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Global bifurcation phenomena for singular one-dimensional p-Laplacian

Yong-Hoon Lee *, Inbo Sim

Department of Mathematics, Pusan National University, Pusan 609-735, Republic of Korea

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Abstract

In this paper, we present global existence results for the following problem

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0, & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, λ a positive parameter and h a nonnegative measurable function on $(0, 1)$ which may be singular at $t = 0$ and/or $t = 1$, and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$. By applying the global bifurcation theorem and figuring the shape of unbounded subcontinua of solutions, we obtain many different types of global existence results of positive solutions. We also obtain existence results of sign-changing solutions for (P_λ) when f is an odd symmetric function.

MSC: 34A37; 34B15

Keywords: Singular boundary value problem; p-Laplacian; Global bifurcation; Existence; Multiplicity

1. Introduction

In this paper, we present global existence results with respect to given parameter λ for the following problem

* Corresponding author.
E-mail address: yhlee@pusan.ac.kr (Y.-H. Lee).

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$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 \quad \text{a.e. } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ}$$

where $\varphi_p(x) = |x|^{p-2}x$, $p > 1$, λ a positive parameter and h a nonnegative measurable function on $(0,1)$ which may be singular at $t = 0$ and/or $t = 1$, and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\mathbb{R}_+ = [0, \infty)$.

The study of existence of positive solutions for problem (P_λ) was initiated by Wang [14]. When $\lambda = 1$ and h satisfies

$$\int_0^{1/2} \varphi_p^{-1}\left(\int_s^{1/2} h(\tau)\,d\tau\right) ds + \int_{1/2}^{1} \varphi_p^{-1}\left(\int_{1/2}^{s} h(\tau)\,d\tau\right) ds < \infty,$$

he proved that if f satisfies either $f_0 = 0$, $f_\infty = \infty$ or $f_0 = \infty$, $f_\infty = 0$, then (P_λ) has at least one positive solution, here we denote

$$f_0 \triangleq \lim_{u \to 0^+} \frac{f(u)}{u^{p-1}} \quad \text{and} \quad f_\infty \triangleq \lim_{u \to \infty} \frac{f(u)}{u^{p-1}}.$$

Kong and Wang [7] studied other types of conditions on f. When $\lambda = 1$ and h satisfies the following condition:

$$h \not\equiv 0 \text{ on any compact subinterval in } (0,1) \quad \text{with } 0 < \int_0^1 h(s)\,ds < \infty,$$

under some additional restrictions on f, they proved that if f satisfies either $f_0 = \infty$, $f_\infty = \infty$ or $f_0 = 0$, $f_\infty = 0$, then (P_λ) has at least two positive solutions.

Our main concern on the condition of f in this paper is when f satisfies $0 < f_0 < \infty$ so let us give the following assumptions first:

(A_1) $0 < f_0 < \infty$,
(A_2) $f_\infty = 0$,
(A_3) $f_\infty = \infty$.

Recently, Agarwal et al. [1] studied this case. Under the same condition on h given in [14] and some additional conditions on f, they proved that if (A_1) and (A_3) are satisfied then (P_λ) has at least one positive solution for $\lambda \in (0, \frac{1}{f_0 \varphi_p(\alpha_1)})$ and if (A_1) and (A_2) are satisfied then (P_λ) has at least one positive solution for $\lambda \in (\frac{1}{f_0 \varphi_p(\alpha_2)}, \infty)$, where constants α_i are given from the integral condition of h. Sánchez [12] also showed similar result when h satisfies the same condition given in Kong and Wang. He proved that if (A_1) and (A_3) are satisfied, then (P_λ) has at least one positive solution for $\lambda \in (0, \frac{1}{f_0 \int_0^1 h(s)\,ds})$. Although the above results give good information on parametric constants for existence, they are local with respect to the parameter. Thus it appears important to extend the local existence results to global ones, that is, the existence, multiplicity and non-existence results according to λ varying on $\mathbb{R}_+$ and this is the main goal of this paper.

We will do the global analysis not only for positive solutions but also for sign-changing solutions when f is an odd function so that we confine our indefinite weight h as follows:

(H) $h(t) \in L^1(0,1)$, $h \geqslant 0$ a.e., with $\int_I h(s)\,ds > 0$ for any compact subinterval I in $(0,1)$.

Proofs of previous results mainly make use of upper and lower solutions method, fixed point theorem and fixed point index theory on cones. For results in this paper, we employ bifurcation argument mainly making use of one of Rabinowitz type global bifurcation theorems [11].

For this purpose, we need to deal with a method called a homotopy along p which was developed by [4] for continuous scalar ODE case and then extended to PDE case by [9] and to vector ODE case by [5]. Thus in this paper, the method will be extended to singular scalar ODE case. By applying the global bifurcation theorem and figuring the shape of unbounded subcontinua of solutions, we obtain many different types of global existence results, for example, Theorems 3.5, 3.6, 3.10, 3.14, 3.15 and 3.18 for positive solutions of (P_λ) and Theorems 4.3, 4.6 and Corollaries 4.7, 4.8 for sign-changing solutions of (P_λ).

This paper is organized as follows: In Section 2, we establish a sequence $\{\mu_k(p)\}$ of eigenvalues of (P_λ) under the assumption (H) on h and $f(u)=\varphi_p(u)$. And we show the alternatives of subcontinuum which is bifurcating from $(\mu_k(p),0)$ in the sense of Rabinowitz. Furthermore, we obtain the unboundedness is the only possibility. In Sections 3 and 4, we apply the results in Section 2 to figure the shape of subcontinuum of positive solutions and sign-changing solutions with help of the generalized Picone-type identity. Finally, we conclude Section 4 applying to the radial solutions of quasilinear elliptic problems defined on annuli or exterior domains.

2. Existence of unbounded continuum

In this section, we prove the existence of unbounded subcontinuum for the following problem:

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t)\varphi_p(u(t)) + \lambda g(t,u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0)=u(1)=0. \end{cases} \tag{G_λ}$$

Let us denote $\mathcal{A} = \{q \in L^1(0,1)\colon\ q \geqslant 0$ a.e. and $\int_I q(s)\,ds > 0$, for any compact interval I in $(0,1)\}$. Throughout this paper, we assume $h \in \mathcal{A}$ without any further mention. Moreover, we assume the following hypotheses:

(H_1) there exist $\beta \in \mathcal{A}$ and $\phi \in C(\mathbb{R},\mathbb{R}_+)$ such that $|g(t,u)| \leqslant \beta(t)\phi(u)$ for all $(t,u) \in (0,1)\times\mathbb{R}$,
(H_2) $\phi(u) = o(|u|^{p-1})$ as $u \to 0$,
(H_3) $g(t,-u) = -g(t,u)$.

We first state the main theorem in this section.

Theorem 2.1. *Assume* (H_1), (H_2) *and* (H_3). *Then for each $k \in \mathbb{N}$, there exists an unbounded subcontinuum $\mathcal{C}_k$ in $\mathcal{S}$ bifurcating from $(\mu_k(p),0)$ where $\mathcal{S}$ is the closure of set of nontrivial solutions for* (G_λ) *and $\mu_k(p)$ is the kth eigenvalue of the problem*

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t)\varphi_p(u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0)=u(1)=0. \end{cases}$$

We introduce the equivalent integral operator form. Consider the problem

$$\begin{cases} \varphi_p(u'(t))' = h & \text{a.e. in } (0,1), \\ u(0)=u(1)=0. \end{cases} \tag{AP}$$

By a solution of problem (AP), we understand a function $u \in C^1[0,1]$ with $\varphi_p(u')$ absolutely continuous which satisfies (AP). Problem (AP) is equivalently written as

$$u(t) = G_p(h)(t) \triangleq \int_0^t \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds,$$

where $a : L^1(0,1) \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) dt = 0. \tag{2.1}$$

It is known that $G_p : L^1(0,1) \to C_0^1[0,1]$ is continuous and maps equi-integrable sets of $L^1(0,1)$ into relatively compact sets of $C_0^1[0,1]$. One may refer to Manásevich and Mawhin [9,10] and García-Huidobro et al. [5] for more details.

The following lemma is known as the generalized Picone identity. Let us consider the following two operators:

$$l_p[y] = \big(\varphi_p(y')\big)' + b_1(t)\varphi_p(y), \tag{2.2}$$

$$L_p[z] = \big(\varphi_p(z')\big)' + b_2(t)\varphi_p(z). \tag{2.3}$$

Lemma 2.2. [8, p. 382] *Let $b_1, b_2 \in L^1(I)$, I an interval and if y and z are functions such that $y, z, \varphi_p(y')$, and $\varphi_p(z')$ are differentiable on I and $z(t) \neq 0$ for $t \in I$. Then we have the following identity*:

$$\frac{d}{dt}\left\{\frac{|y|^p\varphi_p(z')}{\varphi_p(z)} - y\varphi_p(y')\right\} \tag{2.4}$$

$$= (b_1 - b_2)|y|^p \tag{2.5}$$

$$- \left[|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right)\right] \tag{2.6}$$

$$- y l_p(y) + \frac{|y|^p}{\varphi_p(z)} L_p(z). \tag{2.7}$$

Remark 2.3. By Young's inequality, we get

$$|y'|^p + (p-1)\left|\frac{yz'}{z}\right|^p - p\varphi_p(y)y'\varphi_p\left(\frac{z'}{z}\right) \geqslant 0,$$

and the equality holds if and only if $\operatorname{sgn} y' = \operatorname{sgn} z'$ and $|\frac{y'}{y}|^p = |\frac{z'}{z}|^p$.

Since the bifurcation points of (G_λ) is related to the eigenvalues of the problem

$$\varphi_p\big(u'(t)\big)' + \lambda h(t)\varphi_p\big(u(t)\big) = 0 \quad \text{a.e. in } (0,1), \tag{E_λ^p}$$

$$u(0) = 0 = u(1). \tag{D}$$

We summarize the eigenvalue property of problem $(\mathrm{E}_\lambda^p)+(\mathrm{D})$. Define the operator T_λ^p: $C_0^1[0,1]\to C_0^1[0,1]$ by

$$T_\lambda^p(u)(t)=G_p\big(-\lambda h\varphi_p(u)\big)(t)=\int_0^t \varphi_p^{-1}\left(a\big(-\lambda h\varphi_p(u)\big)-\int_0^s \lambda h(\tau)\varphi_p\big(u(\tau)\big)\,d\tau\right)ds.$$

Then T_λ^p is completely continuous and problem $(\mathrm{E}_\lambda^p)+(\mathrm{D})$ is equivalent to

$$u=T_\lambda^p(u).$$

When $p=2$, the eigenvalues of problem $(\mathrm{E}_\lambda^2)+(\mathrm{D})$ is known as follows:

Proposition 2.4. [2] *Let $h\in\mathcal{A}$. Then*

(i) *the set of all eigenvalues of* $(\mathrm{E}_\lambda^2)+(\mathrm{D})$ *is a countable set* $\{\mu_k(2)\mid k\in\mathbb{N}\}$ *satisfying* $0<\mu_1(2)<\mu_2(2)<\cdots<\mu_k(2)<\cdots\to\infty$,
(ii) *for each* k, $\mathrm{Ker}(I-T_{\mu_k(2)}^2)$ *is a subspace of* $C^1[0,1]$ *and its dimension is* 1,
(iii) *let u_k be a corresponding eigenfunction to $\mu_k(2)$, then the number of interior zeros of u_k is* $k-1$.

It is well known that T_λ^2 is completely continuous in $C^1[0,1]$. Thus the Leray–Schauder degree $\mathrm{d_{LS}}(I-T_\lambda^2,B_r(0),0)$ is well defined for arbitrary r-ball $B_r(0)$ and $\lambda\neq\mu_k$, $k\in\mathbb{N}$.

Lemma 2.5. *For $r>0$, we have*

$$\mathrm{d_{LS}}\big(I-T_\lambda^2,B_r(0),0\big)=\begin{cases}1, & \text{if } \lambda<\mu_1(2),\\ (-1)^k, & \text{if } \lambda\in(\mu_k(2),\mu_{k+1}(2)).\end{cases}$$

Proof. Since T_λ^2 is compact and linear, by Theorem 8.10 [3] and Proposition 2.4(ii),

$$\mathrm{d_{LS}}\big(I-T_\lambda^2,B_r(0),0\big)=(-1)^{m(\lambda)}=(-1)^k,$$

where $m(\lambda)$ is the sum of algebraic multiplicity of the eigenvalues μ satisfying $\mu^{-1}\lambda>1$ and the proof is done. □

We now introduce the eigenvalue problem for $(\mathrm{E}_\lambda^p)+(\mathrm{D})$ when $p>1$. We first notice that all eigenvalues of $(\mathrm{E}_\lambda^p)+(\mathrm{D})$ are positive. Indeed, let u be the corresponding eigenfunction to μ. Multiplying by u both sides in (E_λ^p) and integrating, we get

$$\begin{aligned}\int_0^1\big(\varphi_p\big(u'(s)\big)\big)'u(s)\,ds&=\varphi_p\big(u'(t)\big)u(t)\big|_0^1-\int_0^1\varphi_p\big(u'(s)\big)u'(s)\,ds\\&=-\int_0^1\big|u'(s)\big|^p\,ds=-\mu\int_0^1 h(s)\varphi_p\big(u(s)\big)u(s)\,ds.\end{aligned}$$

Therefore, we have

$$\mu = \frac{\int_0^1 |u'(s)|^p\, ds}{\int_0^1 h(s)|u(s)|^p\, ds} \geqslant 0.$$

If $\mu = 0$, then $u'(t) \equiv 0$ a.e. Thus by the uniqueness of initial value problem $u(t) \equiv 0$. This contradiction implies $\mu > 0$.

Combining results of [15,16], we have the following property for the eigenvalues of $(\mathrm{E}_\lambda^p) + (\mathrm{D})$. Proof of (i) is proved in [16], but we give a rough sketch of the proof for reader's convenience.

Proposition 2.6. *Assume $h \in \mathcal{A}$. Then we have*

(i) *the set of all eigenvalues of $(\mathrm{E}_\lambda^p) + (\mathrm{D})$ is a countable set $\{\mu_k(p) \mid k \in \mathbb{N}\}$ satisfying $0 < \mu_1(p) < \mu_2(p) < \cdots < \mu_k(p) < \cdots \to \infty$,*
(ii) *for each k, $\mathrm{Ker}(I - T^p_{\mu_k(p)})$ is a subspace of $C^1[0,1]$ and its dimension is 1,*
(iii) *let u_k be a corresponding eigenfunction to $\mu_k(p)$, then the number of interior zeros of u_k is $k-1$.*

Proof. Assume $\lambda > 0$. Let $\varphi_p(u') = -\lambda^{1/q} v$ in (E_λ^p), where $\frac{1}{p} + \frac{1}{q} = 1$. Then (E_λ^p) is equivalent to the system:

$$u' = -\lambda^{1/p}\varphi_q(v), \qquad v' = \lambda^{1/p} h(t)\varphi_p(u). \tag{2.8}$$

Let $(C_p(t), S_p(t))$ be a unique solution of the following initial value problem

$$\begin{cases} u' = -\varphi_q(v), & v' = \varphi_p(u), \\ u(0) = 1, & v(0) = 0. \end{cases} \tag{2.9}$$

Then introducing the polar coordinates

$$u = r^{1/p} C_p(\theta) \quad \text{and} \quad v = r^{1/q} S_p(\theta),$$

(E_λ^p) is equivalently written as

$$r' = p\lambda^{1/p}\big(h(t) - 1\big)\varphi_p\big(C_p(\theta)\big)\varphi_q\big(S_p(\theta)\big) r \triangleq R(t, \theta, r; \lambda), \tag{2.10}$$

$$\theta' = p\lambda^{1/p}\big(p^{-1} h(t)\big|C_p(\theta)\big|^p + q^{-1}\big|S_p(\theta)\big|^q\big) \triangleq \Theta(t, \theta; \lambda). \tag{2.11}$$

For fixed initial data $\theta_0 \in \mathbb{R}$, Eq. (2.11) has a unique solution $\theta(t; \theta_0, \lambda)$ satisfying $\theta(0; \theta_0, \lambda) = \theta_0$ and it can be extended on $[0, 1]$, since

$$0 \leqslant \Theta(t, \theta; \lambda) \leqslant \lambda^{\frac{1}{p}} \max\big\{h(t), 1\big\},$$

for all $t \in [0, 1]$. By [16, Lemma 2.2], we see that if $\lambda_1 > \lambda_2 > 0$ then $\theta(t; \theta_0, \lambda_1) \geqslant \theta(t; \theta_0, \lambda_2)$ for all $t \in [0, 1]$ and all θ_0. Furthermore,

$$\theta(1; \theta_0, \lambda_1) > \theta(1; \theta_0, \lambda_2). \tag{2.12}$$

In order to consider solutions of (E_λ^p) + (D), it is enough to consider solutions (u, v) of (2.8) satisfying $(u(0), v(0)) = (0, 1)$. Thus μ is an eigenvalue of problem (E_λ^p) + (D) if and only if such solution (u, v) satisfying $u(1) = 0$. Since $u(t) = r(t)^{1/p} C_p(\theta(t; \pi_p/2, \lambda))$, $u(1) = 0$ if and only if

$$\theta(1; \pi_p/2, \lambda) = \pi_p/2 + k\pi_p, \tag{2.13}$$

for some $k \in \mathbb{N}$, where $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$. By using the continuity of θ with respect to λ and the inequalities

$$\theta_0 + \lambda^{1/p} \int_0^1 h_-(s)\,ds \leqslant \theta(1; \theta_0, \lambda) \leqslant \theta_0 + \lambda^{1/p} \int_0^1 h_+(s)\,ds$$

for all $\theta_0 \in \mathbb{R}$ and $\lambda > 0$, where $h_-(t) \triangleq \min\{1, h(t)\}$ and $h_+(t) \triangleq \max\{1, h(t)\}$, we get

$$\lim_{\lambda \to 0+} \theta(1; \pi_p/2, \lambda) = \pi_p/2, \qquad \lim_{\lambda \to +\infty} \theta(1; \pi_p/2, \lambda) = +\infty.$$

By (2.12), the function $\theta(1; \pi_p/2, \lambda)$ is strictly increasing with respect to λ and thus, for each $k \in \mathbb{N}$, (2.13) has a unique solution, say, $\mu_k(p)$. This gives an eigenvalue of (E_λ^p) + (D) and completes the proof of (i).

Since $\theta(0; \pi_p/2, \mu_k(p)) = \pi_p/2$ and $\theta(1; \pi_p/2, \mu_k(p)) = \pi_p/2 + k\pi_p$ and $\theta(t; \pi_p/2, \mu_k(p))$ is strictly increasing with respect to t, $C_p(\theta(t; \pi_p/2, \mu_k(p)))$ has exactly $k-1$ interior zeros. Therefore the corresponding eigenfunction u_k of $\mu_k(p)$ also has exactly $k-1$ interior zeros and this completes the proof of (iii).

The eigenfunctions are of $C^1[0, 1]$, since $h \in L^1(0, 1)$. Suppose that u_1 and u_2 are two eigenfunctions corresponding to the same eigenvalue $\mu_k(p)$. Without loss of generality we may assume that there exists an interval $(c, d) \subset (0, 1)$, such that $u_1(c) = u_1(d) = 0$, and $u_1, u_2 > 0$ on (c, d). Integrating the generalized Picone identity on (c, d) with $y = u_1$, $z = u_2$ and $b_1 = b_2 = \mu_k(p)h(t)$, we get

$$-\left[\frac{|u_1|^p \varphi_p(u_2')}{\varphi_p(u_2)} - u_1 \varphi_p(u_1')\right]_c^d = \int_c^d |u_1'|^p + (p-1)\left|\frac{u_1 u_2'}{u_2}\right|^p - p\varphi_p(u_1) u_1' \varphi_p\left(\frac{u_2'}{u_2}\right) dt.$$

The left-hand side of the equality is 0 from $u_1(c) = 0 = u_1(d)$. Since the integrand of the right-hand side is bigger than 0 by Young's inequality, it should be 0 and thus by Remark 2.3, $\operatorname{sgn} u_1' = \operatorname{sgn} u_2'$ and $|\frac{u_1'}{u_1}|^p = |\frac{u_2'}{u_2}|^p$. This implies $\frac{u_1'}{u_1} = \frac{u_2'}{u_2}$ on (c, d). Thus $(\frac{u_1}{u_2})' \equiv 0$ and $u_1 = \mu u_2$ on (c, d) for some $\mu \in \mathbb{R}$. Since $u_1, u_2 \in C^1[0, 1]$ and u_1 and μu_2 share the same initial condition at c and d, we can extend the identity $u_1 \equiv \mu u_2$ up to the interval $(0, 1)$ by the uniqueness of the initial value problem and this completes the proof of (ii). □

Since T_λ^p is completely continuous, from Proposition 2.6, the Leray–Schauder degree $\mathrm{d_{LS}}(I - T_\lambda^p, B_r(0), 0)$ is well defined for all $r > 0$, and for $\lambda \neq \mu_k(p)$, $k \in \mathbb{N}$. But the operator T_λ^p is not linear and thus we cannot employ the same argument as in the proof of Lemma 2.5

for the computation of $d_{LS}(I - T_\lambda^p, B_r(0), 0)$. We rather use a homotopy along p. The following lemma is essential to define our homotopy.

Lemma 2.7. *For each $k \in \mathbb{N}$, $\mu_k(p)$ as a function of $p \in (1, \infty)$ is continuous.*

Proof. As we see in the proof of Proposition 2.6, $\mu_k(p)$ is defined by the equation

$$\theta\big(1; \pi_p/2, \mu_k(p)\big) = \pi_p/2 + k\pi_p.$$

Since $\theta(1; \pi_p/2, \lambda)$ is continuous and strictly increasing in λ and $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$, $\mu_k(p)$ is continuous in p. $\square$

Now, we compute $d_{LS}(I - T_\lambda^p, B_r(0), 0)$.

Lemma 2.8. *For fixed $p > 1$ and all $r > 0$, we have*

$$d_{LS}\big(I - T_\lambda^p, B_r(0), 0\big) = \begin{cases} 1, & \text{if } \lambda < \mu_1(p), \\ (-1)^k, & \text{if } \lambda \in (\mu_k(p), \mu_{k+1}(p)). \end{cases}$$

Proof. We give the proof for the case $p > 2$. Proof for the case $1 < p < 2$ is similar. We also assume $\lambda \in (\mu_k(p), \mu_{k+1}(p))$ and leave the case $\lambda < \mu_1(p)$ to the reader. By Lemma 2.7, the map $q \mapsto \mu_k(q)$ is continuous. Thus we define a continuous function $\gamma : [2, p] \to \mathbb{R}$ with $\gamma(q) = \lambda$. We denote that

$$\mu_k(q) < \gamma(q) < \mu_{k+1}(q). \tag{2.14}$$

Define

$$\mathcal{T}(q, u) = u - T_{\gamma(q)}^q(u) = u - G_q\big(-\gamma(q)h\varphi_q(u)\big) \triangleq u - \mathcal{G}(q, u).$$

Then from the obvious modification of Proposition 2.4 [5], we see that $\mathcal{G}$ is completely continuous and $\mathcal{T}$ is a compact perturbation of the identity. If there exists $u \in \partial B_r(0)$ such that $\mathcal{T}(q, u) = 0$ for some $q \in [2, p]$, then $u \not\equiv 0$ and u satisfies

$$\begin{cases} \varphi_q(u'(t))' + \gamma(q)h(t)\varphi_q(u(t)) = 0 & \text{a.e. in } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

This implies that u is an eigenvalue of $(\mathrm{E}_\lambda^q) + (\mathrm{D})$ with the corresponding eigenvalue $\gamma(q)$. This contradicts to (2.14) and thus $d_{LS}(\mathcal{T}(q, \cdot), B_r(0), 0)$ is well defined. By the property of homotopy invariance and Lemma 2.5,

$$\begin{aligned} d_{LS}\big(I - T_\lambda^p, B_r(0), 0\big) &= d_{LS}\big(I - T_{\gamma(p)}^p, B_r(0), 0\big) = d_{LS}\big(\mathcal{T}(p, \cdot), B_r(0), 0\big) \\ &= d_{LS}\big(\mathcal{T}(2, \cdot), B_r(0), 0\big) = d_{LS}\big(I - T_{\gamma(2)}^2, B_r(0), 0\big) = (-1)^k, \end{aligned}$$

since $\gamma(2) \in (\mu_k(2), \mu_{k+1}(2))$. This completes the proof. $\square$

For the existence of bifurcation branches for problem (G_λ), we will make use of the following well-known theorem.

Theorem 2.9. [13] *Let $F : \mathbb{R} \times E \to E$ be completely continuous such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that there exist constants $\rho, \eta \in \mathbb{R}$, with $\rho < \eta$, such that $(\rho, 0)$ and $(\eta, 0)$ are not bifurcation points for the equation*

$$u - F(\lambda, u) = 0. \tag{2.15}$$

Furthermore, assume that

$$\mathrm{d_{LS}}\big(I - F(\rho, \cdot), B_r(0), 0\big) \neq \mathrm{d_{LS}}\big(I - F(\eta, \cdot), B_r(0), 0\big),$$

where $B_r(0) = \{u \in E\colon \|u\|_E < r\}$ is an isolating neighborhood of the trivial solution for both constants ρ and η. Let

$$\mathcal{S} = \overline{\big\{(\lambda, u)\colon (\lambda, u) \textit{ is a solution of } (2.15) \textit{ with } u \neq 0\big\}} \cup \big([\rho, \eta] \times \{0\}\big),$$

and let $\mathcal{C}$ be the component of $\mathcal{S}$ containing $[\rho, \eta] \times \{0\}$. Then either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times E$, or*
(ii) *$\mathcal{C} \cap [(\mathbb{R}\backslash[\rho, \eta]) \times \{0\}] \neq \emptyset$.*

Define the Nemitskii operators $H_i : \mathbb{R} \times C_0^1[0, 1] \to L^1(0, 1)$ by

$$H_1(\lambda, u)(t) \triangleq -\lambda h(t)\varphi_p\big(u(t)\big) \quad \text{and} \quad H_2(\lambda, u)(t) \triangleq -\lambda g\big(t, u(t)\big),$$

respectively. Then H_i, $i = 1, 2$, are continuous operators which send bounded sets of $(0, \infty) \times C_0^1[0, 1]$ into equi-integrable sets of $L^1(0, 1)$ and problem (G_λ) can be equivalently written as

$$u = G_p \circ (H_1 + H_2)(\lambda, u) \triangleq F(\lambda, u).$$

F is completely continuous in $\mathbb{R} \times C_0^1[0, 1] \to C_0^1[0, 1]$ and $F(\lambda, 0) = 0$, $\forall \lambda \in \mathbb{R}$.

Theorem 2.10. *Assume* (H_1), (H_2) *and* (H_3). *Then for $p > 1$, each $(\mu_k(p), 0)$ is a bifurcation point of* (G_λ) *and the associated bifurcation branch $\mathcal{C}_k$ satisfies the alternatives in Theorem* 2.9.

Proof. Let $p > 1$ be given. Take $\rho = \mu_k(p) - \delta_k$ and $\eta = \mu_k(p) + \delta_k$ with sufficiently small $\delta_k > 0$ so that ρ and η are not eigenvalues of $(\mathrm{E}_\lambda^p) + (\mathrm{D})$. We shall prove that 0 is an isolated solution of (G_λ) with $\lambda = \rho, \eta$ and for sufficiently small $r > 0$,

$$\mathrm{d_{LS}}\big(I - F(\rho, \cdot), B_r(0), 0\big) = \mathrm{d_{LS}}\big(I - T_\rho^p, B_r(0), 0\big), \tag{2.16}$$

$$\mathrm{d_{LS}}\big(I - F(\eta, \cdot), B_r(0), 0\big) = \mathrm{d_{LS}}\big(I - T_\eta^p, B_r(0), 0\big). \tag{2.17}$$

To show this, specially (2.16) here, we only need to show that there exists $r > 0$, such that, for all $\tau \in [0, 1]$, the equation

$$u = J(\tau, u) \triangleq \tau T_\rho^p(u) + (1 - \tau)F(\rho, u), \tag{2.18}$$

has no nontrivial solution in $\overline{B_r(0)}$. Indeed, suppose on the contrary that there exist sequences $\{u_n\} \subset C_0^1[0,1]$ and $\{\tau_n\} \subset [0,1]$ such that $\|u_n\|_1 \to 0$ as

$$u_n = J(\tau_n, u_n).$$

Then

$$u_n(t) = \tau_n \int_0^t \varphi_p^{-1}\left(a_n + \int_0^s \rho h(\tau)\varphi_p\big(u_n(\tau)\big)\,d\tau\right) ds$$
$$+ (1-\tau_n) \int_0^t \varphi_p^{-1}\left(b_n + \int_0^s \rho h(\tau)\varphi_p\big(u_n(\tau)\big) + \rho g\big(\tau, u_n(\tau)\big)\,d\tau\right) ds,$$

where $a_n = a(\rho h\varphi_p(u_n))$, $b_n = a(\rho h\varphi_p(u_n) + \rho g(\cdot, u_n))$, and the function $a : L^1(0,1) \to \mathbb{R}$ is given at the beginning of this section. Assume $\tau_n \to \tau_0 \in [0,1]$ and let $v_n \triangleq \frac{u_n}{\|u_n\|_1}$, then we get

$$v_n(t) = \tau_n \int_0^t \varphi_p^{-1}\left(\hat{a}_n + \int_0^s \rho h(\tau)\varphi_p\big(v_n(\tau)\big)\,d\tau\right) ds$$
$$+ (1-\tau_n) \int_0^t \varphi_p^{-1}\left(\hat{b}_n + \int_0^s \rho h(\tau)\varphi_p\big(v_n(\tau)\big) + \rho \frac{g(\tau, u_n(\tau))}{\|u_n\|_1^{p-1}}\,d\tau\right) ds,$$

where $\hat{a}_n = \frac{a_n}{\|u_n\|_1^{p-1}}$ and $\hat{b}_n = \frac{b_n}{\|u_n\|_1^{p-1}}$. Since the function a is homogeneous, it is interesting to notice

$$\hat{a}_n = \frac{a_n}{\|u_n\|_1^{p-1}} = \frac{a(\rho h\varphi_p(u_n))}{\|u_n\|_1^{p-1}} = a\left(\rho h \frac{\varphi_p(u_n)}{\|u_n\|_1^{p-1}}\right) = a\big(\rho h\varphi_p(v_n)\big). \tag{2.19}$$

Similarly,

$$\hat{b}_n = a\left(\rho h\varphi_p(v_n) + \rho \frac{g(\cdot, u_n)}{\|u_n\|_1^{p-1}}\right). \tag{2.20}$$

Now we have

$$v_n'(t) = \tau_n \varphi_p^{-1}\left(\hat{a}_n + \int_0^t \rho h(s)\varphi_p\big(v_n(s)\big)\,ds\right)$$
$$+ (1-\tau_n)\varphi_p^{-1}\left(\hat{b}_n + \int_0^t \rho h(s)\varphi_p\big(v_n(s)\big) + \rho \frac{g(s, u_n(s))}{\|u_n\|_1^{p-1}}\,ds\right).$$

Since all $\{\tau_n\}, \{\hat{a}_n\}, \{\hat{b}_n\}$ are bounded, $\{v_n\}, \{u_n\}$ are uniformly bounded, from (H_1) and (H_2), we conclude that $\{v_n'\}$ is also uniformly bounded. Thus by Arzela–Ascoli theorem, $\{v_n\}$ has a uniformly convergent subsequence in $C[0,1]$. Let $v_n \to v$. By (H_2), (2.19) and (2.20) imply

$$\lim_{n\to\infty} \hat{a}_n = a\big(\rho h \varphi_p(v)\big) = \lim_{n\to\infty} \hat{b}_n.$$

By using the Lebesgue dominated convergence theorem,

$$v(t) = \int_0^t \varphi_p^{-1}\left(a\big(\rho h\varphi_p(v)\big) + \int_0^s \rho h(\tau)\varphi_p\big(v(\tau)\big)\,d\tau\right) ds.$$

This implies that ρ is an eigenvalue of (E_λ^p) + (D). This contradiction shows that there exists $r > 0$, such that (2.18) has only trivial solution in $\overline{B_r(0)}$, for all $\tau \in [0,1]$. Thus $\mathrm{d_{LS}}(I - J(\tau,\cdot), B_r(0), 0)$ is well defined for all $\tau \in [0,1]$ and by the property of homotopy invariance, we get

$$\begin{aligned}\mathrm{d_{LS}}\big(I - T_\rho^p, B_r(0), 0\big) &= \mathrm{d_{LS}}\big(I - J(1,\cdot), B_r(0), 0\big) = \mathrm{d_{LS}}\big(I - J(0,\cdot), B_r(0), 0\big)\\ &= \mathrm{d_{LS}}\big(I - F(\rho,\cdot), B_r(0), 0\big).\end{aligned} \tag{2.21}$$

Furthermore we know by Lemma 2.8 that

$$\mathrm{d_{LS}}\big(I - T_\rho^p, B_r(0), 0\big) = (-1)^{k-1}, \quad \text{for } \rho = \mu_k(p) - \delta_k.$$

Similar equality in (2.21) holds for $\lambda = \eta$ with the same ball $B_r(0)$. Since $\eta = \mu_k(p) + \delta_k$, again by Lemma 2.8, we get

$$\mathrm{d_{LS}}\big(I - T_\eta^p, B_r(0), 0\big) = (-1)^k.$$

Therefore

$$\mathrm{d_{LS}}\big(I - F(\rho,\cdot), B_r(0), 0\big) \neq \mathrm{d_{LS}}\big(I - F(\eta,\cdot), B_r(0), 0\big),$$

and the theorem is a consequence of Theorem 2.9. □

We finally prove that the first choice of the alternative of Theorem 2.9 is the only possibility. Let us denote $N_k^+ = \{u \in C_0^1[0,1]$: u has exactly $k-1$ simple zeros in $(0,1)$, $u > 0$ near 0 and all zeros of u in $[0,1]$ are simple$\}$ and let $N_k^- = -N_k^+$ and $N_k = N_k^- \cup N_k^+$.

Lemma 2.11. *If* (μ, u) *is a solution of* (G_μ) *and* u *has a double zero* (*i.e.*, $u(t) = 0 = u'(t)$ *for some* $t \in [0,1]$), *then* $u \equiv 0$.

Proof. Let u be a solution of (G_μ) and $t^* \in [0,1]$ be a double zero. Then

$$u(t) = \int_t^{t^*} \varphi_p^{-1}\left(-\lambda \int_s^{t^*} h(\tau)\big[\varphi_p\big(u(\tau)\big) + g\big(\tau, u(\tau)\big)\big]\,d\tau\right) ds. \tag{2.22}$$

By (H_1), we may choose $C_{\phi,u} > 0$ such that

$$\phi(v) \leqslant C_{\phi,u}|v|^{p-1}, \tag{2.23}$$

for all $|v| \in [0, \|u\|_\infty + 1]$. First, we consider $t \in (0, t^*)$. If

$$u(s) \leqslant \varphi_p^{-1}\left(-\lambda \int_s^{t^*} h(\tau)\left[\varphi_p\big(u(\tau)\big) + g\big(\tau, u(\tau)\big)\right] d\tau\right),$$

then from (2.23) and (H_1), we obtain

$$\varphi_p\big(\big|u(t)\big|\big) \leqslant |\lambda| \int_s^{t^*} \left[h(\tau) + C_{\phi,u} h(\tau)\right] \varphi_p\big(\big|u(\tau)\big|\big)\, d\tau.$$

By Gronwell's inequality, we get $u \equiv 0$ on $[0, t^*]$. Otherwise, we get

$$\big|u(t)\big| \leqslant \int_t^{t^*} \big|u(s)\big|\, ds.$$

Again by Gronwell's inequality, we get $u \equiv 0$ on $[0, t^*]$. Similarly, we can get $u \equiv 0$ on $[t^*, 1]$ and the proof is complete. □

By the similar arguments in Lemma 3.2 [6], we get the following lemma.

Lemma 2.12. *Let $\mathcal{C}_k$ be a subcontinuum of solutions for* (G_{μ_k}) *bifurcating from* $(\mu_k, 0)$. *Then* $\mathcal{C}_k \cap \mathbb{R} \times \{0\} \subset \bigcup_{j=1}^{\infty} \{(\mu_j, 0)\}$.

Lemma 2.13. *For each $k > 0$, there is a neighborhood $\mathcal{O}_k$ of $(\mu_k, 0)$ such that $(\lambda, u) \in \mathcal{O}_k \cap \mathcal{S}$ and $u \not\equiv 0$ implies $u \in N_k$.*

Proof. If not, then there would be a sequence $\{(\lambda_n, u_n)\} \in \mathcal{S}$ such that $u_n \not\equiv 0, u_n \notin N_k$ and $(\lambda_n, u_n) \to (\mu_k, 0)$. Thus we have

$$u_n(t) = \int_0^t \varphi_p^{-1}\left(b_n + \lambda_n \int_0^s \left[h(\tau)\varphi_p\big(u_n(\tau)\big) + g\big(\tau, u_n(\tau)\big)\right] d\tau\right) ds.$$

Let $v_n \triangleq \frac{u_n}{\|u_n\|_1}$. Then v_n satisfies

$$v_n(t) = \int_0^t \varphi_p^{-1}\left(\hat{b}_n + \lambda_n \int_0^s \left[h(\tau)\varphi_p\big(v_n(\tau)\big) + \frac{g(\tau, u_n(\tau))}{\|u_n\|_1^{p-1}}\right] d\tau\right) ds.$$

Hence, as in the proof of Theorem 2.10, $\{v_n\}$ contains a uniformly convergent subsequence which relabel as the original sequence. Let $\lim v_n = v$, with $\|v\|_1 = 1$. Then we obtain

$$v(t) = \int_0^t \varphi_p^{-1}\left(a\big(\mu_k h \varphi_p(v)\big) + \mu_k \int_0^s h(\tau)\varphi_p\big(v(\tau)\big)\, d\tau\right) ds.$$

Consequently, v is a kth eigenfunction corresponding to μ_k. Hence, $v \in N_k$. Since N_k is open, there is $0 < i_0 < k$ such that $v_n \in N_{i_0}$ for all n. Consequently, $v_n \in N_{i_0}$, $\forall n$, $v \in N_k$, $i_0 < k$ and $v_n \to v$. This is impossible and the proof is complete. □

Proof of Theorem 2.1. Since we proved the existence of the alternatives of subcontinuum, it remains to show that $\mathcal{C}_k$ is unbounded. If we show $\mathcal{C}_k \subset (\mathbb{R} \times N_k) \cup \{(\mu_k, 0)\}$, then $\mathcal{C}_k$ is unbounded by Lemma 2.13, Theorem 2.10 and by the fact $N_j \cap N_k = \emptyset$, for $j \neq k$. Suppose $\mathcal{C}_k \not\subset (\mathbb{R} \times N_k) \cup \{(\mu_k, 0)\}$. Then there exists $(\lambda, u) \in \mathcal{C}_k \cap (\mathbb{R} \times \partial N_k)$ such that $(\lambda, u) \neq (\mu_k, 0)$, $u \notin N_k$, and $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times N_k)$. Since $u \in \partial N_k$, by Lemma 2.11, $u \equiv 0$. Thus by Lemma 2.12, $\lambda = \mu_j$, $j \neq k$, and this contradicts to Lemma 2.13, since $(\lambda_n, u_n) \to (\lambda, u)$ with $(\lambda_n, u_n) \in \mathcal{C}_k \cap (\mathbb{R} \times N_k)$. □

3. The shape of subcontinuum $\mathcal{C}_1$

In this section, we sketch the shape of unbounded subcontinuum $\mathcal{C}_1$ of positive solutions for the following problem which is known to exist in Theorem 2.1:

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ}$$

where λ is a positive real parameter, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $h \in \mathcal{A}$. Assume

(A_1) $0 < f_0 < \infty$,
(A_2) $f_\infty = 0$,
(A_3) $f_\infty = \infty$.

Let us define $k : \mathbb{R} \to \mathbb{R}$ by

$$k(u) = \begin{cases} f(u), & u \geqslant 0, \\ -f(-u), & u < 0, \end{cases}$$

and consider the following problem:

$$\begin{cases} \varphi_p(u'(t))' + \lambda f_0 h(t) \varphi_p(u(t)) + \lambda h(t)[k(u(t)) - f_0 \varphi_p(u(t))] = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{H_λ}$$

It is obvious that a positive solution of problem (H_λ) is a positive solution of problem (P_λ). Assume (A_1). Then problem (H_λ) satisfies conditions (H_1)–(H_3) with $h(t) = f_0 h(t)$, $\beta(t) = h(t)$ and $\phi(u) = k(u) - f_0 \varphi_p(u)$. Thus by Theorem 2.1, ($H_\lambda$) has an unbounded subcontinuum $\mathcal{C}_k$ bifurcating from $(\mu_k(p), 0)$, where $\mu_k(p)$ is the kth eigenvalue of problem:

$$\begin{cases} \varphi_p(u'(t))' + \lambda f_0 h(t) \varphi_p(u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{E_λ}$$

We need the following lemmas to get various existence results.

Lemma 3.1. *If u is a positive solution of* (P_λ)*, then u is concave.*

Proof. Let u be a positive solution of (P_λ), then $\varphi_p(u')' < 0$. This implies $\varphi(u')$ is decreasing, and thus u' is also decreasing. It follows that u is concave. □

Lemma 3.2. *Let $b_i(t) > 0$ for $t \in (0,1)$ and $y, z \in C^1[0,1]$ satisfy the following inequalities*:

$$\varphi_p(y')' + b_1(t)\varphi_p(y) \geqslant 0, \tag{3.1}$$

$$\varphi_p(z')' + b_2(t)\varphi_p(z) \leqslant 0. \tag{3.2}$$

If $y(0) = z(0) = 0 = y(1) = z(1)$, and $y(t), z(t) > 0$, $t \in (0,1)$, then

$$\int_0^1 \left\{ \frac{|y|^p \varphi_p(z')}{\varphi_p(z)} \right\}' dt = 0. \tag{3.3}$$

Proof. We compute that

$$\int_0^1 \left\{ \frac{|y|^p \varphi_p(z')}{\varphi_p(z)} \right\}' dt = \lim_{t \to 1-} \frac{|y(t)|^p \varphi_p(z'(t))}{\varphi_p(z(t))} - \lim_{t \to 0+} \frac{|y(t)|^p \varphi_p(z'(t))}{\varphi_p(z(t))} \triangleq H_1 - H_0.$$

We prove $H_1 = H_0 = 0$. By uniqueness and concavity, $z'(1) < 0$. By L'Hospital's rule, we have

$$\begin{aligned} H_1 &= \lim_{t \to 1-} \frac{|y(t)|^p \varphi_p(z'(t))}{\varphi_p(z(t))} \geqslant \lim_{t \to 1-} \frac{p\varphi_p(y(t))y'(t)\varphi_p(z'(t)) - b_2(t)|y(t)|^p \varphi_p(z(t))}{(p-1)|z(t)|^{p-2} z'(t)} \\ &\geqslant \lim_{t \to 1-} \frac{p\varphi_p(y(t))y'(t)\varphi_p(z'(t))}{(p-1)|z(t)|^{p-2}} = \frac{p|z'(1)|^{p-2} y'(1)}{(p-1)} \lim_{t \to 1-} \frac{\varphi_p(y(t))}{|z(t)|^{p-2}}. \end{aligned}$$

Since $H_1 \leqslant 0$, if $1 < p \leqslant 2$, then $H_1 = 0$. If $2 < p \leqslant 3$, then we apply L'Hospital's rule again and we obtain

$$\lim_{t \to 1-} \frac{\varphi_p(y(t))}{|z(t)|^{p-2}} = \lim_{t \to 1-} \frac{(p-1)|y(t)|^{p-2} y'(t)}{(p-2)\varphi_{p-2}(z(t))z'(t)} = \frac{(p-1)y'(1)}{(p-2)z'(1)} \lim_{t \to 1-} \frac{|y(t)|^{p-2}}{\varphi_{p-2}(z(t))}.$$

This implies that $H_1 = 0$. If $k < p \leqslant k+1$, then we continue this process k times to obtain $H_1 = 0$.

Similarly, since $z'(0) > 0$, we have

$$\begin{aligned} H_0 &= \lim_{t \to 0+} \frac{|y(t)|^p \varphi_p(z'(t))}{\varphi_p(z(t))} \leqslant \lim_{t \to 0+} \frac{p\varphi_p(y(t))y'(t)\varphi_p(z'(t)) - b_2(t)|y(t)|^p \varphi_p(z(t))}{(p-1)|z(t)|^{p-2} z'(t)} \\ &\leqslant \lim_{t \to 0+} \frac{p\varphi_p(y(t))y'(t)\varphi_p(z'(t))}{(p-1)|z(t)|^{p-2}} = \frac{p|z'(0)|^{p-2} y'(0)}{(p-1)} \lim_{t \to 0+} \frac{\varphi_p(y(t))}{|z(t)|^{p-2}}. \end{aligned}$$

Since $H_0 \geqslant 0$, if $1 < p \leqslant 2$, then $H_0 = 0$. The repeated process completes the proof. □

From the assumption (A_2), we know that there exists $L_f > 0$ such that

$$f(u) \leqslant L_f \varphi_p(u), \quad \forall u > 0.$$

Lemma 3.3. *Assume* (A_1) *and* (A_2)*. Let u be a positive solution of* (P_λ)*. Then* $\lambda \geqslant \frac{\mu_1(p) f_0}{L_f} \triangleq \lambda_f$*, where $\mu_1(p)$ is the first eigenvalue of* (E_λ)*.*

Proof. Assume (A_1) and (A_2). Then there exists $L_f > 0$ such that $f(u) \leqslant L_f \varphi_p(u)$, for all $u > 0$. Let u be a positive solution of (P_λ). Then

$$0 = \varphi_p\bigl(u'(t)\bigr)' + \lambda h(t) f\bigl(u(t)\bigr) \leqslant \varphi_p\bigl(u'(t)\bigr)' + \lambda L_f h(t) \varphi_p\bigl(u(t)\bigr).$$

Let ϕ be an eigenfunction corresponding to the first eigenvalue $\mu_1(p)$ of (E_λ) with $\phi > 0$ on $(0,1)$. Taking $y = u, b_1(t) = \lambda L_f h(t)$ and $z = \phi, b_2(t) = \mu_1(p) f_0 h(t)$ in Lemma 2.2 and integrating (2.4)–(2.7), we have

$$\int_0^1 \bigl(\mu_1(p) f_0 h(t) - \lambda L_f h(t)\bigr)\bigl|u(t)\bigr|^p \, dt \leqslant 0.$$

Hence we have

$$\mu_1(p) f_0 - \lambda L_f \leqslant 0,$$

that is,

$$\lambda \geqslant \frac{\mu_1(p) f_0}{L_f}. \qquad \square$$

The following lemma is a priori estimate of positive solutions in $\|\cdot\|_1$ for (P_λ). It is interesting to notice that the boundedness in $\|\cdot\|_\infty$ implies the boundedness in $\|\cdot\|_1$ when $\lambda \in J$, where J is a compact interval, in fact, if u is a positive solution of (P_λ), for any $\zeta \in [0,1]$, we get

$$-\varphi_p\bigl(u'(\zeta)\bigr) = \int_y^\zeta \lambda h(s) f\bigl(u(s)\bigr) \, ds,$$

where $u'(y) = 0, \lambda \in J$.

Lemma 3.4. *Assume* (A_1) *and* (A_2)*. Let J be a compact interval in $(0,\infty)$. Then for all $\lambda \in J$, there exists $M_J > 0$ such that all possible positive solutions u of* (P_λ) *satisfy* $\|u\|_\infty \leqslant M_J$*.*

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of positive solutions of (P_λ) with $\{\lambda_n\} \subset J \triangleq [a,b]$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Let

$$\alpha \in \left(0, \frac{1}{b\varphi_p(\gamma_p Q)}\right), \quad \text{where } \gamma_p = \max\bigl\{1, 2^{\frac{2-p}{p-1}}\bigr\}, \qquad Q = \varphi_p^{-1}\left(\int_0^1 h(s)\, ds\right).$$

Then by (A_2), there exists $u_\alpha > 0$ such that $u > u_\alpha$ implies $f(u) < \alpha u^{p-1}$.

Let $m_\alpha \triangleq \max_{u\in[0,u_\alpha]} f(u)$ and let $A_n \triangleq \{t \in [0,1]\colon\ u_n(t) \leqslant u_\alpha\}$ and $B_n \triangleq \{t \in [0,1]\colon u_n(t) > u_\alpha\}$. Put $u_n(\delta_n) = \max_{u\in[0,1]} u_n(t)$. Then we have, for $0 \leqslant s \leqslant \delta_n$,

$$\begin{aligned} u_n(\delta_n) &= \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \leqslant \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_0^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \\ &\leqslant \varphi_p^{-1}(\lambda_n) \int_0^{\delta_n} \varphi_p^{-1}\left(\int_{A_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau + \int_{B_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \\ &\leqslant \varphi_p^{-1}(\lambda_n) \int_0^{\delta_n} \varphi_p^{-1}\left(m_\alpha \int_{A_n} h(\tau)\,d\tau + \int_{B_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds. \end{aligned}$$

Thus

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \leqslant \gamma_p \int_0^{\delta_n} \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}\left(\int_{B_n} \frac{h(\tau) f(u_n(\tau))}{\|u_n\|_\infty^{p-1}}\,d\tau\right)\right] ds.$$

On B_n, $u_n(s) > u_\alpha$ implies $\frac{f(u_n(s))}{\|u_n\|_\infty^{p-1}} \leqslant \frac{f(u_n(s))}{u_n^{p-1}(s)} \leqslant \alpha$. Thus

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \leqslant \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

Since $0 < a \leqslant \lambda_n \leqslant b$ for all n, we have $\frac{1}{\varphi_p^{-1}(\lambda_n)} \geqslant \frac{1}{\varphi_p^{-1}(b)}$ for all n and thus

$$\frac{1}{\varphi_p^{-1}(b)} \leqslant \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha)Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha)Q\right].$$

By the fact $\|u_n\|_\infty \to \infty$ as $n \to \infty$, we get

$$\frac{1}{\varphi_p^{-1}(b)} \leqslant \gamma_p \varphi_p^{-1}(\alpha) Q < \gamma_p \varphi_p^{-1}\left(\frac{1}{b\varphi_p(\gamma_p Q)}\right) Q < \frac{1}{\varphi_p^{-1}(b)}.$$

This contradiction completes the proof. □

Now we have the first existence result for the case (A_2).

Theorem 3.5. *Assume* (A_1) *and* (A_2). *Then there exist* $\lambda^* \geqslant \lambda_* > 0$ *such that* (P_λ) *has at least one positive solution for* $\lambda \geqslant \lambda^*$ *and no positive solution for* $\lambda < \lambda_*$.

Under the assumptions (A_1) and (A_2), it is hard to know whether $\lambda^* = \lambda_*$ or not which means the existence result of Theorem 3.5 is local. But if we add more conditions on f, then we may get some global existence results. For this purpose, we consider the following two cases for f:

(D_1) $f(u) < f_0 u^{p-1}$, $\forall u > 0$,
(D_2) there exists $\tilde{u} > 0$ such that $f(u) > f_0 u^{p-1}$, $\forall u \in (0, \tilde{u})$.

Theorem 3.6. *Assume* (A_1), (A_2) *and* (D_1). *Then* (P_λ) *has at least one positive solution for* $\lambda > \mu_1(p)$ *and no positive solution for* $\lambda \leqslant \mu_1(p)$.

Proof. Assume (A_1), (A_2) and (D_1). Then using the generalized Picone identity with $y = u$, $b_1(t) = \lambda f_0 h(t)$ and $z = \phi$, $b_2(t) = \mu_1(p) f_0 h(t)$, we can prove that (P_λ) has no positive solution for $\lambda \leqslant \mu_1(p)$. □

For the case (D_2), we assume $f(u) > 0$, $\forall u > 0$.

Lemma 3.7. *Assume* (A_1), (A_2) *and* (D_2). *If u is a positive solution of* (P_λ) *with* $\|u\|_\infty < \tilde{u}$, *then* $\lambda < \mu_1(p)$.

Proof. Using the generalized Picone identity with $z = u$, $b_2(t) = \lambda L_f h(t)$ and $y = \phi$, $b_1(t) = \mu_1(p) f_0 h(t)$, where $L_f > f_0$, we get the conclusion. □

Since $\mathcal{C}_1$ is unbounded, the situation $(\lambda, u_\lambda) \in \mathcal{C}_1$ and $\lambda \to \infty$ should be occurred by Lemma 3.4.

Lemma 3.8. *Assume* (A_1), (A_2) *and* (D_2). *If* $(\lambda, u) \in \mathcal{C}_1$ *with* $\lambda \to \infty$, *then* $\|u\|_\infty \to \infty$.

Proof. Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\} \in \mathcal{C}_1$ such that $\lambda_n \to \infty$ and $\|u_n\|_\infty$ uniformly bounded. It is enough to consider the following two cases: $\|u_n\|_\infty \to 0$ and $\|u_n\|_\infty \to d > 0$.

First, we assume $\|u_n\|_\infty \to 0$. Then by (A_1), there exists $0 < \epsilon < \frac{1}{2}$ such that $|u| \leqslant \epsilon$ implies $f(u) \geqslant \frac{f_0}{2} u^{p-1}$. Also by the property of solutions of (P_λ) [12], for any $u_n(t) \geqslant \epsilon^2 \|u_n\|_\infty$ for all $t \in [\epsilon, 1-\epsilon]$. We will get a contradiction with

$$\epsilon \triangleq \min\left\{\frac{1}{8}, \frac{\inf \delta_n}{2}\right\},$$

where $u_n(\delta_n) \triangleq \max_{t\in[0,1]} u_n(t)$. Without loss of generality, we may assume $\inf \delta_n > 0$, otherwise we can analyze exactly the same way on $[\delta_n, 1]$.

$$\begin{aligned}
u_n(\delta_n) &= \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \geqslant \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\lambda_n \int_s^{2\epsilon} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \\
&\geqslant \varphi_p^{-1}(\lambda_n) \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\int_s^{2\epsilon} h(\tau) \frac{f_0}{2} u_n^{p-1}(\tau)\,d\tau\right) ds \\
&\geqslant \varphi_p^{-1}(\lambda_n) \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\int_s^{2\epsilon} \frac{f_0}{2} \epsilon^{2(p-1)} \|u_n\|_\infty^{p-1} h(\tau)\,d\tau\right) ds \\
&\geqslant \varphi_p^{-1}(\lambda_n) \varphi_p^{-1}\left(\frac{f_0}{2}\epsilon^{2(p-1)}\right) \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\int_s^{2\epsilon} h(\tau)\,d\tau\right) ds \|u_n\|_\infty.
\end{aligned}$$

Therefore we have

$$1 \geqslant \varphi_p^{-1}(\lambda_n)\varphi_p^{-1}\left(\frac{f_0}{2}\epsilon^{2(p-1)}\right)C_h,$$

where $C_h = \min_{t\in[\epsilon,1-\epsilon]}\int_\epsilon^{2\epsilon}\varphi_p^{-1}(\int_s^{2\epsilon}h(\tau)\,d\tau)\,ds$. This is a contradiction since $\lambda_n \to \infty$.

Finally, we assume $\|u_n\|_\infty \to d > 0$. First, assume $\delta_n \to \delta \in (0,1)$. For those large n with $\|u_n\| > \frac{d}{2}$ and $\delta_n > \frac{\delta}{2}$, consider the triangle with vertices $(0,0)$, $(\frac{5}{4}\delta, \frac{d}{2})$ and $(1,0)$. Then $u_n(\frac{\delta}{4}) \geqslant \frac{d}{10}$ and by the concavity of u_n and $\|u_n\| \to d$, we have $\frac{d}{10} \leqslant u_n(t) \leqslant 2d$, $t \in [\frac{\delta}{4}, \frac{\delta}{2}]$ (this is true whether $\frac{5}{4}\delta \leqslant 1$ or not). Thus we have

$$\begin{aligned} u_n(\delta_n) &= \int_0^{\delta_n}\varphi_p^{-1}\left(\lambda_n\int_s^{\delta_n}h(\tau)f\big(u_n(\tau)\big)\,d\tau\right)ds \geqslant \int_{\delta/4}^{\delta/2}\varphi_p^{-1}\left(\lambda_n\int_s^{\delta/2}h(\tau)f\big(u_n(\tau)\big)\,d\tau\right)ds \\ &\geqslant \varphi_p^{-1}(C_d)\int_{\delta/4}^{\delta/2}\varphi_p^{-1}\left(\int_s^{\delta/2}h(\tau)\,d\tau\right)ds\cdot\varphi_p^{-1}(\lambda_n), \end{aligned}$$

where $C_d = \min_{\frac{d}{10}\leqslant u\leqslant 2d} f(u)$. This is a contradiction. Next, assume $\delta = 0$ (the proof for the case of $\delta = 1$ is similar). Then for those large n with $\|u_n\|_\infty > \frac{d}{2}$ and $\delta_n < \frac{1}{8}$, we get $\frac{d}{16} \leqslant u_n(t) \leqslant 2d$, $t \in [\frac{3}{4}, \frac{7}{8}]$, by using the triangle with vertices $(0,0)$, $(0, \frac{d}{2})$ and $(1,0)$. Thus

$$\begin{aligned} u_n(\delta_n) &= \int_0^{\delta_n}\varphi_p^{-1}\left(\lambda_n\int_s^{\delta_n}h(\tau)f\big(u_n(\tau)\big)\,d\tau\right)ds \geqslant \int_{3/4}^{7/8}\varphi_p^{-1}\left(\lambda_n\int_s^{7/8}h(\tau)f\big(u_n(\tau)\big)\,d\tau\right)ds \\ &\geqslant \varphi_p^{-1}(\widehat{C}_d)\int_{3/4}^{7/8}\varphi_p^{-1}\left(\int_s^{7/8}h(\tau)\,d\tau\right)ds\cdot\varphi_p^{-1}(\lambda_n), \end{aligned}$$

where $\widehat{C}_d = \min_{\frac{d}{16}\leqslant u\leqslant 2d} f(u)$. This contradiction completes the proof. □

It is interesting to notice that the shape of $\mathcal{C}_1$ is not effected by the shape of f once f satisfies (A_2).

Lemma 3.9. *Assume* (A_1), (A_2) *and* (D_2). *If* $(P_{\bar\lambda})$ *has a positive solution for some* $\bar\lambda > 0$, *then* (P_λ) *also has a positive solution for all* $\lambda \in (\bar\lambda, \infty)$.

Proof. It is enough to show that (P_λ) has a positive solution for $\lambda \in (\bar\lambda, \lambda^*)$ by Theorem 3.5. Let $\lambda \in (\bar\lambda, \lambda^*)$ and $u_{\bar\lambda}$ be a positive solution of $(P_{\bar\lambda})$. Then obviously $u_{\bar\lambda}$ is a lower solution of (P_λ). Since $u_{\bar\lambda}$ is positive, concave and of $C^1[0,1]$, we may assume $u'_{\bar\lambda}(0^+) = a > 0$, and $u'_{\bar\lambda}(1^-) = b < 0$. Then we can choose N big enough so that $N > \lambda$ and $\|u_N\|_\infty > \max\{a, -b\}$ by Lemma 3.8 and we get $u_N(t) > \bar u(t)$, $\forall t \in (0,1)$ by comparing two triangles with vertices

$(0,0)$, $(\delta_n, \|u_\lambda\|_\infty)$, $(1,0)$ and $(0,0)$, $(\frac{-b}{a-b}, \frac{-ab}{a-b})$, $(1,0)$, respectively. Obviously, u_N is an upper solution of (P_λ) and the proof is done. □

Theorem 3.10. *Assume* (A_1), (A_2) *and* (D_2). *Then there exist* $\lambda_0 \leqslant \lambda^* < \mu_1(p)$ *such that* (P_λ) *has at least two positive solutions for* $\lambda \in (\lambda^*, \mu_1(p))$, *one positive solution for* $\lambda \in (\mu_1(p), \infty) \cup [\lambda_0, \lambda^*]$, *and no positive solutions for* $\lambda \in (0, \lambda_0)$.

Proof. Let $\lambda_0 \triangleq \inf\{\lambda > 0\colon (\mathrm{P}_\lambda)$ has a positive solution$\}$. Then $\lambda^* \geqslant \lambda_0 > \frac{\mu_1(p) f_0}{L_f}$ by Lemma 3.3. If $\lambda^* = \lambda_0$, then the proof is done. If $\lambda_0 < \lambda^*$, then by Lemma 3.9, we know that (P_λ) has at least one positive solution for all $\lambda > \lambda_0$. We complete the proof by showing the existence of positive solution at λ_0. Once again, by Lemma 3.9, there exist a sequence $\{\lambda_n\}$ and $\{u_n\}$ such that $\lambda_n \to \lambda_0$ and u_n which is a corresponding solution to λ_n of (P_λ) satisfies $u_n = G_p(\lambda_n h f(u_n))$. It follows from Lemma 3.4 that $\{u_n\}$ is bounded. Since G_p is compact, $\{u_n\}$ has convergent subsequence and we may suppose that converges to u_0. Since G_p is continuous, we have $u_0 = G_p(\lambda_0 h f(u_0))$. This completes the proof. □

Now we consider the case that f satisfies (A_3). In what is to follow, we assume $f(u) > 0$, for all $u > 0$.

Lemma 3.11. *Assume* (A_1) *and* (A_3). *Let u be a positive solution of* (P_λ). *Then* $\lambda \leqslant \lambda_*$ *for some* $\lambda_* \geqslant \mu_1(p)$.

Proof. From the conditions on f, we may choose $0 < \widetilde{L}_f \leqslant f_0$ such that $f(u) \geqslant \widetilde{L}_f u^{p-1}$, for all $u > 0$. Let u be a positive solution of (P_λ). Then

$$0 = \varphi_p\big(u'(t)\big)' + \lambda h(t) f\big(u(t)\big) \geqslant \varphi_p\big(u'(t)\big)' + \lambda \widetilde{L}_f h(t) \varphi_p\big(u(t)\big).$$

Once again, let ϕ be an eigenfunction corresponding to the first eigenvalue $\mu_1(p)$ of (E_λ) with $\phi > 0$ on $(0,1)$. Taking $y = \phi$, $b_1(t) = \mu_1(p) f_0 h(t)$ and $z = u$, $b_2(t) = \lambda \widetilde{L}_f h(t)$ in Lemma 3.2 and integrating (2.4)–(2.7), we have

$$\int_0^1 \big(\lambda \widetilde{L}_f h(t) - \mu_1(p) f_0 h(t)\big) \big|u(t)\big|^p \, dt \leqslant 0.$$

Hence we have

$$\lambda \widetilde{L}_f - \mu_1(p) f_0 \leqslant 0,$$

that is,

$$\lambda \leqslant \frac{\mu_1(p) f_0}{\widetilde{L}_f} \triangleq \lambda_*.$$

Since $f_0 \geqslant \widetilde{L}_f$, $\lambda_* \geqslant \mu_1(p)$ and this completes the proof. □

Lemma 3.12. *Assume* (A_1) *and* (A_3). *Let J be a compact interval in* $(0, \infty)$. *Then for all* $\lambda \in J$, *there exists* $b_J > 0$ *such that all possible positive solutions u of* (P_λ) *satisfy* $\|u\|_\infty \leqslant b_J$.

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of positive solutions for (P_{λ_n}) with $\{\lambda_n\} \subset J$ and $\|u_n\|_\infty \to \infty$. By property (b) of Lemma 1 in [12], for any $0 < \epsilon < \frac{1}{2}$, $u_n(t) \geqslant \epsilon^2 \|u_n\|_\infty$ for all $t \in [\epsilon, 1-\epsilon]$. Choose

$$\epsilon \triangleq \min\left\{\frac{1}{8}, \frac{\inf \delta_n}{2}\right\},$$

where $u_n(\delta_n) \triangleq \max_{t\in[0,1]} u_n(t)$. As in the proof of Lemma 3.8, we may suppose $\inf \delta_n > 0$. Then from (A_3), we may choose $R_1 > 0$ such that $f(u) \geqslant \eta u^{p-1}$, for $u \geqslant R_1$ and for some $\eta > 0$ and $\varphi_p^{-1}(\lambda_n)\varphi_p^{-1}(\eta\epsilon^{2(p-1)})C_h$, where $C_h = \min_{t\in[\epsilon,1-\epsilon]} \int_\epsilon^{2\epsilon} \varphi_p^{-1}(\int_\epsilon^{2\epsilon} h(\tau)\,d\tau)\,ds$. Since $\|u_n\|_\infty \to \infty$, $\|u_n\|_\infty > \frac{R_1}{\epsilon^2}$ for sufficiently large n. Thus $u_n(t) \geqslant \epsilon^2\|u_n\|_\infty > R_1$, for $t \in [\epsilon, 1-\epsilon]$ and we get for $0 \leqslant s \leqslant \delta_n$,

$$u_n(\delta_n) = \int_0^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \geqslant \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\lambda_n \int_s^{2\epsilon} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds$$

$$\geqslant \varphi_p^{-1}\big(\eta\lambda_n\epsilon^{2(p-1)}\big) \int_\epsilon^{2\epsilon} \varphi_p^{-1}\left(\int_s^{2\epsilon} h(\tau)\,d\tau\right) ds \cdot \|u_n\|_\infty.$$

Therefore we have

$$1 \geqslant \varphi_p^{-1}(\lambda_n)\varphi_p^{-1}\big(\eta\epsilon^{2(p-1)}\big)C_h.$$

This is a contradiction to the choice of η. □

Remark 3.13. Assume (A_1) and (A_3). Then $(\lambda_n, u_n) \in \mathcal{C}_1$ with $\|u_n\|_\infty \to \infty$ implies $\lambda_n \to 0$.

We have obtained the shape of subcontinuum $\mathcal{C}_1$ from Lemmas 3.11 and 3.12, and Remark 3.13.

Theorem 3.14. *Assume* (A_1) *and* (A_3). *Then there exist* $\lambda_* \geqslant \lambda^* > 0$ *such that* (P_λ) *has at least one positive solution for* $\lambda < \lambda^*$ *and no positive solution for* $\lambda > \lambda_*$.

For some global existence results, we consider the following two cases on f:

(S_1) $f(u) > f_0 u^{p-1}, \forall u > 0$,
(S_2) there exists $\tilde{u} > 0$ such that $f(u) < f_0 u^{p-1}, \forall u \in (0, \tilde{u})$.

Theorem 3.15. *Assume* (A_1), (A_3) *and* (S_1). *Then* $\lambda_* = \lambda^* = \mu_1(p)$ *in Theorem* 3.14. *More precisely,* (P_λ) *has at least one positive solution for* $\lambda \in (0, \mu_1(p))$ *and no positive solution for* $\lambda \in [\mu_1(p), \infty)$.

Proof. Using the generalized Picone identity with $y = \phi$, $b_1(t) = \mu_1(p) f_0 h(t)$ and $z = u$, $b_2(t) = \lambda f_0 h(t)$, we get the conclusion. □

Lemma 3.16. *Assume* (A_1), (A_3) *and* (S_2). *If* (P_λ) *has a positive solution with* $\|u\|_\infty < \tilde{u}$, *then* $\lambda > \mu_1(p)$.

Proof. Using the generalized Picone identity with $y = \phi$, $b_1(t) = \mu_1(p) f_0 h(t)$ and $z = u$, $b_2(t) = \lambda \widetilde{L}_f h(t)$, where $\widetilde{L}_f < f_0$, we get the conclusion. □

Lemma 3.17. *Assume* (A_1), (A_3), *and* (S_2). *If* $(P_{\bar\lambda})$ *has a positive solution for some* $\bar\lambda > 0$. *Then* (P_λ) *has a positive solution for all* $\lambda \in (0, \bar\lambda)$.

Proof. For fixed $\lambda_0 \in (\lambda_*, \bar\lambda)$, it is enough to show that (P_{λ_0}) has a positive solution. Obviously, $u_{\bar\lambda}$ is an upper solution of (P_λ). We will find a lower solution of (P_λ) less than $u_{\bar\lambda}$. By Lemma 3.16, we may consider a positive solution u_λ of (P_λ) such that $\|u_\lambda\|_\infty \to 0$, as $\lambda \to \mu_1$. Let $u'_{\bar\lambda}(0^+) = a > 0$ and $u'_{\bar\lambda}(1^-) = b < 0$. Then for A_u the maximum point of u_λ on $[0,1]$, we get

$$u'_\lambda(t) = \varphi_p^{-1}\left(\lambda \int_t^{A_{u_\lambda}} h(s) f\big(u_\lambda(s)\big)\, ds\right) \leqslant \varphi_p^{-1}\left(\lambda \int_0^1 h(s)\, ds \cdot \|f \circ u_\lambda\|_\infty\right),$$
$$\text{for } 0 \leqslant t \leqslant A_{u_\lambda},$$

and $u'_\lambda(t) \geqslant -\varphi_p^{-1}(\lambda \int_0^1 h(s)\, ds \cdot \|f \circ u_\lambda\|_\infty)$, for $A_{u_\lambda} \leqslant t \leqslant 1$. Since $\|u_\lambda\|_\infty$ small enough for λ near $\mu_1(p)$ and f continuous with $f(0) = 0$, $u'_\lambda(0^+) < a$, and $u'_\lambda(1^-) > b$, for λ close enough to $\mu_1(p)$. Thus by the continuity of u'_λ and $u'_{\bar\lambda}$, we get

$$u_\lambda(t) < u_{\bar\lambda}(t), \quad \text{for } t \in (0, \delta_1] \cup [1 - \delta_1, 1). \tag{3.4}$$

Since $\|u_\lambda\|_\infty \to 0$ as $\lambda \to \mu_1(p)$, we may choose u_λ sufficiently small so that

$$u_\lambda(t) < u_{\bar\lambda}(t), \quad \text{on } [\delta_1, 1 - \delta_1]. \tag{3.5}$$

Choosing λ close enough to $\mu_1(p)$ at which the solution u_λ satisfying both (3.4) and (3.5), we get a lower solution of (P_{λ_0}) such that $u_\lambda < u_{\bar\lambda}$ and the proof is done. □

Theorem 3.18. *Assume* (A_1) *and* (A_3). *Also assume* (S_2) *then there exist* $\lambda_0 \geqslant \lambda^* \geqslant \mu_1(p)$ *such that* (P_λ) *has at least two positive solutions for* $\lambda \in (\mu_1(p), \lambda^*)$, *one positive solution for* $\lambda \in (0, \mu_1(p)] \cup [\lambda^*, \lambda_0]$, *and no positive solutions for* $\lambda \in (\lambda_0, \infty)$.

Proof. Let $\lambda_0 \triangleq \sup\{\lambda > 0\colon (P_\lambda)$ has a positive solution$\}$. Then $\lambda^* \leqslant \lambda_0 < \frac{\mu_1(p) f_0}{\widetilde{L}_f}$ by Lemma 3.11. If $\lambda^* = \lambda_0$, then the proof is done. If $\lambda_0 > \lambda^*$, then by Lemma 3.16, we know that (P_λ) has at least one positive solution for all $\lambda < \lambda_0$. The remaining part of proof is the same as in the proof of Theorem 3.10. □

Remark 3.19. The results in Theorems 3.10 and 3.18 are partial, we leave a question when $\lambda^* = \lambda_0$.

4. The shape of subcontinuum $\mathcal{C}_k$, $k \geqslant 2$

In this section, we sketch the shape of unbounded subcontinuum $\mathcal{C}_k$ which was guaranteed to exist in Section 2 of the solutions for the following problem:

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{Q_λ}$$

where λ is a positive real parameter, $f \in C(\mathbb{R}, \mathbb{R})$ and f is an odd function with $f(u) \geqslant 0$ for all $u \geqslant 0$ and $h \in \mathcal{A}$. Assume

(A$_1$) $0 < f_0 < \infty$,
(A$_2$) $f_\infty = 0$,
(A$_3$) $f_\infty = \infty$.

Changing the problem (Q$_\lambda$), we have

$$\begin{cases} \varphi_p(u'(t))' + \lambda f_0 h(t) \varphi_p(u(t)) + \lambda h(t)[f(u(t)) - f_0 \varphi_p(u(t))] = 0 & \text{a.e. in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{R_λ}$$

Assume (A$_1$). Then problem (R$_\lambda$) satisfies conditions (H$_1$)–(H$_3$) with $h(t) = f_0 h(t)$, $\beta(t) = h(t)$ and $\phi(u) = f(u) - f_0 \varphi_p(u)$. Thus by Theorem 2.1 (H$_\lambda$) has an unbounded subcontinuum $\mathcal{C}_k$ bifurcating from $(\mu_k(p), 0)$, where $\mu_k(p)$ is the kth eigenvalue of problem (E$_\lambda$). From assumption (A$_2$), we know that there exists $L_f > 0$ such that

$$f(u) \leqslant L_f u^{p-1}, \quad \forall u \geqslant 0.$$

Lemma 4.1. *Assume* (A$_1$) *and* (A$_2$). *Let u_k be a solution in $\mathcal{C}_k$ of* (Q$_\lambda$). *Then $\lambda \geqslant \frac{\mu_k(p) f_0}{L_f}$, for each $k \geqslant 2$.*

Proof. Let u_k be a solution in $\mathcal{C}_k$ of (Q$_\lambda$) and ϕ_k be an eigenfunction corresponding to the kth eigenvalue $\mu_k(p)$ of (E$_\lambda$). And let t_1^* and t_1 be the first zero of ϕ_k and u_k, with $\phi_k > 0$ in $(0, t_1^*)$ and $u_k > 0$ in $(0, t_1)$, and t_{k-1}^* and t_{k-1} be the last zero of ϕ_k and u_k, respectively.

Case (I), $k = 2$. Suppose $t_1 \leqslant t_1^*$. Then it is easy to show these equalities:

$$\int_0^{t_1} \left\{ \frac{|u_2|^p {\phi'}_2^{(p-1)}}{\phi_2^{(p-1)}} \right\}' dt = 0 \quad \text{and} \quad \int_0^{t_1} -\left\{ u_2 {u'}_2^{(p-1)} \right\}' dt = 0.$$

Since

$$0 = \varphi_p\big(u_2'(t)\big)' + \lambda h(t) f\big(u_2(t)\big) \leqslant \varphi_p\big(u_2'(t)\big)' + \lambda L_f h(t) \varphi_p\big(u_2(t)\big) \quad \text{a.e. in } (0, t_1), \quad \text{and}$$
$$0 = \varphi_p\big(\phi_2'(t)\big)' + \mu_2(p) f_0 h(t) \varphi_p\big(\phi_2(t)\big) \quad \text{a.e. in } (0, t_1),$$

if we take $y = u_2, b_1(t) = \lambda L_f g(t)$ and $z = \phi_2$, $b_2(t) = \mu_2(p) f_0 g(t)$ and integrate (2.4)–(2.7) from 0 to t_1, as in the proof of Lemma 3.3, then we obtain

$$\mu_2(p) f_0 - \lambda L_f \leqslant 0, \quad \text{that is,}$$
$$\lambda \geqslant \frac{\mu_2(p) f_0}{L_f}.$$

Suppose $t_1^* \leqslant t_1$. Then it is easy to show

$$\int_{t_1}^{1} \left\{ \frac{|u_2|^p {\phi'}_2^{(p-1)}}{\phi_2^{(p-1)}} \right\}' dt = 0 \quad \text{and} \quad \int_{t_1}^{1} -\left\{ u_2 {u'}_2^{(p-1)} \right\}' dt = 0.$$

Since

$$0 = \varphi_p\big(u_2'(t)\big)' + \lambda h(t) f\big(u_2(t)\big) \geqslant \varphi_p\big(u_2'(t)\big)' + \lambda L_f h(t) \varphi_p\big(u_2(t)\big) \quad \text{a.e. in } (t_1, 1),$$
$$0 = \varphi_p\big(\phi_2'(t)\big)' + \lambda f_0 h(t) \varphi_p\big(\phi_2(t)\big), \quad \text{a.e. in } (t_1, 1), \quad \text{and}$$
$$-u_2\big[\varphi_p\big(u_2'(t)\big)' + \lambda L_f h(t) \varphi_p\big(u_2(t)\big)\big] \leqslant 0 \quad \text{a.e. in } (t_1, 1),$$

in (2.7), if we take $y = u_2, b_1(t) = \lambda L_f g(t)$ and $z = \phi_2, b_2(t) = \mu_2(p) f_0 g(t)$ and integrate (2.4)–(2.7) from t_1 to 1, then we obtain

$$\lambda \geqslant \frac{\mu_2(p) f_0}{L_f}.$$

Case (II), $k \geqslant 3$. If $t_1 \leqslant t_1^*$ or $t_{k-1}^* \leqslant t_{k-1}$, then we obtain $\lambda \geqslant \frac{\mu_k(p) f_0}{L_f}$ by the same process as in case (I). If $t_1^* < t_1$ and $t_{k-1} < t_{k-1}^*$, then there exists an interval $(t_i, t_{i+1}) \subset (t_i^*, t_{i+1}^*)$ for some i, $1 < i < k$, and we have

$$\int_{t_i}^{t_{i+1}} \left\{ \frac{|u_k|^p {\phi'}_k^{(p-1)}}{\phi_k^{(p-1)}} - u_2 {u'}_2^{(p-1)} \right\}' dt = 0.$$

Either $u_k > 0$ in (t_i, t_{i+1}) or $u_k < 0$ in (t_i, t_{i+1}), we have

$$-u_k\big[\varphi_p\big(u_k'(t)\big)' + \lambda L_f h(t) \varphi_p\big(u_k(t)\big)\big] \leqslant 0.$$

Thus following the argument in case (I), we get

$$\lambda \geqslant \frac{\mu_k(p) f_0}{L_f}. \qquad \square$$

Lemma 4.2. *Assume* (A_1) *and* (A_2). *Let* $J = [a, b]$ *be a compact interval in* $(0, \infty)$. *Then for all* $\lambda \in J$, *there exists* $M_J > 0$ *such that all possible solutions* u *in* $\mathcal{C}_k$ *of* (Q_λ) *satisfy* $\|u\|_\infty \leqslant M_J$.

Proof. Suppose on the contrary that there exists a sequence $\{u_n\}$ of solutions of (Q_λ) with $\{\lambda_n\} \subset J = [a, b]$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Let $\alpha \in (0, \frac{1}{b\varphi_p(\gamma_p Q)})$, where $\gamma_p = \max\{1, 2^{\frac{2-p}{p-1}}\}$, $Q = \varphi_p^{-1}(\int_0^1 h(s)\,ds)$. Then by (A_2), there exists $u_\alpha > 0$ such that $u > u_\alpha$ implies $f(u) < \alpha u^{p-1}$.

Let $m_\alpha \triangleq \max_{u\in[0,u_\alpha]} f(u)$ and let $z_{1,n}, z_{2,n}, \ldots, z_{k-1,n}$ denote the zeros of u_n in $(0, 1)$ and let $A_n \triangleq \{t \in [0, 1]\colon\ u_n(t) \leqslant u_\alpha\}$ and $B_n \triangleq \{t \in [0, 1]\colon\ u_n(t) > u_\alpha\}$. Put $u_n(\delta_n) = \max_{u\in[0,1]} u_n(t)$ (δ_n may not be unique). Then we can choose $[z_{j,n}, z_{j+1,n}] \ni \delta_n$, for some $j \in \{0, 1, \ldots, k-1\}$ and $f(u([z_{j,n}, z_{j+1,n}])) \geqslant 0$. For $z_{j,n} \leqslant s \leqslant \delta_n$, we have

$$\begin{aligned}
u_n(\delta_n) &= \int_{z_{j,n}}^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \leqslant \int_{z_{j,n}}^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_{z_{j,n}}^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \\
&\leqslant \varphi_p^{-1}(\lambda_n) \int_{z_{j,n}}^{\delta_n} \varphi_p^{-1}\left(\int_{A_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau + \int_{B_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \\
&\leqslant \varphi_p^{-1}(\lambda_n) \int_{z_{j,n}}^{\delta_n} \varphi_p^{-1}\left(m_\alpha \int_{A_n} h(\tau)\,d\tau + \int_{B_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds.
\end{aligned}$$

Thus

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \leqslant \gamma_p \int_{z_{j,n}}^{\delta_n} \varphi_p^{-1}\left(\frac{\varphi_p^{-1}(m_\alpha) Q}{\|u_n\|_\infty}\right) + \varphi_p^{-1}\left(\int_{B_n} \frac{h(\tau) f(u_n(\tau))}{\|u_n\|_\infty^{p-1}}\,d\tau\right) ds.$$

On B_n, $u_n(s) > u_\alpha$ implies $\frac{f(u_n(s))}{\|u_n\|_\infty^{p-1}} \leqslant \frac{f(u_n(s))}{u_n^{p-1}(s)} \leqslant \alpha$. Thus

$$\frac{1}{\varphi_p^{-1}(\lambda_n)} \leqslant \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha) Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha) Q\right].$$

Since $0 < a \leqslant \lambda_n \leqslant b$ for all n, we have $\frac{1}{\varphi_p^{-1}(\lambda_n)} \geqslant \frac{1}{\varphi_p^{-1}(b)}$ for all n and thus

$$\frac{1}{\varphi_p^{-1}(b)} \leqslant \gamma_p \left[\frac{\varphi_p^{-1}(m_\alpha) Q}{\|u_n\|_\infty} + \varphi_p^{-1}(\alpha) Q\right].$$

By the fact $\|u_n\|_\infty \to \infty$ as $n \to \infty$, we get

$$\frac{1}{\varphi_p^{-1}(b)} \leqslant \gamma_p \varphi_p^{-1}(\alpha) Q < \gamma_p \varphi_p^{-1}\left(\frac{1}{b\varphi_p(\gamma_p Q)}\right) Q < \frac{1}{\varphi_p^{-1}(b)}.$$

This contradiction completes the proof. □

Now we have the first existence result of problem (Q_λ).

Theorem 4.3. *Assume* (A_1) *and* (A_2). *Then there exists* λ^* *with* $0 < \lambda^* \leqslant \mu_k(p)$ *such that problem* (Q_λ) *has at least one sign-changing solution for all* $\lambda > \lambda^*$.

Let us consider the case (A_3).

Lemma 4.4. *Assume* (A_1) *and* (A_3). *Let u be a solution of* (Q_λ). *Then there exists* $\lambda_0 \geqslant \mu_k(p)$ *such that* $\lambda \leqslant \lambda_0$.

Proof. Using arguments in Lemma 4.1 and the generalized Picone identity with $y = \phi_k$, $b_1(t) = \mu_k(p) f_0 h(t)$ and $z = u_k$, $b_2(t) = \lambda f_0 g(t)$ in (2.4)–(2.7), we get the conclusion. □

Lemma 4.5. *Assume* (A_1) *and* (A_3). *Let J be a compact interval in* $(0, \infty)$. *Then for all* $\lambda \in J$, *there exists* $b_J > 0$ *such that all possible solutions u of* (Q_λ) *satisfy* $\|u\|_\infty \leqslant b_J$.

Proof. Suppose on the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ of solutions of (Q_λ) with $\lambda_n \in J$, $u_n \in \mathcal{C}_k$, and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Let $z_{1(n),n}, z_{2(n),n}, \ldots, z_{(k-1)(n),n}$ denote the zeros of u_n in $(0, 1)$. At least one subinterval $(z_{j(n),n}, z_{(j+1)(n),n}) \triangleq I_n$ is of length at least $\frac{1}{k}$. In fact, $\{\max_{I_n} |u_n|\}$ is an unbounded sequence. Assume that $\{\max_{I_n} |u_n|\}$ is uniformly bounded. Clearly, since u_n is concave (or convex) in I_n, u_n' has one zero y_n in I_n. Integrating (Q_λ), for any $\zeta \in I_n$,

$$\int_{y_n}^{\zeta} -\varphi_p\big(u_n'(s)\big)'\,ds = \int_{y_n}^{\zeta} \lambda_n h(s) f\big(u_n(s)\big)\,ds, \quad \text{that is,}$$

$$-\varphi_p\big(u_n'(\zeta)\big) = \int_{y_n}^{\zeta} \lambda_n h(s) f\big(u_n(s)\big)\,ds.$$

Hence $\{\max_{I_n} |u_n'|\}$ is uniformly bounded. Consider an interval

$$Q_n = (z_{(j-1)(n),n}, z_{j(n),n}) \quad \text{or} \quad Q_n = (z_{(j+1)(n),n}, z_{(j+2)(n),n}).$$

By convexity (or concavity) of u_n on Q_n and the uniform boundedness of $\{u_n'\}$ on I_n, $\{\max_{Q_n} |u_n|\colon\ Q_n = (z_{(j-1)(n),n}, z_{j(n)})$ or $Q_n = (z_{(j+1)(n),n}, z_{(j+2)(n),n})\}$ is uniformly bounded. In $k-1$ steps, this procedure shows that $\|u_n\|_\infty$ is uniformly bounded. Put $\lim z_{j(n),n} = z_{j_0}$ and $\lim z_{(j+1)(n),n} = z_{j_0+1}$ and $\lim \delta_n = \delta$, where $u_n(\delta_n) = \max_{I_n} u_n(t)$. Without loss of generality, we may assume that $z_{j_0} < \delta < z_{j_0+1}$ (the cases of $z_{j_0} = \delta$ or $z_{j_0+1} = \delta$ can be considered similarly as in the proof of Lemma 3.8). Then the concavity of u_n implies the property (b) of Lemma 1 in [12], for any $0 < \epsilon < \frac{\delta - z_{j_0}}{4}$, $u_n(t) \geqslant m\epsilon^2 \|u_n\|_\infty$ (from now on, $\|u_n\|_\infty$ denotes $\|u_n\|_\infty$ on I_n) for all $t \in [z_{j(n),n} + \epsilon, z_{(j+1)(n),n} - \epsilon] \triangleq J_n$, where $m = \min\{\frac{2}{k(\delta - z_{j_0})}, \frac{2}{k(z_{j_0+1} - \delta)}\}$. From (A_3), we may choose $R_1 > 0$ such that $f(u) \geqslant \eta u^{p-1}$, for $u \geqslant R_1$ and for some $\eta > 0$. Since $\|u_n\|_\infty \to \infty$, $\|u_n\|_\infty > \frac{R_1}{m\epsilon^2}$ for sufficiently large n. Thus $u_n(t) \geqslant m\epsilon^2 \|u_n\|_\infty > R_1$, for $t \in J_n$ and we get

$$u_n(\delta_n) = \int_{z_{j(n),n}}^{\delta_n} \varphi_p^{-1}\left(\lambda_n \int_s^{\delta_n} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds \geqslant \int_{z_{j_0}+\epsilon}^{z_{j_0}+2\epsilon} \varphi_p^{-1}\left(\lambda_n \int_s^{z_{j_0}+2\epsilon} h(\tau) f\big(u_n(\tau)\big)\,d\tau\right) ds$$

$$\geqslant \varphi_p^{-1}(\lambda_n) \int\limits_{z_{j_0}+\epsilon}^{z_{j_0}+2\epsilon} \varphi_p^{-1}\left(\int\limits_s^{z_{j_0}+2\epsilon} \eta u_n^{p-1} h(\tau)\, d\tau \right) ds$$

$$\geqslant \varphi_p^{-1}\big(\eta m^{(p-1)} \epsilon^{2(p-1)} \lambda_n\big) \int\limits_{z_{j_0}+\epsilon}^{z_{j_0}+2\epsilon} \varphi_p^{-1}\left(\int\limits_s^{z_{j_0}+2\epsilon} h(\tau)\, d\tau \right) ds \cdot \|u_n\|_\infty.$$

This is a contradiction since $\lambda_n \to \infty$ and completes the proof. □

We have obtained the shape of subcontinuum $\mathcal{C}_k$ as in Theorem 3.14.

Theorem 4.6. *Assume* (A_1) *and* (A_3). *Then there exist* $\lambda_* \geqslant \lambda^* > 0$ *such that* (Q_λ) *has at least one positive solution for* $\lambda < \lambda^*$ *and no positive solution for* $\lambda > \lambda_*$.

We have some global results.

Corollary 4.7. *Assume* (A_1) *and* (A_3). *If* f *satisfies* $f(u) > f_0 u^{p-1}$ *for all* $u > 0$, *then* $\lambda_* = \lambda^* = \mu_k(p)$. *Moreover,* (Q_λ) *has at least one solution for* $0 < \lambda < \mu_k(p)$ *and no solution for* $\lambda \geqslant \mu_k(p)$.

Corollary 4.8. *Assume* (A_1) *and* (A_3). *If* f *satisfies that there exists* $\tilde{u} > 0$ *such that* $f(\tilde{u}) = f_0 \tilde{u}^{p-1}$ *and* $f(u) < f_0 u^{p-1}$ *for all* $u \in (0, \tilde{u})$, *then* (Q_λ) *has a solution with* $\|u\|_\infty < \tilde{u}$ *for* $\lambda > \mu_k(p)$. *Moreover,* (Q_λ) *has at least two, one or no solutions for* $0 < \lambda < \mu_k(p)$ *according to* $\lambda \in (\mu_k(p), \lambda^*)$, $\lambda \in (0, \mu_k(p)] \cup \{\lambda^*\}$, *or* $\lambda \in (\lambda_*, \infty)$, *respectively.*

We conclude this section applying previous results to the radial solutions of quasilinear elliptic problems. Consider, first, the problem on annular domain.

$$\begin{cases} \Delta_p(u) + \lambda K(|x|) f(u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{A_λ}$$

where $\Omega = \{x \in \mathbb{R}^N\colon\ l_1 < |x| < l_2\}$, $l_1, l_2 > 0$, $N \geqslant 3$, $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ and $K \in C([l_1, l_2], (0, \infty))$.

Radial problem (A_λ) is equivalent to the boundary value problem of ODE by $r = |x|$ as follows:

$$\begin{cases} (|u'(r)|^{p-2} u'(r))' + \frac{N-1}{r} |u'(r)|^{p-2} u'(r) + \lambda K(r) f(u(r)) = 0, & r \in (l_1, l_2), \\ u(l_1) = 0 = u(l_2). \end{cases}$$

By consecutive changes of variables, $s = -\int_r^{l_2} t^{\frac{1-N}{p-1}}\, dt$ and $t = \frac{m-s}{m}$ with $m = -\int_{l_1}^{l_2} t^{\frac{1-N}{p-1}}\, dt$, we get

$$\begin{cases} \varphi_p(u'(t))' + \lambda h(t) f(u(t)) = 0, & t \in (0, 1), \\ u(0) = 0 = u(1), \end{cases}$$

where

$$h(t)=|m|^p\Big[l_1^{\frac{p-N}{p-1}}+\Big(l_2^{\frac{p-N}{p-1}}-l_1^{\frac{p-N}{p-1}}\Big)t\Big]^{\frac{1-N}{N-p}}K\Big(\Big[l_1^{\frac{p-N}{p-1}}+\Big(l_2^{\frac{p-N}{p-1}}-l_1^{\frac{p-N}{p-1}}\Big)t\Big]^{\frac{1-N}{N-p}}\Big).$$

Since $h\in C([0,1],(0,\infty))$, Theorems 3.5, 3.10, 3.15 and 3.18 are valid for problem (A_λ) on an annular domain.

Finally, consider problem (A_λ) on an exterior domain $\Omega=\{x\in\mathbb{R}^N\colon |x|>r_0\}$, $r_0>0$, $N\geqslant 3$ and $1<p<N$. By changes of variables, $r=|x|$, $s=r^{\frac{p-N}{p-1}}$ and $t=r_0^{\frac{N-p}{p-1}}s$, we get the following equivalence of ODE problem:

$$\begin{cases}\varphi_p(u'(t))'+\lambda h(t)f(u(t))=0, & 0<t<1,\\ u(0)=0=u(1),\end{cases}$$

where $h(t)=\frac{(p-1)^p}{(p-N)^p}r_0^p t^{\frac{(N-1)p}{p-N}}K(r_0t^{\frac{p-1}{p-N}})$.

Assume

$$\int_{r_0}^{\infty} r^{N-1}K(r)\,dr<\infty.$$

Then $h\in L^1(0,1)$ and Theorems 3.5, 3.10, 3.15 and 3.18 are valid for problem (A_λ) on an exterior domain.

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References

[1] R.P. Agarwal, H. Lü, D. O'Regan, Eigenvalues and the one-dimensional p-Laplacian, J. Math. Anal. Appl. 266 (2002) 383–400.
[2] H. Asakawa, Nonresonant singular two-point boundary value problems, Nonlinear Anal. 47 (2001) 4849–4860.
[3] K. Diemling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1987.
[4] M. del Pino, M. Elgueta, R. Manásevich, A homotopic deformation along p of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, J. Differential Equations 80 (1989) 1–13.
[5] M. García-Huidobro, R. Manásevich, J.R. Ward, A homotopy along p for systems with a vector p-Laplace operator, Adv. Differential Equations 8 (2003) 337–356.
[6] B. Im, E. Lee, Y.H. Lee, A global bifurcation phenomena for second order singular boundary value problems, J. Math. Anal. Appl. 308 (2005) 61–78.
[7] L. Kong, J. Wang, Multiple positive solutions for the one-dimensional p-Laplacian, Nonlinear Anal. 42 (2000) 1327–1333.
[8] T. Kusano, T. Jaros, N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, Nonlinear Anal. 40 (2000) 381–395.
[9] R. Manásevich, J. Mawhin, Periodic solutions of nonlinear systems with p-Laplacian-like operators, J. Differential Equations 145 (1998) 367–393.
[10] R. Manásevich, J. Mawhin, Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators, J. Korean Math. Soc. 37 (2000) 665–685.
[11] P.H. Rabinowitz, Some aspects of nonlinear eigenvalue problems, Rocky Mountain J. Math. 3 (1973) 161–202.

[12] J. Sánchez, Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian, J. Math. Anal. Appl. 292 (2004) 401–414.
[13] K. Schmitt, R. Thompson, Nonlinear Analysis and Differential Equations: An Introduction, Univ. of Utah Lecture Notes, Univ. of Utah Press, Salt Lake City, 2004.
[14] J. Wang, The existence of positive solutions for the one-dimensional p-Laplacian, Proc. Amer. Math. Soc. 125 (1997) 2275–2283.
[15] X. Yang, Sturm type problems for singular p-Laplacian boundary value problems, Appl. Math. Comput. 136 (2003) 181–193.
[16] M. Zhang, Nonuniform nonresonance of semilinear differential equations, J. Differential Equations 166 (2000) 33–50.