual constraints (4.1)–(4.4). From (4.1) we have for \((x^*, q^*)\)

\[
\eta(x^*, u^*) \sum_{i=1}^{p} \nabla \lambda_i f_i(u^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(u^*) = 0.
\]  

(4.9)
Arguing along the lines of the proof of Theorem 3.2, (4.9), along with Theorem 2.1, yields

\[
\sum_{i=1}^{p} b_i(x^*, u^*) \left[ \left( \lambda_i^* f_i(x^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(x^*) \right) - \left( \lambda_i^* f_i(u^*) + \sum_{j=1}^{m} y_{ij}^* h_{ij}(u^*) \right) \right] > 0. \tag{4.10}
\]

As in Theorem 3.2, (4.10) yields \( (q^* - v^*) \sum_{i=1}^{P} b_i(x^*, u^*) \lambda_i^* > 0 \). Since \( \sum_{i=1}^{P} b_i(x^*, u^*) \lambda_i^* > 0 \), we obtain \( q^* > v^* \), which contradicts (4.8). Therefore, \( (x^*, q^*) = (u^*, v^*) \).

We now introduce another dual program (D-1) which yields, as a special case, Wolfe’s dual [24] for a single objective optimization problem:

\[
(D-1) \quad \max \sum_{i=1}^{P} \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right]
\]

subject to

\[
\sum_{i=1}^{P} \nabla \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] = 0
\]

\[
\sum_{i=1}^{P} \lambda_i = 1
\]

\[
u \in X_0, \quad u \in R, \quad \lambda \in R^P, \quad Y \in R^{P \times m}, \quad \lambda \geq 0, \quad Y \geq 0.
\]

We now obtain (D-1) from (D). In (D), summing the constraint (4.2) over \( i \) and using (4.3),

\[
\max v \tag{4.11}
\]

subject to

\[
\sum_{i=1}^{P} \nabla \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] = 0
\]

\[
\sum_{i=1}^{P} \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] \geq v \sum_{i=1}^{P} \lambda_i = v \tag{4.12}
\]

\[
\sum_{i=1}^{P} \lambda_i = 1
\]

\[
u \in X, \quad v \in R, \quad \lambda \in R^P, \quad Y \in R^{P \times m}, \quad \lambda \geq 0, \quad Y \geq 0.
\]
The objective function (4.11) and the constraint (4.12) yield

$$\max \sum_{i=1}^{p} \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] \sum_{i=1}^{p} \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right]$$

subject to \( \sum_{i=1}^{p} \nabla \left[ \lambda_i f_i(u) + \sum_{j=1}^{m} y_{ij} h_{ij}(u) \right] = 0 \)

\( \sum_{i=1}^{p} \lambda_i = 1 \)

\( u \in X, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, Y \geq 0, \)

which is the same as (D-1).

5. GENERALIZED FRACTIONAL PROGRAMMING

Before we discuss in this section the applications of the results obtained in Sections 3 and 4 to a class of generalized fractional programming problems \([5–7, 14, 16, 17, 19, 25]\), we state the following results that we shall use in the sequel.

**Lemma 5.1.** Let \( i = 1, 2, \ldots, p, \) and \( \phi_i, \psi_i : R^n \rightarrow R \) be differentiable functions. Let

(i) \( \phi_i \) be \( \eta \)-invex and nonnegative on \( X_0 \),

(ii) \( \psi_i \) be \( \eta \)-incave and strictly positive on \( X_0 \),

then for every \( x, u \in R^n \) we have

$$\frac{\psi_i(x)}{\psi_i(u)} \frac{\phi_i(x)}{\psi_i(x)} - \frac{\phi_i(u)}{\psi_i(u)} \geq \eta(x, u) \nabla \left[ \frac{\phi_i(u)}{\psi_i(u)} \right].$$

Setting

$$b_i(x, u) = \frac{\psi_i(x)}{\psi_i(u)} > 0, \quad f_i(x) = \frac{\phi_i(x)}{\psi_i(x)},$$

we obtain \( b_i(x, u) [f_i(x) - f_i(u)] \geq \eta(x, u) \nabla f_i(u) \) which shows that for \( i = 1, 2, \ldots, p, f_i(x) = \phi_i(x) / \psi_i(x) \) is a properly \((b_i, \eta)\)-invex function.

**Remark 5.1.** In Lemma 5.1, if

(i) \( \psi_i \) is both \( \eta \)-invex and \( \eta \)-incave, then \( \phi_i \) need not be restricted to be nonnegative;
(ii) either at least one of the functions \( \phi_i \) and \(-\psi_i\) be strictly \( \eta\)-invex, then for every \( x, u \in \mathbb{R}^n \) we have
\[
\frac{\psi_i(x)}{\psi_i(u)} \left[ \frac{\phi_i(x)}{\psi_i(x)} - \frac{\phi_i(u)}{\psi_i(u)} \right] > \eta(x, u) \nabla \left[ \frac{\phi_i(u)}{\psi_i(u)} \right].
\]
Setting \( h_i(x, u) = \psi_i(x)/\psi_i(u) > 0, f_i(x) = \phi_i(x)/\psi_i(x) \), we obtain that for \( i = 1, 2, \ldots, p \), \( f_i(x) = \phi_i(x)/\psi_i(x) \) is a properly strictly \( (b_i, \eta)\)-invex function.

**Generalized Fractional Programming.** We consider the following generalized fractional programming problem [5–7, 14, 16, 17, 19, 25]:

\[
(GFP) \quad \min_x \max_{1 \leq i \leq p} \left[ \frac{\phi_i(x)}{\psi_i(x)} \right]
\]

subject to \( h_j(x) \leq 0, \quad j = 1, 2, \ldots, m, \ x \in X_0, \)

where,

1. \( X_0 \subset \mathbb{R}^n \) is an open convex set,
2. \( \phi_i, \psi_i: \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, 2, \ldots, p \), are differentiable functions,
3. \( \psi_i \) is \( \eta\)-invex and strictly positive on \( X_0 \), \( \phi_i \) is \( \eta\)-invex and nonnegative on \( X_0 \) (\( \phi_i \) need not be nonnegative on \( X_0 \) when \( \psi_i \) is both \( \eta\)-invex and \( \eta\)-incave on \( X_0 \)),
4. \( h_j: \mathbb{R}^n \to \mathbb{R} \) for \( j = 1, 2, \ldots, m \), are differentiable \( \eta\)-invex functions on \( X_0 \),
5. \( \lambda_i \phi_i + y_i h_i(x) \geq 0 \) on \( X_0 \) \( \forall \lambda_i \geq 0, \ y_{ij} \geq 0, \ i = 1, 2, \ldots, p \), and \( j = 1, 2, \ldots, m \) \( \lambda_i \phi_i + y_i h_i(x) \) need not be nonnegative on \( X_0 \) when \( \psi_i \) is both \( \eta\)-invex and \( \eta\)-incave on \( X_0 \).

From (GFP) we obtain the following transformed generalized fractional programming problem (TGFP).

\[
(TGFP) \quad \min_x \max_{1 \leq i \leq p} \left[ \frac{\phi_i(x)}{\psi_i(x)} \right]
\]

subject to \( \left[ h_j(x)/\psi_i(x) \right] \leq 0, \quad j = 1, 2, \ldots, m, \ x \in X_0. \)

Setting \( h_j(x) = h_j(x)/\psi_i(x) \), and \( f_j(x) = \phi_i(x)/\psi_i(x) \), we observe that (TGFP) is of the same form as (P).

The following lemma relates the sets of feasible solutions of (GFP) and (TGFP).
Lemma 5.2. (i) $x$ is (GFP)-feasible if and only if it is (TGFP)-feasible.

(ii) $x^*$ is (GFP)-optimal if and only if it is (TGFP)-optimal.

Using (B5) and Lemma 5.1 we have the following.

Lemma 5.3. The function, $\lambda_i f_i(x) + y_i h_i(x) = (\lambda_i \phi_i(x) + y_i h_i(x))/\psi_i(x)$, for $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, m$ is $(b_i, \eta)$-invex with $b_i(x, u) = \psi_i(x)/\psi(u) > 0$, $i = 1, 2, \ldots, p$.

In view of Lemma 5.1, 5.2, and 5.3, we see that the results of Sections 2, 3, and 4 become applicable to (GFP) and we have the following (GD) and (GFD-1) duals to (GFP):

\[
\begin{align*}
(GD) & \quad \max u \\
& \quad \text{subject to} \\
& \quad \sum_{i=1}^{p} v \left[ \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_j h_j(u)}{\psi_i(u)} \right] = 0 \quad (5.1) \\
& \quad \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_j h_j(u)}{\psi_i(u)} \geq \lambda_i v, \quad i = 1, 2, \ldots, p. \quad (5.2) \\
& \quad \sum_{i=1}^{p} \lambda_i = 1 \quad (5.3) \\
& \quad u \in X_0, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, y \geq 0. \quad (5.4)
\end{align*}
\]

and

\[
\begin{align*}
(GFD-1) & \quad \max \sum_{i=1}^{p} \left[ \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_j h_j(u)}{\psi_i(u)} \right] \\
& \quad \text{subject to} \sum_{i=1}^{p} v \left[ \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_j h_j(u)}{\psi_i(u)} \right] = 0 \\
& \quad \sum_{i=1}^{p} \lambda_i = 1 \\
& \quad u \in X_0, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, Y \geq 0.
\end{align*}
\]

In the present case we can drive from (GD) another dual (GFD-2) as follows.
Multiplying both sides of the constraint (5.2) by $\psi_i(u)$ and then summing both sides over $i$, we have

$$\sum_{i=1}^{p} \left[ \lambda_i \phi_i(u) + \sum_{j=1}^{m} y_{ij} h_j(u) \right] \geq \psi_i(u) \sum_{i=1}^{p} \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_{ij} h_j(u)}{(\sum_{i=1}^{p} \lambda_i \psi_i(u))} \geq v.$$ 

Thus from (GD) we obtain

$$\max v$$

subject to

$$\sum_{i=1}^{p} \psi_i(u) \left[ \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_{ij} h_j(u)}{(\sum_{i=1}^{p} \lambda_i \psi_i(u))} \right] = 0$$

$$\sum_{i=1}^{p} \lambda_i = 1$$

$$u \in X_0, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, Y \geq 0,$$

which is equivalent to

$$(GFD-2) \quad \max \frac{\sum_{i=1}^{p} \lambda_i \phi_i(u) + \sum_{j=1}^{m} y_{ij} h_j(u)}{(\sum_{i=1}^{p} \lambda_i \psi_i(u))}$$

subject to

$$\sum_{i=1}^{p} \psi_i(u) \left[ \frac{\lambda_i \phi_i(u) + \sum_{j=1}^{m} y_{ij} h_j(u)}{(\sum_{i=1}^{p} \lambda_i \psi_i(u))} \right] = 0$$

$$\sum_{i=1}^{p} \lambda_i = 1$$

$$u \in X_0, v \in R, \lambda \in R^p, Y \in R^{p \times m}, \lambda \geq 0, y \geq 0.$$

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REFERENCES


