JOURNAL OF DIFFERENTIAL EQUATIONS **141**, 218–235 (1997)
ARTICLE NO. DE973333

On the Cauchy Problem for the Periodic Camassa–Holm Equation

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Received September 17, 1996; revised March 18, 1997

1. INTRODUCTION

In this paper we consider the Cauchy problem for the recently derived shallow water equation (see [3])

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where u describes the free surface of the water above a flat bottom. Unlike the Korteweg-de Vries equation (which is an approximation to the equations of motion, cf. [17]), this model is obtained by approximating directly in the Hamiltonian for Euler's equations in the shallow water regime, cf. [4].

With $m = u - u_{xx}$, the equation can be written as

$$\begin{cases} m_t = -2mu_x - m_x u, & t > 0, \quad x \in \mathbb{R}, \\ m(0) = \phi, & x \in \mathbb{R}, \end{cases} \tag{1}$$

and this is the form of the equation with which we are going to work.

For more than 15 years, Eq. (1) was known, being derived by Fuchssteiner and Fokas, cf. [8, 9], as a bi-Hamiltonian generalization of the Korteweg-de Vries equation. Actually, in the Fuchssteiner-Fokas derivation there is a computational error: going through it again, one comes up with the exact form of (1), see [10]. As noted in [15], the novelty of Camassa and Holm's work was that they gave a physical derivation of Eq. (1) and found that the solitary waves interact like solitons. The existence of solitons and various other types of special solutions to Eq. (1) is studied extensively (see [1, 2, 7]). For related problems we also refer to [11, 14] and the citations therein.

We are interested in the periodic problem for (1), that is, we look for solutions of (1) which are spatially of period 1. We will prove a local

0022-0396/97 $25.00

existence theorem for (1) in the Sobolev space H^2 of functions with period 1, denoted H^2_p, using Kato's method for abstract quasi-linear equations [12, 13].

To obtain global results from the local ones is in general a matter of a priori estimates. Although (1) is known to have an abundance of conservation laws, cf. [3, 4], we will see that the nature of the conservation laws helps us only partially in the search for a priori estimates. There are two main problems that we encounter: one is that we do not even control by conservation laws the H^2_p-norm of u as time varies and the other problem is of a technical nature: in the computation of $(d/dt)\,\|m\|^2_{H^2_p}$ we need that m is smoother than H^2 for fixed $t>0$.

If the initial data $\phi \in H^2_p$ does not change sign ($\phi \geqslant 0$ or $\phi \leqslant 0$), we overcome both difficulties and we prove that (1) is globally well-posed in H^2_p. We also show that for a large class of initial data $\phi \in H^2_p$ taking both strictly positive and strictly negative values, the solution of (1) blows-up in finite time.

2. THE LOCAL PROBLEM IN H^2_P

In this section we will prove that the Cauchy problem for (1) is locally well-posed in the space H^2_p. To obtain the local existence and uniqueness result we apply Kato's method [12, 13], for the Cauchy problem for abstract quasi-linear equations of evolution. For convenience we state the relevant theorem in the simplest form sufficient for the present purposes.

Consider the Cauchy problem for the quasi-linear equation of evolution

$$\begin{cases} \dfrac{dv}{dt} + A(v)v = f(v), & t>0, \\ v(0)=\phi. \end{cases} \tag{2}$$

Let X, Y be reflexive Banach spaces with Y continuously and densely imbedded in X. Let S be an isomorphism (bi-continuous linear map) of Y onto X. Assume that the function A, defined on Y, satisfies the following conditions:

(i) $A(y)$ is a linear operator in X; it is quasi-m-accretive, uniformly for $\|y\|_Y$ bounded. In other words, for every $M>0$ there is a real number β such that for every $y \in Y$ with $\|y\|_Y \leqslant M$, $[-A(y)]$ generates a C_0-semigroup $\{e^{-tA(y)}\}_{t\geqslant 0}$ with

$$\|e^{-tA(y)}\|_{X\to X} \leqslant e^{\beta t}, \qquad t \geqslant 0;$$

(ii) $A(y)$ is a bounded linear operator from Y to X for every $y \in Y$ and

$$\|(A(y)-A(z))w\|_X \leqslant \mu_A \|y-z\|_X \|w\|_Y, \qquad y, z, w \in Y,$$

for some constant μ_A depending only on $\max\{\|y\|_Y, \|z\|_Y\}$;

(iii) for any $M>0$, the inequality

$$\|(SA(y)-A(y)S)S^{-1}w\|_X \leqslant \mu_1(M)\|w\|_X, \qquad y \in Y, \quad \|y\|_Y \leqslant M$$

holds for all $w \in Y$ (here $\mu_1(M)>0$ is a constant);

(iv) for each $M>0$, f is a bounded function from $\{y \in Y: \|y\|_Y \leqslant M\}$ to Y. Also, we have

$$\|f(y)-f(z)\|_X \leqslant \mu_2 \|y-z\|_X, \qquad y, z \in Y,$$

and

$$\|f(y)-f(z)\|_Y \leqslant \mu_3 \|y-z\|_Y, \qquad y, z \in Y,$$

for some constants μ_2 and μ_3 depending only on $\max\{\|y\|_x, \|z\|_x\}$ and $\max\{\|y\|_Y, \|z\|_Y\}$, respectively.

THEOREM 1 [12]. *Assume conditions* (i), (ii), (iii) *and* (iv). *For any* $\phi \in Y$ *there is a* $T>0$, *depending only on* $\|\phi\|_Y$, *and a unique solution* v *to* (2) *such that*

$$v \in C([0, T]; Y) \cap C^1([0, T]; X).$$

Moreover, $v(t)$ *depends continuously on* $\phi = v(0)$ *in the* Y*-norm.*

Remark. By the fact that T depends only on $\|\phi\|_Y$ it is understood that for any $M>0$ there is a $T(M)>0$ for all $\phi \in Y$ with $\|\phi\|_Y \leqslant M$. The appearance of a closed interval $[0, T]$ in the statement of Theorem 1 is unnatural since the solution v exists on a larger interval $[0, T+\varepsilon)$ but it is convenient in the formulation of the continuous dependence (the solution exists on the same interval when ϕ is changed slightly).

THEOREM 2. *If* $\phi \in H^2_p$, *there is a* $T>0$, *depending only on* $\|\phi\|_{H^2_p}$, *such that* (1) *has a unique solution*

$$m \in C([0, T]; H^2_p) \cap C^1([0, T]; L^2[0, 1])$$

and $m(t)$ *depends continuously on* ϕ *in the* H^2_p*-norm.*

Proof. To prove Theorem 2 we have to check that the hypotheses (i), (ii), (iii) and (iv) above hold for the case of Eq. (1) with $\phi \in H^2_p$. In order to simplify notation, from now on all integrals are considered with respect to the spatial variable.

Let $X = L^2[0, 1]$, $Y = H^2_p$. As an isometric isomorphism of Y onto X we choose $S = Id - D^2$, $D = (D/dx)$ (we take the usual norm on $L^2[0, 1]$ and let $\|y\|^2_Y = \|Sy\|^2_X = \|y\|^2_X + 2\|y_x\|^2_X + \|y_{xx}\|^2_X$ for $y \in Y$). We denote by $c > 0$ a generic constant.

Write (1) in the form

$$\begin{cases} \dfrac{dm}{dt} = -2m(S^{-1}m)_x - m_x(S^{-1}m), \\ m(0) = \phi. \end{cases}$$

We choose

$$A(y) = (S^{-1}y)D, \qquad f(y) = -2y(S^{-1}y)_x, \quad y \in Y.$$

To verify condition (i), fix $M > 0$ and let $y \in Y$, $\|y\|_Y \leqslant M$. We have

$$S^{-1}y \in Y, \qquad (S^{-1}y)_{xx} = (S^{-1}y) - y.$$

Observe that $(S^{-1}y)(0) = (S^{-1}y)(1)$. Thus, there is an $\xi \in (0, 1)$ with $(S^{-1}y)_x(\xi) = 0$. For $x \in [\xi, \xi + 1]$ we therefore have

$$\begin{aligned} |(S^{-1}y)_x(x)|^2 &= 2\int_\xi^x (S^{-1}y)_{xx}(S^{-1}y)_x = 2\int_\xi^x ((S^{-1}y)(S^{-1}y)_x - y(S^{-1}y)_x) \\ &\leqslant 2\|S^{-1}y\|_X \|(S^{-1}y)_x\|_X + 2\|y\|_X \|(S^{-1}y)_x\|_X \end{aligned}$$

by Schwarz's inequality. Since $\|y\|_X \leqslant \|y\|_Y \leqslant M$ and

$$\|S^{-1}y\|_X + \|(S^{-1}y)_x\|_X \leqslant 2\|S^{-1}y\|_Y = 2\|y\|_X \leqslant 2\|y\|_Y \leqslant 2M,$$

we find

$$\sup_{x \in [0,1]} |(S^{-1}y)_x(x)|^2 \leqslant 12M^2$$

thus (see [12], p. 38) $A(y)$ is quasi-m-accretive with the corresponding β equal to $\sqrt{3}\,M$.

It is easy to check that for any $y \in Y$, $A(y)$ is a bounded linear operator from Y to X: if $v \in Y$,

$$\|A(y)v\|_X^2 = \int_0^1 v_x^2(S^{-1}y)^2 \leqslant \|v_x\|_{L^\infty}^2 \|S^{-1}y\|_X^2$$

$$\leqslant \|v_x\|_{L^\infty[0,1]}^2 \|S^{-1}y\|_Y^2 = \|v_x\|_{L^\infty[0,1]}^2 \|y\|_X^2 \leqslant c\,\|v\|_Y^2 \|y\|_Y^2$$

taking into account the Sobolev inequality corresponding to the inclusion $H_p^1 \subset L^\infty[0,1]$.

To complete the checking of (ii), let $y, z, w \in Y$. We have

$$\|(A(y) - A(z))w\|_X^2 = \int_0^1 w_x^2(S^{-1}y - S^{-1}z)^2$$

$$\leqslant \|w_x\|_{L^\infty[0,1]}^2 \|S^{-1}(y-z)\|_X^2 \leqslant c\,\|w\|_Y^2 \|y-z\|_X^2.$$

Let us now verify (iii). For $v = S^{-1}w$ with $w \in X$, we compute

$$SA(y)v - A(y)\,Sv = -2v_{xx}(S^{-1}y)_x - v_x(S^{-1}y)_{xx}$$

and recall the formulas

$$(S^{-1}w)_{xx} = (S^{-1}w) - w, \qquad (S^{-1}y)_{xx} = (S^{-1}y) - y.$$

Therefore

$$\begin{aligned}\|(SA(y) - A(y)S)\,S^{-1}w\|_X &= \|2(S^{-1}w)(S^{-1}y)_x - 2w(S^{-1}y)_x \\ &\quad + (S^{-1}w)_x\,(S^{-1}y) - (S^{-1}w)_x y\|_X \\ &\leqslant 2\,\|(S^{-1}y)_x\|_{L^\infty[0,1]}\,(\|S^{-1}w\|_X + \|w\|_X) \\ &\quad + \|(S^{-1}w)_x\|_X\,(\|S^{-1}y\|_{L^\infty[0,1]} + \|y\|_{L^\infty[0,1]}).\end{aligned}$$

In view of the relations (recall c stands for a generic constant)

$$\|(S^{-1}w)_x\|_X \leqslant \|S^{-1}w\|_Y = \|w\|_X, \qquad \|S^{-1}w\|_X \leqslant \|S^{-1}w\|_Y = \|w\|_X,$$

$$\|(S^{-1}y)_x\|_{L^\infty[0,1]} + \|S^{-1}y\|_{L^\infty[0,1]} \leqslant c\,\|S^{-1}y\|_Y = c\,\|y\|_x \leqslant c\,\|y\|_Y,$$

$$\|y\|_{L^\infty[0,1]} \leqslant c\,\|y\|_Y,$$

we obtain

$$\|(SA(y) - A(y)S)\,S^{-1}w\|_X \leqslant 6c\,\|w\|_X\,\|y\|_Y$$

and condition (iii) is satisfied.

The same type of estimates as above can also be used to check condition (iv). The proof is completed. ▮

3. SOME ESTIMATES

It is known (see [3, 4]) that (1) has an abundance of conservation laws, among which

$$I_1 = \int_0^1 u = \int_0^1 m, \; I_2 = \int_0^1 (u^2 + u_x^2).$$

The next conservation laws (see [5]) are not useful for our purposes since they do not give much more information about the H_p^2-norm of a solution to (1) than the first two do. Actually, the blow-up result in the last section shows that in general one can not expect to control the H_p^2-norm of u. Therefore some restrictions on the initial data have to be placed in order to get global solutions.

Let $\phi \in H_p^2$ with $\|\phi\|_{H_p^2}^2 = M$. As long as the solution $m(t) \in H_p^2$ exists, the following estimates hold:

$$\left|\int_0^1 u\right| = \left|\int_0^1 m\right| = \left|\int_0^1 \phi\right| \leqslant \left(\int_0^1 \phi^2\right)^{1/2} \leqslant \|\phi\|_{H_p^2} = \sqrt{M},$$

$$\int_0^1 (u^2 + u_x^2) = \int_0^1 (\phi^2 + \phi_x^2) \leqslant \|\phi\|_{H_p^2}^2 = M.$$

Observe that

$$u^2(x) - u^2(y) = 2\int_x^y u u_x \leqslant \int_0^1 (u^2 + u_x^2) \leqslant M, \qquad x, y \in [0, 1].$$

The inequality $\int_0^1 u^2 \leqslant \int_0^1 (u^2 + u_x^2) \leqslant M$ implies

$$\min_{x \in [0,1]} u^2(x) \leqslant M.$$

Combining this with the above estimates, we obtain

$$u^2(x) \leqslant 2M, \qquad x \in [0, 1],$$

as long as the solution $m(t) \in H_p^2$ exists.

If we obtain a constant $K > 0$ (depending on M) such that

$$\|m(t)\|_{H_p^2}^2 \leqslant \|\phi\|_{H_p^2}^2 e^{Kt}$$

as long as the solution $m(t) \in H_p^2$ exists, we would have global existence of the solution in the space H_p^2.

It is not hard to check that as long as the solution $m(t) \in H_p^2$ exists,

$$\frac{d}{dt} \int_0^1 m^2 = -3 \int_0^1 m^2 u_x, \tag{3}$$

and (recall $m = u - u_{xx}$)

$$\frac{d}{dt} \int_0^1 m_x^2 = -4 \int_0^1 m m_x u - 6 \int_0^1 m_x^2 u_x - 2 \int_0^1 m_x m_{xx} u. \tag{4}$$

We have (formally)

$$\begin{aligned} \frac{d}{dt} \int_0^1 m_{xx}^2 = & -4 \int_0^1 m m_{xx} u_{xxx} - 10 \int_0^1 m_x m_{xx} u_{xx} \\ & -8 \int_0^1 m_{xx}^2 u_x - 2 \int_0^1 m_{xx} m_{xxx} u, \end{aligned} \tag{5}$$

but to justify this we need to know that m is smoother than H^2 ($m \in H^3$ would be fine). Moreover, trying to get from (3), (4) and (5) that

$$\frac{d}{dt} \|m(t)\|_{H_p^2}^2 \leqslant K \|m(t)\|_{H_p^2}^2$$

appears not at all obvious.

The rest of the paper is devoted to the proof that in the case when ϕ does not change sign these difficulties can be overcome.

A key point in our arguments is the fact that if we consider the spectral problem

$$f'' = \tfrac{1}{4} f + \lambda m f, \qquad 0 \leqslant x \leqslant 1, \tag{6}$$

(here t is a parameter and we look for those values of λ for which this equation has a nontrivial periodic or anti-periodic solution on [0,1]), then the periodic and anti-periodic spectrum of (6)—with $m(t) \in H_p^2$ solution of (1) and t as the time parameter—are integrals of motion of (1): they do not depend on the parameter t, see [3].

If we consider (6) with some $m \in C(\mathbb{R})$ of period 1, $m \not\equiv 0$, we have the following:

LEMMA [6]. *If $m \geqslant 0$ ($m \leqslant 0$), then the periodic and anti-periodic spectrum of* (6) *are formed by infinitely many strictly negative* (*respectively strictly positive*) *numbers. If m takes both strictly positive and strictly* *negative values, we have infinitely many elements of the periodic and anti-periodic spectrum of* (6) *on both sides of* $\lambda = 0$ *on the real axis.*

As a consequence, observe that if $\phi \geqslant 0$ (or $\phi \leqslant 0$) on $[0, 1]$, then, as long as the solution $m(t) \in H_P^2$ to (1) with initial data $m(0) = \phi$ exists, we have $m \geqslant 0$ (respectively $m \leqslant 0$).

If $\phi \in H_p^2$ satisfies $\phi \geqslant 0$, let $\xi \in (0, 1)$ be such that $u_x(\xi) = 0$ (we have $u(0) = u(1)$). Since $m(t) \geqslant 0$ as long as the solution exists in H_p^2, we have

$$\int_\xi^x m = -u_x(x) + \int_\xi^x u \leqslant \int_\xi^{\xi+1} m = \int_0^1 m, \qquad x \in [\xi, \xi+1],$$

A simple application of the maximum principle shows that $u \geqslant 0$ if $m \geqslant 0$, so that we get

$$-u_x(x) \leqslant \int_0^1 m \leqslant \sqrt{M}, \qquad x \in [0, 1],$$

where $\|\phi\|^2_{H_p^2} = M$. Using (3), we find

$$\frac{d}{dt} \int_0^1 m^2 \leqslant 3\sqrt{M} \int_0^1 m^2, \tag{7}$$

as long as the solution to (1) exists in H_p^2.

If $\phi \in H_p^2$ satisfies $\phi \leqslant 0$, let $\xi \in (0, 1)$ be such that $u_x(\xi) = 0$. Then (we have that $m(t) \leqslant 0$ and $u(t) \leqslant 0$ as long as the solution exists in H_p^2),

$$0 \geqslant \int_x^\xi m = \left(\int_x^\xi u \right) - u_x(x), \qquad x \in [\xi - 1, \xi],$$

thus

$$-u_x(x) \leqslant \int_0^1 |u| \leqslant \sqrt{M}, \qquad x \in [0, 1],$$

where $\|\phi\|^2_{H_p^2} = M$, and by (3) we get again (7).

From (7) we obtain by Gronwall's inequality that

$$\int_0^1 m^2 \leqslant e^{3t\sqrt{M}} \int_0^1 \phi^2, \tag{8}$$

as long as the solution $m(t) \in H_p^2$ with initial data $m(0) = \phi$ exists.

4. A SPECIAL CASE

Let us now prove the following

THEOREM 3. *Assume that* $\phi \in C^\infty(\mathbb{R})$ *is of period* 1 *and has no zeros. Then there is a unique global solution* $m(t) \in H_p^2$ *to* (1) *with* $m(0) = \phi$ *and for any fixed* $t \geqslant 0$, $m(t) \in C^\infty(\mathbb{R})$.

Proof. Assume that $\phi > 0$ (the other case is similar).

Let $T > 0$ (given by Theorem 2) be such that (1) has a unique solution $m(t) \in H_p^2$ for $t \in [0, T]$. We intend to prove first that $m(t) > 0$ for $t \in [0, T]$; by the discussion in the previous section we only know that $m(t) \geqslant 0$ for $t \in [0, T]$.

Assume that for some $t \in [0, T]$ we have that $m(t)$ has a zero. Since $m \in C([0, T]; H_p^2)$, we deduce that there is a smallest such t, denoted t_0. Clearly $t_0 > 0$. For $t \in [0, t_0)$, the quantities

$$I_{-1} = \int_0^1 \sqrt{m},\ I_{-2} = \int_0^1 \left(\frac{m_x^2}{4m^{5/2}} + \frac{1}{\sqrt{m}} \right),$$

are conserved in time (see [5]).

For $t \in (0, t_0)$ we have (using (1) and the periodicity)

$$\frac{d}{dt} \int_0^1 \frac{1}{m} = -\int_0^1 \frac{m_t}{m^2} = \int_0^1 \frac{m_x u}{m^2} + 2 \int_0^1 \frac{u_x}{m}$$

$$= -\int_0^1 u \left(\frac{1}{m} \right)_x + 2 \int_0^1 \frac{u_x}{m} = 3 \int_0^1 \frac{u_x}{m}.$$

Since $m \in C([0, t_0]; H_p^2)$ and $m = u - u_{xx}$, it is easy to see that there is a constant $L > 0$ such that

$$|u_x|_{L^\infty[0,1]} \leqslant L, \qquad t \in [0, t_0].$$

Recall $m(t) > 0$ on $[0, t_0)$. Combining the above estimates, we find

$$\frac{d}{dt} \int_0^1 \frac{1}{m} \leqslant 3L \int_0^1 \frac{1}{m}, \qquad t \in (0, t_0),$$

and Gronwall's inequality implies that $(m(0) = \phi)$

$$\int_0^1 \frac{1}{m} \leqslant \left(\int_0^1 \frac{1}{\phi} \right) e^{3Lt}, \qquad t \in [0, t_0].$$

A simple application of Schwarz's inequality therefore yields

$$\int_0^1 \frac{1}{\sqrt{m}} \leqslant \left(\int_0^1 \frac{1}{m}\right)^{1/2} \leqslant \left(\int_0^1 \frac{1}{\phi}\right)^{1/2} e^{(3/2)Lt}, \qquad t \in [0, t_0].$$

As long as $t \in [0, t_0)$, we have for $0 \leqslant y \leqslant x \leqslant 1$,

$$\begin{aligned}\left(\frac{1}{m(x)}\right)^{1/4} - \left(\frac{1}{m(y)}\right)^{1/4} &= -\frac{1}{4}\int_x^y \frac{m_x}{m^{5/4}} \leqslant \frac{1}{4}\left(\int_0^1 \frac{m_x^2}{m^{5/2}}\right)^{1/2} \\ &\leqslant \frac{1}{2}\left|\int_0^1 \left(\frac{m_x^2}{4m^{5/2}} + \frac{1}{\sqrt{m}}\right)\right|^{1/2} \\ &\leqslant \sqrt{I_{-2}}, \qquad t \in [0, t_0),\end{aligned}$$

thus for every $t \in [0, t_0)$, we have

$$\max_{x \in [0,1]} \left(\frac{1}{m(x)}\right)^{1/4} - \min_{x \in [0,1]} \left(\frac{1}{m(x)}\right)^{1/4} \leqslant \sqrt{I_{-2}}.$$

On the other hand, we have that for every $t \in [0, t_0)$,

$$\min_{x \in [0,1]} \left(\frac{1}{m(x)}\right)^{1/4} \leqslant \left(\int_0^1 \frac{1}{\sqrt{m}}\right)^{1/2} \leqslant \left(\int_0^1 \frac{1}{\phi}\right)^{1/4} e^{(3/4)Lt},$$

and, combining the two last inequalities, we find that for every $t \in [0, t_0)$,

$$\max_{x \in [0,1]} \left(\frac{1}{m(x)}\right)^{1/4} \leqslant \sqrt{I_{-2}} + \left(\int_0^1 \frac{1}{\phi}\right)^{1/4} e^{(3/4)Lt}.$$

We proved therefore the existence of an $\varepsilon > 0$ such that

$$m(t, x) \geqslant \varepsilon, \qquad x \in [0, 1], \quad t \in [0, t_0).$$

Since

$$\lim_{t \uparrow t_0} \left(\max_{x \in [0,1]} |m(t, x) - m(t_0, x)| \right) = 0$$

we find it impossible that $m(t_0)$ has a zero.

The previous argument shows that as long as the solution $m(t) \in H^2_p$ exists, we have $m(t) > 0$ on $[0, 1]$. Therefore the Liouville substitution

$$z(r) = (m(x))^{1/4} y(x), \qquad \text{where} \quad r = \frac{\int_0^x \sqrt{m}}{\int_0^1 \sqrt{m}}$$

is perfectly valid and transforms (6) into

$$\frac{d^2 z}{dr^2} = \left(\lambda + \frac{1}{4m(x)} + \frac{m_{xx}(x)}{4m^2(x)} - \frac{5m_x^2(x)}{16m^3(x)} \right) \left(\int_0^1 \sqrt{m} \right)^2 z. \tag{9}$$

Equation (9) is a Hill's equation and we know (see [16]) that the spectrum is formed by a periodic ground state λ_0 preceded alternately by anti-periodic and periodic pairs of simple or double eigenvalues

$$\cdots < \lambda_4 \leqslant \lambda_3 < \lambda_2 \leqslant \lambda_1 < \lambda_0$$

accumulating at $-\infty$.

At $t = 0$ we have $m(0) = \phi \in C^\infty(\mathbb{R})$ of period 1 and therefore (see [16])

$$|\lambda_{2n} - \lambda_{2n-1}| = O(n^{-p}) \qquad \text{as} \quad n \to \infty, \tag{10}$$

for every $p = 1, 2, \ldots$. But, as $m(t) \in H^2_p$ satisfies (1), the periodic and anti-periodic spectrum do not change (recall I_{-1} is a constant of motion) so that (10) will hold at any time $t \in [0, T]$ and since (10) is a necessary and sufficient condition for the potential to be in $C^\infty(\mathbb{R})$ with period 1, cf. [16], we find that for any fixed $t \in [0, T]$,

$$\frac{1}{4m(x)} + \frac{m_{xx}(x)}{4m^2(x)} - \frac{5m_x^2(x)}{16m^3(x)} \in C^\infty(\mathbb{R})$$

which implies that $m(t) \in C^\infty(\mathbb{R})$ with period 1 for any $t \in [0, T]$.

Assume now that the maximal interval of existence of the solution $m(t) \in H^2_p$ of (1) with $m(0) = \phi$ is $[0, T_0)$ for some $T_0 \in \mathbb{R}$. Then, as we saw, $m(t) \in C^\infty(\mathbb{R})$ with period 1 and $m(t) > 0$ for any $t \in [0, T_0)$. Therefore (see the previous section)

$$\int_0^1 m^2 \leqslant e^{3t\sqrt{M}} \int_0^1 \phi^2, \qquad t \in [0, T_0),$$

where $M = \|\phi\|^2_{H^2_p}$. Also, the expressions (4) and (5) take the more pleasant form

$$\frac{d}{dt}\int_0^1 m_x^2 = -4\int_0^1 mm_x u - 5\int_0^1 m_x^2 u_x, \tag{11}$$

$$\frac{d}{dt}\int_0^1 m_{xx}^2 = -4\int_0^1 mm_{xx}u_{xxx} - 10\int_0^1 m_x m_{xx} u_{xx} - 7\int_0^1 m_{xx}^2 u_x. \tag{12}$$

It is not hard to see (repeat the arguments from the beginning of the previous section) that the boundedness of $m(t)$ on $[0, T_0)$ in the $L^2[0, 1]$-norm implies the existence of a constant $\alpha > 0$ (depending on M and T_0) such that for all $t \in [0, T_0)$,

$$u^2(x) + u_x^2(x) \leqslant \alpha^2, \qquad x \in [0, 1].$$

From (11) we therefore get,

$$\frac{d}{dt}\int_0^1 m_x^2 \leqslant 2\alpha\int_0^1 (m^2 + m_x^2) + 5\alpha\int_0^1 m_x^2, \qquad t \in [0, T_0).$$

Combining this with (3), we obtain

$$\frac{d}{dt}\int_0^1 (m^2 + m_x^2) \leqslant 7\alpha\int_0^1 (m^2 + m_x^2), \qquad t \in (0, T_0). \tag{13}$$

An application of Gronwall's inequality yields

$$\int_0^1 (m^2 + m_x^2) \leqslant e^{7\alpha t}\int_0^1 (\phi^2 + \phi_x^2), \qquad t \in [0, T_0).$$

We may now repeat the method: we find a constant $\gamma > \alpha$ (depending on M and T_0) such that for all $t \in [0, T_0)$,

$$u^2(x) + u_x^2(x) + u_{xx}^2(x) \leqslant \gamma^2, \qquad x \in [0, 1].$$

Since $m = u - u_{xx}$, we have

$$-\int_0^1 mm_{xx}u_{xxx} = \int_0^1 mm_{xx}m_x - \int_0^1 mm_{xx}u_x$$

$$= \int_0^1 um_{xx}m_x - \int_0^1 u_{xx}m_{xx}m_x - \int_0^1 mm_{xx}u_x$$

and, using (12) and (13), we find that on $(0, T_0)$,

$$\frac{d}{dt}\int_0^1 (m^2+2m_x^2+m_{xx}^2) \leqslant 37\gamma \int_0^1 (m^2+2m_x^2+m_{xx}^2), \tag{14}$$

since if $f \in L^\infty[0, 1]$, $g, h \in L^2[0, 1]$, we have

$$2\left|\int_0^1 fgh\right| \leqslant \|f\|_{L^\infty[0, 1]} \int_0^1 (g^2+h^2).$$

An application of Gronwall's inequality to (14) yields

$$\|m\|_{H_p^2}^2 \leqslant e^{37\gamma t}\,\|\phi\|_{H_p^2}^2, \qquad t \in [0, T_0),$$

and, since $[0, T_0)$ was supposed to be the maximal interval of existence of the solution $m(t) \in H_p^2$, we obtain a contradiction (we assumed $T_0 \in \mathbb{R}$).

This proves that the maximal interval of existence of the solution $m(t) \in H_p^2$ with initial data $m(0)=\phi$ is $[0, \infty)$. ∎

5. MAIN RESULT

We have now all the necessary means to prove

THEOREM 4. *Assume that $\phi \in H_p^2$ does not change sign ($\phi \geqslant 0$ or $\phi \leqslant 0$ on $[0, 1]$). Then Eq. (1) is globally well-posed in H_p^2.*

Proof. Let us consider the case $\phi \geqslant 0$ (the other case is similar).

Choose $\phi^n \in C^\infty(\mathbb{R})$ of period 1 such that $\phi^n(t) > 0$, $n \geqslant 1$, and

$$\lim_{n\to\infty} \|\phi^n - \phi\|_{H_p^2} = 0.$$

By Theorem 3 we know that the solution $m^n(t) \in H_p^2$ of (1) with initial data $m^n(0) = \phi^n$ exists for all $t > 0$ and $m^n(t) \in C^\infty(\mathbb{R})$ of period 1, $m^n(t) > 0$ for every $t \geqslant 0$, $n \geqslant 1$.

Assume that the solution $m(t) \in H_p^2$ of (1) with initial data $m(0)=\phi$ is defined on a maximal interval $[0, T_0)$ with $T_0 \in \mathbb{R}$. We would have

$$\lim_{t \uparrow T_0} \|m(t)\|_{H_p^2} = \infty$$

in view of Theorem 2.

A closer look at the proof of Theorem 3 enables us to say that there is a constant $K_1 > 0$ (depending only on $\|\phi\|_{H_p^2}$ and T_0) such that

$$\max_{0 \leqslant t \leqslant T_0} \|m^n(t)\|^2_{H_p^2} \leqslant K_1. \tag{15}$$

Let $T \in (0, T_0)$ be such that

$$\|m(T)\|^2_{H_p^2} > K_1 + 1. \tag{16}$$

Since $\lim_{n \to \infty} \|\phi^n - \phi\|_{H_p^2} = 0$ we deduce by the continuous dependence on intial data (see Theorem 2) that

$$\lim_{n \to \infty} \|m^n(T) - m(T)\|_{H_p^2} = 0$$

which is impossible in view of (15) and (16).

The contradiction that we obtained proves that the solution $m(t) \in H_p^2$ of (1) with $m(0) = \phi$ is global. ∎

We will prove that there are smooth initial data for which the corresponding solution to (1) does not exist globally. It is worth to note that neither the smoothness nor the size of the initial data influence the life-span but the shape of the initial data. In particular, there are smooth initial data with arbitrary support for which the resulting solution does not exist globally.

Let us first derive a useful identity satisfied by a solution to (1).

The operator S^{-1} can be represented as the following convolution operator:

$$S^{-1}f(x) = \int_0^1 G(x-y)\, f(y)\, dy, \qquad f \in L_2[0, 1],$$

where G is the Green's function

$$G(x) = \frac{\operatorname{ch}(x - [x] - (1/2))}{2 \operatorname{sh}(1/2)}, \qquad x \in \mathbb{R}.$$

Given $\phi \in H_p^2$, let $u_0 \in H_p^4$ be such that $u_0 - u_{0,\,xx} = \phi$. If

$$m \in C([0, T); H_p^2) \cap C^1([0, T); L^2[0, 1])$$

is the solution of (1) with initial data ϕ, define for every $t \in [0, T)$ the function $u(t) \in H_p^4$ such that $u(t) - u(t)_{xx} = m(t)$. Using Eq. (1) it is not difficult to verify that

$$S(u_t + uu_x) = -2uu_x - u_x u_{xx} = -\partial_x(u^2 + \tfrac{1}{2}u_x^2).$$

Hence

$$u_t + uu_x = -\partial_x(G * (u^2 + \tfrac{1}{2}u_x^2)) \qquad \text{in} \quad C([0, T); H_p^3),$$

where $*$ stands for the convolution with respect to the spatial variable. Differentiating with respect to x,

$$\begin{aligned} u_{tx} + u_x^2 + uu_{xx} &= -\partial_x^2(G * (u^2 + \tfrac{1}{2}u_x^2)) \\ &= (S - Id)(G * (u^2 + \tfrac{1}{2}u_x^2)) \\ &= u^2 + \tfrac{1}{2}u_x^2 - G * (u^2 + \tfrac{1}{2}u_x^2), \end{aligned}$$

and therefore we have

$$u_{tx} + uu_{xx} = u^2 - \tfrac{1}{2}u_x^2 - G * (u^2 + \tfrac{1}{2}u_x^2) \tag{17}$$

in the space $C([0, T); H_p^2)$.

THEOREM 5. *Assume that* $u_0 \in H_p^4$ *is odd,* $u_0 \not\equiv 0$. *Then the solution of* (1) *with initial data* $\phi = u_0 - u_{0,xx}$ *does not exist globally.*

Proof. Let $[0, T)$ be the maximal interval of existence of the solution $m \in C([0, T); H_p^2) \cap C^1([0, T); L^2[0, 1])$ of (1) with initial data $m(0) = \phi$. As before, we define $u \in H_p^4$ by $u - u_{xx} = m$ at any time $t \in [0, T)$. Note that $m(t)$ and $u(t)$ are equivalent pieces of information and we can regard

$$u \in C([0, T); H_p^4) \cap C^1([0, T); H_p^2)$$

as a solution to the Camassa-Holm equation in the original form

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{18}$$

As one can check, the function

$$v(t, x) := -u(t, -x), \qquad t \in [0, T), \qquad x \in \mathbb{R},$$

is also a solution of (18) in $C([0, T); H_p^4) \cap C^1([0, T); H_p^2)$ with initial data u_0. By uniqueness we conclude that $v \equiv u$ and therefore $u(t, \cdot)$ is odd for any $t \in [0, T)$. In particular, by continuity with respect to the spatial variable of u and u_{xx}, we get

$$u(t, 0) = u_{xx}(t, 0) = 0 \qquad \text{for} \quad t \in [0, T). \tag{19}$$

Define $g(t) := u_x(t, 0)$ for $t \in [0, T)$ and note that $g \in C^1([0, T), \mathbb{R})$. From (17) and (19) we get

$$\frac{dg}{dt}(t) = -\frac{1}{2}\, g^2(t) - \int_0^1 G(x-y)\left(u^2 + \frac{1}{2}\, u_x^2\right) dy$$

$$\leqslant -\frac{1}{2}\, g^2(t), \qquad t \in (0, T). \tag{19}$$

Consequently,

$$\frac{1}{g(t)} \geqslant \frac{1}{g(0)} + \frac{t}{2}, \qquad t \in [0, T).$$

If $g(0) < 0$ we obtain $T < -(2/g(0))$ and the solution does not exist globally.

If $g(0) = 0$, note that

$$\frac{dg}{dt}(t) \leqslant -\int_0^1 G(x-y)\left(u^2 + \frac{1}{2}\, u_x^2\right) dy < 0, \qquad t \in (0, T).$$

Indeed, since $G * (u^2 + \frac{1}{2}\, u_x^2)$ having zeros for some time $t_0 \in (0, T)$ would imply that we would have $u(t_0) \equiv 0$, thus $m(t_0) \equiv 0$. However, backward uniqueness holds for the equation (it is just a repetition of the arguments in Section 2) and $u_0 \not\equiv 0$. The previous relation shows that $g(t)$ is strictly decreasing and therefore it becomes negative immediately after time zero and we fall in the previously considered case.

To prove the theorem, we have to consider the case $g(0) > 0$.

Assume $g(0) > 0$ and suppose that the solution $m(t)$ to (1) with the corresponding initial data exists globally in H_p^2.

We have

$$\frac{1}{2\,\mathrm{sh}(1/2)} \leqslant G(x), \qquad x \in [0, 1],$$

so that, in view of (19) and recalling the constants of motion from Section 3,

$$\frac{dg}{dt}(t) \leqslant -\frac{1}{2\,\mathrm{sh}(1/2)} \int_0^1 \left(u^2 + \frac{1}{2}\, u_x^2\right)$$

$$\leqslant -\frac{1}{4\,\mathrm{sh}(1/2)} \int_0^1 (u^2 + u_x^2) = -\frac{I_2}{4\,\mathrm{sh}(1/2)}, \qquad t > 0.$$

Integration yields

$$g(t) \leqslant g(0) - \frac{I_2}{4\,\mathrm{sh}(1/2)}\,t, \qquad t > 0,$$

and therefore at some instant $t_0 > 0$ we will have $g(t_0) < 0$. As above, the inequality

$$\frac{dg}{dt}(t) \leqslant -\frac{1}{2}\,g^2(t), \qquad t > t_0,$$

holds and therefore $g(t)$ must blow-up in finite time. This contradiction settles the case $g(0) > 0$ and the proof is complete. ▮

ACKNOWLEDGMENTS

I am grateful to G. DaPrato for helpful discussions and to the referee for useful suggestions. This paper was written while the author was visiting the Scuola Normale Superiore di Pisa, Italy.

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