J. Math. Anal. Appl. 348 (2008) 461–468

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Unexpected local extrema for the Sendov conjecture

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ARTICLE INFO

Article history:
Received 22 February 2008
Available online 26 July 2008
Submitted by S. Ruscheweyh

Keywords:
Sendov
Critical points
Polynomial
Derivative
Extremal

ABSTRACT

Let $S(n)$ be the set of all polynomials of degree n with all roots in the unit disk, and define $d(P)$ to be the maximum of the distances from each of the roots of a polynomial P to that root's nearest critical point. In this notation, Sendov's conjecture asserts that $d(P)$ is at most 1 for every P in $S(n)$. Define P in $S(n)$ to be locally extremal if $d(P)$ is at least $d(Q)$ for all nearby Q in $S(n)$. In this paper, we determine sufficient conditions for real polynomials of degree n with a root strictly between 0 and 1 and a real critical point of order $n-3$ to be locally extremal, and we use these conditions to find locally extremal polynomials of this form of degrees 8, 9, 12, 13, 14, 15, 19, 20, and 26.

1. Introduction

In 1958, Sendov conjectured that if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. This conjecture has yet to be settled, although it has been the subject of more than 80 papers over the intervening years, and so has been verified for many special cases. These have been documented by Sendov [8], Schmeisser [6], Sheil-Small [9, Chapter 6] and Rahman and Schmeisser [5, Section 7.3].

Let $n \geqslant 2$ be an integer and let $S(n)$ be the set of all complex polynomials of degree n with all roots in the unit disk. For a polynomial P with roots $z_1, \ldots, z_n$ and critical points $\zeta_1, \ldots, \zeta_{n-1}$, define

$$d(P) = \max_{1 \leqslant i \leqslant n} \left\{ \min_{1 \leqslant j \leqslant n-1} |z_i - \zeta_j| \right\}.$$

If $P \in S(n)$, then the Gauss–Lucas Theorem [5, Theorem 2.1.1] implies that $d(P) \leqslant 2$, and Sendov's conjecture asserts that $d(P) \leqslant 1$.

We will say that a polynomial P is *expected* if it is of the form $P(z) = c(z^n + a)$ with $|a| = 1$. In 1972, Phelps and Rodriguez defined a polynomial $P \in S(n)$ to be *extremal* if $d(P) = \sup\{d(Q)\colon\ Q \in S(n)\}$, and conjectured [4, after Theorem 5] that extremal polynomials are all expected. Since any expected polynomial P has $d(P) = 1$, this conjecture implies Sendov's conjecture. The Phelps–Rodriguez conjecture has also been verified for a number of special cases, as documented by Rahman and Schmeisser [5, Section 7.3].

Define an ϵ-*neighborhood* of $P \in S(n)$ to be the set of all the polynomials $Q \in S(n)$ whose roots are within ϵ of the roots of P (in the sense that the roots of Q can be paired with the roots of P so that in each pair, the difference of the roots has a modulus less than ϵ). Define a polynomial $P \in S(n)$ to be *locally extremal* if $d(P) \geqslant d(Q)$ for all Q in some ϵ-neighborhood of P, and note that maximizing $d(P)$ over all locally extremal polynomials P would settle the Sendov conjecture.

The expected polynomials are all locally extremal, as was demonstrated by Vâjâitu and Zaharescu [10] and Miller [3, Theorem 3]. Given this and the Phelps–Rodriguez conjecture, it is tempting to approach Sendov's conjecture by trying

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.07.049

to show that all locally extremal polynomials must be expected. Indeed, Schmieder has made several such attempts [7], although Borcea has shown [2, Section 1] that each contains a technical flaw.

In this paper, we prove

Theorem 1. *For each $n \in \{8, 9, 12, 13, 14, 15, 19, 20, 26\}$, there are locally extremal polynomials $P \in S(n)$ of the form*

$$P(z) = \int_{\beta}^{z} (w-a)^{n-3}\big(w^2 + bw + c\big)\,dw \quad \textit{with } 0 < \beta < 1.$$

Note that polynomials of this form would not be expected, as each has a root β that is not on the unit circle. Thus Theorem 1 implies that there are locally extremal polynomials that are not expected.

Theorem 1 has several consequences. First, it shows that the technical flaws in Schmieder's approach cannot be patched. Second, the structure of these polynomials (in particular, the multiple critical point) identifies circumstances that potential variational proofs of Sendov's conjecture will need to address. Finally, finding these polynomials is a significant new step in identifying all local extrema, which would settle the fate of Sendov's conjecture.

In Section 2, we list eight properties that we claim are sufficient for a polynomial to be locally extremal, and in Sections 3 and 4 we verify that these properties suffice. In Section 5, we describe how to construct polynomials that satisfy all these properties, and in Section 6 we list the resulting polynomials, thereby verifying Theorem 1.

2. Properties

In Section 4 we prove that for a real polynomial P of degree $n \geqslant 5$ to be locally extremal, it suffices for it to satisfy 8 properties, beginning with the following.

A: All roots of P lie in the closed unit disk.
B: All roots of P that are on the unit circle are simple.
C: P has a root at β, with $0 < \beta < 1$.
D: All critical points of P lie on a circle of positive radius centered at β.
E: P has a real critical point $a < \beta$ of order $n-3$.

Let $z_1, \dots, z_n$ be the roots of P, numbered so that $z_1, \dots, z_m$ are on the unit circle. Let $\zeta_1, \dots, \zeta_{n-1}$ be the critical points of P, numbered (as allowed by property E) so that $\zeta_j = a$ for $j \geqslant 3$.

Note that property D implies that P has a simple root at β. Our next property is

F: The roots and critical points of P satisfy the inequality

$$\max_{z_i \neq \beta} \min_{1 \leqslant j \leqslant n-1} |z_i - \zeta_j| < \min_{1 \leqslant j \leqslant n-1} |\beta - \zeta_j| < 1.$$

To examine the effects of changing β and the ζ_j by small amounts, denoted by $\Delta\beta$ (which we will require to be real) and $\Delta\zeta_j$, we will use the following notation.

$$\begin{aligned}
E_k &= -\Re\left[\frac{\Delta\zeta_k - \Delta\beta}{\zeta_k - \beta}\right] \quad \text{for } k=1 \text{ and } k=2,\\
F_k &= 0 \quad \text{for } k=1 \text{ and } k=2,\\
E_3 &= \frac{\Re[\sum_{j=3}^{n-1}\Delta\zeta_j] - (n-3)\Delta\beta}{\beta - a}, \quad \text{and}\\
F_3 &= -\frac{\sum_{j=3}^{n-1}(\Im[\Delta\zeta_j])^2}{2(\beta-a)^2}.
\end{aligned} \tag{2.1}$$

Since (up to a constant multiple) $P'(z) = \prod_{j=1}^{n-1}(z-\zeta_j)$ and by property C we have $P(z) = \int_\beta^z P'(w)\,dw$, then the roots of P are functions of β and the ζ_j. By property B, the roots of P that are on the unit circle (being simple) are differentiable functions of β and the ζ_j.

Recall that $z_1, \dots, z_m$ are on the unit circle. For $i = 1, \dots, m$ define

$$E_{i+3} = \Re\left[\frac{1}{z_i}\frac{\partial z_i}{\partial\beta}\right]\Delta\beta + \sum_{j=1}^{2}\Re\left[\frac{1}{z_i}\frac{\partial z_i}{\partial\zeta_j}\Delta\zeta_j\right] + \Re\left[\frac{1}{z_i}\frac{\partial z_i}{\partial\zeta_3}\sum_{j=3}^{n-1}\Delta\zeta_j\right], \quad \text{and}$$

$$F_{i+3} = -\Re\left[\frac{1}{2z_i P'(z_i)} \int_{\beta}^{z_i} \frac{P'(w)\,dw}{(w-a)^2}\right] \sum_{j=3}^{n-1} \left(\Im[\Delta\zeta_j]\right)^2.$$

Note that for a fixed polynomial P, the $m+3$ expressions E_k are all linear in the 7 real "variables" $\Delta\beta$, $\Re[\Delta\zeta_1]$, $\Im[\Delta\zeta_1]$, $\Re[\Delta\zeta_2]$, $\Im[\Delta\zeta_2]$, $\Re[\sum_{j=3}^{n-1}\Delta\zeta_j]$ and $\Im[\sum_{j=3}^{n-1}\Delta\zeta_j]$, and that the $m+3$ expressions F_k are all constant multiples of the real "variable" $\sum_{j=3}^{n-1}(\Im[\Delta\zeta_j])^2$. Our final two properties are

G: There are constants $c_k > 0$ (that depend on P, but are independent of the 8 "variables") so that the sums $\sum_{k=1}^{m+3} c_k E_k = 0$ and $\sum_{k=1}^{m+3} c_k F_k = \sum_{j=3}^{n-1}(\Im[\Delta\zeta_j])^2$.

H: The coefficient matrix of the system $\{E_k = 0\colon\ k = 1,\dots,m+3\}$ in our 7 "variables" is of rank 7.

3. Preliminary calculations

We will show that a real polynomial P of degree $n \geqslant 5$ that satisfies properties A–H is locally extremal, as follows.

Define r to be the radius of the circle in property D and recall that (up to a constant multiple) $P(z) = \int_\beta^z \prod_{j=1}^{n-1}(w-\zeta_j)\,dw$. From our properties A and D we know that P has all its roots in the closed unit disk and all its critical points on a circle of radius $r > 0$ centered at β. Define the n-tuple $(\Delta\beta, \Delta\zeta_1, \dots, \Delta\zeta_{n-1}) \in C^n$ to be an *improvement* of P if $\Delta\beta$ is real and if the polynomial

$$\int_{\beta+\Delta\beta}^{z} \prod_{j=1}^{n-1}\left[w - (\zeta_j + \Delta\zeta_j)\right] dw \tag{3.1}$$

has all its roots in the closed unit disk and all its critical points strictly outside the circle of radius r centered at $\beta + \Delta\beta$. By property F we know that $r < 1$, so there is at least one improvement, namely $(1-\beta, -\zeta_1, \dots, -\zeta_{n-1})$.

For an improvement $I = (\Delta\beta, \Delta\zeta_1, \dots, \Delta\zeta_{n-1})$, define

$$\|I\| = \left(|\Delta\beta|^2 + \sum_{j=1}^{n-1} |\Delta\zeta_j|^2\right)^{1/2}.$$

Note that $\|I\| > 0$, for if $\|I\| = 0$, then the critical points of (3.1) would be the critical points of P, hence on (and thus not strictly outside) the circle of radius r centered at $\beta + \Delta\beta$.

We now show that if P is not locally extremal, then there are arbitrarily small improvements of P. Let

$$\epsilon_0 = \min_{1\leqslant j\leqslant n-1} |\beta - \zeta_j| - \max_{z_i \neq \beta} \min_{1\leqslant j\leqslant n-1} |z_i - \zeta_j|$$

and note that $\epsilon_0 > 0$ by property F. Take any ϵ with $0 < \epsilon < \min(\epsilon_0/4, \beta/2)$ and recall our definition of ϵ-neighborhood from Section 1. The roots of a monic polynomial depend continuously on the coefficients of that polynomial [5, Theorem 1.3.1], so the critical points of P are continuous functions of the roots of P. Thus there is a $\delta > 0$ such that for all polynomials Q in a δ-neighborhood of P, the roots and critical points of Q are within ϵ of the roots and critical points of P.

By property F, we know that $d(P) = r$, so if P is not locally extremal, then there is a polynomial $\hat{P} \in S(n)$ in the δ-neighborhood of P with $d(\hat{P}) > r$. For $i = 1,\dots,n$, let $\hat{z}_i$ be the root of $\hat{P}$ paired with z_i (so $|\hat{z}_i - z_i| < \epsilon$), and for $j = 1,\dots,n-1$, let $\hat{\zeta}_j$ be the critical point of $\hat{P}$ paired with ζ_j (so $|\hat{\zeta}_j - \zeta_j| < \epsilon$). Since the roots and critical points of $\hat{P}$ differ from the roots and critical points of P by at most $\epsilon_0/4$, then by property F we have

$$\max_{\hat{z}_i \neq \hat{\beta}} \min_{1\leqslant j\leqslant n-1} |\hat{z}_i - \hat{\zeta}_j| < \min_{1\leqslant j\leqslant n-1} |\hat{\beta} - \hat{\zeta}_j|$$

so $d(\hat{P}) = \min_{1\leqslant j\leqslant n-1} |\hat{\beta} - \hat{\zeta}_j|$ and thus $\hat{P}$ has all its critical points strictly outside the circle of radius r centered at $\hat{\beta}$.

Note that $|\hat{\beta}| \geqslant |\beta| - |\hat{\beta} - \beta| \geqslant \beta/2 > 0$, define $u = |\hat{\beta}|/\hat{\beta}$ and note that $|u - 1| = ||\hat{\beta}| - \hat{\beta}|/|\hat{\beta}| \leqslant 2\epsilon/(\beta/2)$. Since $|u| = 1$, then the transformation $z \to uz$ is a rotation about the origin, so $I = (u\hat{\beta} - \beta, u\hat{\zeta}_1 - \zeta_1, \dots, u\hat{\zeta}_{n-1} - \zeta_{n-1})$ is an improvement of P. Note that each $|u\hat{\zeta}_j - \zeta_j| \leqslant |u| \cdot |\hat{\zeta}_j - \zeta_j| + |\zeta_j| \cdot |u - 1| \leqslant C\epsilon$ for some constant C (and similarly for $|u\hat{\beta} - \beta|$), so $\|I\|$ can be made arbitrarily small.

We have just seen that if P is not locally extremal, then there are improvements I of P with $\|I\|$ arbitrarily small. This means that we can show that P is locally extremal by proving

Theorem 2. *If P is a real polynomial of degree $n \geqslant 5$ that satisfies properties* A–H, *then there is a constant $A > 0$ so that every improvement I of P has $\|I\| \geqslant A$.*

We will prove this theorem by approximating the roots and critical points of P. To keep track of the errors in these approximations, we introduce a variant of the Landau "Big Oh" notation by defining the bound

$$E \leqslant \mathcal{B}(t^k)$$

to mean that there is a number A (independent of t) so that $E \leqslant At^k$ for all $t > 0$ in some neighborhood of 0. The notation $E \geqslant \mathcal{B}(t^k)$ is defined similarly. Finally, we define $E = \mathcal{B}(t^k)$ to mean that $|E| \leqslant \mathcal{B}(t^k)$, and $E = F + \mathcal{B}(t^k)$ to mean that $E - F = \mathcal{B}(t^k)$. Given this, we have

Lemma 3. *If $z = \mathcal{B}(t)$, then*

(1) $|1 + z| = 1 + \Re[z] + \mathcal{B}(t^2)$, *and*

(2) $|1 + z| = 1 + \Re[z] + (1/2)(\Im[z])^2 + \mathcal{B}(t^3)$.

Proof. Note that $\Re[z] = \mathcal{B}(t)$ and $|z|^2 = \mathcal{B}(t^2)$. The results follow from the equality $|1 + z| = (1 + 2\Re[z] + |z|^2)^{1/2}$, the Taylor series $(1 + s)^{1/2} = 1 + s/2 - s^2/8 + \mathcal{B}(s^3)$ and the substitution $s = 2\Re[z] + |z|^2$. $\square$

We will be examining the relationships between the roots and critical points of P. We calculate the partial derivatives of these relationships with

Lemma 4. *Write $P'(z) = \prod_{j=1}^{n-1}(z - \zeta_j)$ and suppose that z_i is a simple root of $P(z) = \int_\beta^z P'(w)\,dw$. Then*

(1) *z_i is an analytic function of $\beta, \zeta_1, \ldots, \zeta_{n-1}$,*

(2) $$\frac{\partial z_i}{\partial \beta} = \frac{P'(\beta)}{P'(z_i)},$$

(3) $$\frac{\partial z_i}{\partial \zeta_j} = \frac{1}{P'(z_i)} \int_\beta^{z_i} \frac{P'(w)\,dw}{w - \zeta_j},$$

(4) $$\frac{\partial^2 z_i}{\partial \zeta_j^2} = \frac{2}{z_i - \zeta_j}\frac{\partial z_i}{\partial \zeta_j} - \frac{P''(z_i)}{P'(z_i)}\left(\frac{\partial z_i}{\partial \zeta_j}\right)^2, \quad \textit{and}$$

(5) $$\frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} = \frac{1}{z_i - \zeta_k}\frac{\partial z_i}{\partial \zeta_j} + \frac{1}{z_i - \zeta_j}\frac{\partial z_i}{\partial \zeta_k} - \frac{P''(z_i)}{P'(z_i)}\frac{\partial z_i}{\partial \zeta_j}\frac{\partial z_i}{\partial \zeta_k} - \frac{1}{P'(z_i)}\int_\beta^{z_i} \frac{P'(w)\,dw}{(w - \zeta_j)(w - \zeta_k)} \quad \textit{for } j \neq k.$$

Proof. We may assume that $z_i \neq \beta$ (else the results would be trivially true). Proofs of parts 1–3 can be found in [1, Lemmas 2.1 and 2.3].

Part 4 can be established by writing $P'(z) = (z - \zeta_j)Q(z)$ (with Q independent of ζ_j) and calculating the second partial derivative (with respect to ζ_j) of

$$0 = \int_\beta^{z_i} wQ(w)\,dw - \zeta_j \int_\beta^{z_i} Q(w)\,dw.$$

This gives us

$$0 = (z_i - \zeta_j)Q(z_i)\frac{\partial^2 z_i}{\partial \zeta_j^2} - 2Q(z_i)\frac{\partial z_i}{\partial \zeta_j} + \left[(z_i - \zeta_j)Q'(z_i) + Q(z_i)\right]\left(\frac{\partial z_i}{\partial \zeta_j}\right)^2$$

from which part 4 follows.

Part 5 can likewise be established by writing $P'(z) = (z - \zeta_j)(z - \zeta_k)Q(z)$ and calculating the mixed second partial derivative of the corresponding integral. This gives us

$$0 = \left[z_i^2 - (\zeta_j + \zeta_k)z_i + \zeta_j\zeta_k\right]Q(z_i)\frac{\partial^2 z_i}{\partial \zeta_j \partial \zeta_k} + \left\{\left[z_i^2 - (\zeta_j + \zeta_k)z_i + \zeta_j\zeta_k\right]Q'(z_i) + \left[2z_i - (\zeta_j + \zeta_k)\right]Q(z_i)\right\}\frac{\partial z_i}{\partial \zeta_j}\frac{\partial z_i}{\partial \zeta_k}$$
$$- (z_i - \zeta_j)Q(z_i)\frac{\partial z_i}{\partial \zeta_j} - (z_i - \zeta_k)Q(z_i)\frac{\partial z_i}{\partial \zeta_k} + \int_\beta^{z_i} Q(w)\,dw$$

from which part 5 follows. $\square$

4. Proof of Theorem 2

Recall that P is a real polynomial of degree $n \geqslant 5$ that satisfies properties A–H. We will prove Theorem 2 in two stages, first by estimating the roots and critical points of P with linear approximations, and then by improving our estimates with quadratic approximations.

We begin by examining the conclusions that can be drawn from linear approximations to the roots and critical points of P with

Proposition 5. *If $I = (\Delta\beta, \Delta\zeta_1, \dots, \Delta\zeta_{n-1})$ is an improvement of P and $t = \|I\|$, then $\Delta\beta = \mathcal{B}(t^2)$, $\Delta\zeta_j = \mathcal{B}(t^2)$ for $j \leqslant 2$, $\Re[\Delta\zeta_j] = \mathcal{B}(t^2)$ for $j \geqslant 3$ and $\sum_{j=3}^{n-1} \Delta\zeta_j = \mathcal{B}(t^2)$.*

Proof. From the definition of t we know that $\Delta\beta = \mathcal{B}(t)$ and that each $\Delta\zeta_j = \mathcal{B}(t)$. Since I is an improvement, then each $|(\zeta_j - \beta) + (\Delta\zeta_j - \Delta\beta)| > r$, so $|1 + (\Delta\zeta_j - \Delta\beta)/(\zeta_j - \beta)| > r/|\zeta_j - \beta| = 1$, so using part 1 of Lemma 3 gives us that each

$$-\Re\left[\frac{\Delta\zeta_j - \Delta\beta}{\zeta_j - \beta}\right] \leqslant \mathcal{B}(t^2). \tag{4.1}$$

Recalling the expressions E_k defined in (2.1), this gives us two inequalities $E_1 \leqslant \mathcal{B}(t^2)$ and $E_2 \leqslant \mathcal{B}(t^2)$. Adding (4.1) for $j = 3, \dots, n-1$ and recalling that $\zeta_j = a$ is real for $j \geqslant 3$ gives us a third inequality $E_3 \leqslant \mathcal{B}(t^2)$.

By part 1 of Lemma 4 the roots of our improved polynomial (3.1) that originate from the simple roots z_i of P are analytic functions of $\beta, \zeta_1, \dots, \zeta_{n-1}$, so each such root is of the form $z_i + \Delta z_i$, with

$$\Delta z_i = \frac{\partial z_i}{\partial\beta}\Delta\beta + \sum_{j=1}^{2} \frac{\partial z_i}{\partial\zeta_j}\Delta\zeta_j + \frac{\partial z_i}{\partial\zeta_3}\sum_{j=3}^{n-1}\Delta\zeta_j + \mathcal{B}(t^2).$$

Note that each $\Delta z_i = \mathcal{B}(t)$. Since I is an improvement, then each $|z_i + \Delta z_i| \leqslant 1$. If $|z_i| = 1$, then $|1 + \Delta z_i/z_i| \leqslant 1/|z_i| = 1$ so using part 1 of Lemma 3 gives us $\Re[\Delta z_i/z_i] \leqslant \mathcal{B}(t^2)$ and thus we have inequalities $E_{i+3} \leqslant \mathcal{B}(t^2)$ for $i = 1, \dots, m$.

By property G, there are constants $c_k > 0$ so that $\sum_{k=1}^{m+3} c_k E_k = 0$. Since each $E_i \leqslant \mathcal{B}(t^2)$, then each

$$E_i = \frac{1}{c_i}\left[\sum_{k=1}^{m+3} c_k E_k + \sum_{i \neq k=1}^{m+3} c_k(-E_k)\right] \geqslant \mathcal{B}(t^2)$$

so each $E_i = \mathcal{B}(t^2)$.

Thus we consider the system $\{E_k = \mathcal{B}(t^2)\colon\ k = 1, \dots, m+3\}$. By definition, this system is equivalent to a system of linear inequalities of the form

$$\{-A_k t^2 \leqslant E_k \leqslant A_k t^2\colon\ k = 1, \dots, m+3\}$$

for some numbers $A_1, \dots, A_{m+3}$. By property H, the coefficient matrix of this system is of rank 7, and so solving this system (using elementary row operations on the inequalities) shows that the values of our 7 "variables" are constrained by similar inequalities, hence they are all $\mathcal{B}(t^2)$. Thus we can conclude that $\Delta\beta = \mathcal{B}(t^2)$, that $\Delta\zeta_j = \mathcal{B}(t^2)$ for $j \leqslant 2$, and that $\sum_{j=3}^{n-1} \Delta\zeta_j = \mathcal{B}(t^2)$.

Suppose that $j \geqslant 3$. Since by property E we know that $\zeta_j - \beta = a - \beta < 0$, and since $\Delta\beta = \mathcal{B}(t^2)$, then (4.1) implies that each $\Re[\Delta\zeta_j] \leqslant \mathcal{B}(t^2)$. Since $\sum_{k=3}^{n-1} \Re[\Delta\zeta_k] = \mathcal{B}(t^2)$, then each

$$\Re[\Delta\zeta_j] = \sum_{k=3}^{n-1} \Re[\Delta\zeta_k] + \sum_{j \neq k=3}^{n-1} \left(-\Re[\Delta\zeta_k]\right) \geqslant \mathcal{B}(t^2),$$

so each $\Re[\Delta\zeta_j] = \mathcal{B}(t^2)$. This finishes the proof of Proposition 5. □

At this point, for $j \geqslant 3$ we know only that each $\Delta\zeta_j = \mathcal{B}(t)$. We can improve this estimate by looking at quadratic approximations to the roots and critical points of P with

Proposition 6. *If $I = (\Delta\beta, \Delta\zeta_1, \dots, \Delta\zeta_{n-1})$ is an improvement of P and $t = \|I\|$, then each $\Delta\zeta_j = \mathcal{B}(t^{3/2})$.*

Proof. Note that the hypotheses of Proposition 5 are satisfied, so we may use all of its conclusions. In particular, we know that $\Delta\zeta_j = \mathcal{B}(t^2)$ for $j \leqslant 2$ and that $\Re[\Delta\zeta_j] = \mathcal{B}(t^2)$ for $j \geqslant 3$, so to verify Proposition 6 we need only show that $\Im[\Delta\zeta_j] = \mathcal{B}(t^{3/2})$ for $j \geqslant 3$. We will do this by repeating the calculations of Proposition 5, but working now to $\mathcal{B}(t^3)$.

Since I is an improvement, then each $|1 + (\Delta\zeta_j - \Delta\beta)/(\zeta_j - \beta)| > 1$, so using part 2 of Lemma 3 gives us

$$-\Re\left[\frac{\Delta\zeta_j - \Delta\beta}{\zeta_j - \beta}\right] - \frac{1}{2}\left(\Im\left[\frac{\Delta\zeta_j - \Delta\beta}{\zeta_j - \beta}\right]\right)^2 \leqslant \mathcal{B}\big(t^3\big). \tag{4.2}$$

Define inequalities 1 and 2 by evaluating (4.2) for $j = 1$ and $j = 2$ respectively. By Proposition 5, for $j \leqslant 2$ we have $\Delta\zeta_j - \Delta\beta = \mathcal{B}(t^2)$, so inequalities 1 and 2 are $E_k + F_k \leqslant \mathcal{B}(t^3)$ for $k = 1, 2$.

Define inequality 3 to be the sum of (4.2) for $j = 3, \dots, n-1$. Now each $\zeta_j - \beta = -(\beta - a)$ is real for $j \geqslant 3$, and each $\Im[\Delta\zeta_j - \Delta\beta] = \Im[\Delta\zeta_j]$, so inequality 3 can be written as $E_3 + F_3 \leqslant \mathcal{B}(t^3)$.

Each root of our improved polynomial is of the form $z_i + \Delta z_i$. A quadratic approximation to Δz_i includes terms of the form $\Delta\beta\Delta\zeta_j = \mathcal{B}(t^3)$ and (for $j \leqslant 2$ or $k \leqslant 2$) $\Delta\zeta_j\Delta\zeta_k = \mathcal{B}(t^3)$, which are absorbed into the $\mathcal{B}(t^3)$ when we write

$$\Delta z_i = \frac{\partial z_i}{\partial\beta}\Delta\beta + \sum_{j=1}^{2}\frac{\partial z_i}{\partial\zeta_j}\Delta\zeta_j + \frac{\partial z_i}{\partial\zeta_3}\sum_{j=3}^{n-1}\Delta\zeta_j + \frac{1}{2}\sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k}\Delta\zeta_j\Delta\zeta_k + \mathcal{B}\big(t^3\big).$$

Note that Proposition 5 implies that each $\Delta z_i = \mathcal{B}(t^2)$. Now $\zeta_j = a$ for $j \geqslant 3$, so Lemma 4 shows that for $j \geqslant 3$ and $k \geqslant 3$ we have

$$\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k} - \frac{\partial^2 z_i}{\partial\zeta_3\partial\zeta_4} = \begin{cases} \frac{1}{P'(z_i)}\int_\beta^{z_i}\frac{P'(w)\,dw}{(w-a)^2} & \text{if } j = k,\\ 0 & \text{if } j \neq k.\end{cases}$$

From Proposition 5 we know that

$$\sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\Delta\zeta_j\Delta\zeta_k = \left(\sum_{j=3}^{n-1}\Delta\zeta_j\right)^2 = \mathcal{B}\big(t^4\big)$$

so

$$\sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k}\Delta\zeta_j\Delta\zeta_k = \sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\left(\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k} - \frac{\partial^2 z_i}{\partial\zeta_3\partial\zeta_4}\right)\Delta\zeta_j\Delta\zeta_k + \mathcal{B}\big(t^4\big)$$

and thus

$$\sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k}\Delta\zeta_j\Delta\zeta_k = \frac{1}{P'(z_i)}\int_\beta^{z_i}\frac{P'(w)\,dw}{(w-a)^2}\sum_{j=3}^{n-1}(\Delta\zeta_j)^2 + \mathcal{B}\big(t^4\big).$$

From Proposition 5 we also know that each $(\Delta\zeta_j)^2 = -(\Im[\Delta\zeta_j])^2 + \mathcal{B}(t^3)$ for $j \geqslant 3$, so

$$\sum_{j=3}^{n-1}\sum_{k=3}^{n-1}\frac{\partial^2 z_i}{\partial\zeta_j\partial\zeta_k}\Delta\zeta_j\Delta\zeta_k = \frac{-1}{P'(z_i)}\int_\beta^{z_i}\frac{P'(w)\,dw}{(w-a)^2}\sum_{j=3}^{n-1}\big(\Im[\Delta\zeta_j]\big)^2 + \mathcal{B}\big(t^3\big).$$

Recall that each $|z_i + \Delta z_i| \leqslant 1$ and that each $\Delta z_i = \mathcal{B}(t^2)$. If $|z_i| = 1$, then $|1 + \Delta z_i/z_i| \leqslant 1$, so using part 1 of Lemma 3 gives us $\Re[\Delta z_i/z_i] \leqslant \mathcal{B}(t^4)$ and so $E_{i+3} + F_{i+3} \leqslant \mathcal{B}(t^3)$ for $i = 1, \dots, m$.

Thus we have $E_k + F_k \leqslant \mathcal{B}(t^3)$ for $k = 1, \dots, m+3$. From property G there are constants $c_k > 0$ so that $\sum_{j=3}^{n-1}(\Im[\Delta\zeta_j])^2 = \sum_{k=1}^{m+3} c_k(E_k + F_k) \leqslant \mathcal{B}(t^3)$ and thus $\Im[\Delta\zeta_j] = \mathcal{B}(t^{3/2})$ for $j \geqslant 3$, which completes the proof of Proposition 6. □

Using our estimates from Propositions 5 and 6, we can now write the

Proof of Theorem 2. Let $I = (\Delta\beta, \Delta\zeta_1, \dots, \Delta\zeta_{n-1})$ be any improvement of P, and let $t = \|I\|$. From Propositions 5 and 6, we know that $\Delta\beta = \mathcal{B}(t^2)$ and that each $\Delta\zeta_j = \mathcal{B}(t^{3/2})$, so

$$t = \|I\| = \left(|\Delta\beta|^2 + \sum_{j=1}^{n-1}|\Delta\zeta_j|^2\right)^{1/2} = \mathcal{B}\big(t^{3/2}\big).$$

Thus there are positive constants ϵ and K (depending only on P) so that $t \leqslant Kt^{3/2}$ for all $t \in (0, \epsilon)$. Letting $A = \min\{\epsilon, 1/K^2\}$, we have $\|I\| = t \geqslant A$. □

5. Calculations

Recall that Theorem 2 implies that any real polynomial of degree $n \geqslant 5$ that satisfies Properties A–H is locally extremal. In this section, we show how to find locally extremal polynomials by describing how to construct polynomials that satisfy our properties A–H. Maple code for these calculations can be found in the supplementary material of this article, available at doi: 10.1016/j.jmaa.2008.07.049.

Recall that P is to be a real polynomial of degree $n \geqslant 5$. To satisfy property E we must have $P'(z) = (z-a)^{n-3}(z^2+bz+c)$ for some real numbers a, b and c. To satisfy property C, we must have $P(z) = \int_\beta^z P'(w)\,dw$, so the coefficients of P will be polynomials in $\{\beta, a, b, c\}$.

To satisfy property D (where the critical points of P are the real number a and the complex roots of z^2+bz+c), we generate our first equation $\beta^2 + b\beta + c = (\beta - a)^2$.

Note that property G implies that the rows of the linear system $\{E_k = 0\colon\ k = 1, \ldots, m+3\}$ are linearly dependent. Since property H states that the coefficient matrix of this system is of rank 7, this means that there must be at least 8 equations in the system, so $m \geqslant 5$ and thus we must look for polynomials with at least 5 roots on the unit circle.

If n is odd, we will look for polynomials with three pairs of complex conjugate roots on the unit circle. Each such pair will be the roots of a quadratic of the form z^2+d_iz+1, so the remainders upon dividing P by each of these three quadratics are linear polynomials with both coefficients equal to 0. This generates an additional six equations in the seven variables $\{\beta, a, b, c, d_1, d_2, d_3\}$. Thus we have a nonlinear system of 7 equations in 7 unknowns.

If n is even, we will look for polynomials with two pairs of complex conjugate roots on the unit circle and a root at -1. The conjugate roots generate an additional 4 equations in the six variables $\{\beta, a, b, c, d_1, d_2\}$ (as above), and the equation $P(-1) = 0$ generates a sixth equation. Thus we have a nonlinear system of 6 equations in 6 unknowns.

Thus in either case we get a nonlinear system of equations, with the same number of equations as unknowns, so we can try to solve this system. Note that there may be more than one solution, so we will need to choose the "correct" one, by specifying appropriate initial estimates. For each such solution, we verify properties A–H by checking the following assertions. (Details of these computations may be found in the Maple code referenced above.)

Property A: The maximum modulus of the roots of P is equal to 1 (to the accuracy calculated).

Property B: The minimum distance between any two roots of P is greater than 0.1. (This shows that all roots of P are simple.)

Property C: We have $0.7 < \beta < 0.9$. (We know that $P(\beta) = 0$, since $P(z) = \int_\beta^z P'(w)\,dw$ by construction.)

Property D: The distances between β and the critical points of P are all equal (to the accuracy calculated), and this common distance is greater than 0.9.

Property E: The quantity $\beta - a > 0.9$. (Note that the critical point a is real and of order $n-3$ by construction.)

Property F: If we define

$$R = \max_{z_i \neq \beta} \min_{1 \leqslant j \leqslant n-1} |z_i - \zeta_j| \quad \text{and} \quad r = \min_{1 \leqslant j \leqslant n-1} |\beta - \zeta_j|,$$

then $r < 0.97$ and $r - R > 0.02$.

Property G: The linear system given by the sums has a solution in which every $c_k > 0.3$.

Property H: The seventh largest singular value of the coefficient matrix is greater than 0.04.

Once we have verified properties A–H for a specific polynomial P, we know by Theorem 2 that P is locally extremal and we are done.

6. Proof of Theorem 1

For the values of β and $P'(z)$ given below, one can verify that the polynomials $P(z) = \int_\beta^z P'(w)\,dw$ satisfy properties A–H, and thus by Theorem 2 are locally extremal. (Details of these computations may be found with the Maple code referenced above.)

For $n = 8$, we take $\beta = 0.7290857513$ and

$$P'(z) = (z + 0.2035409790)^5\bigl(z^2 - 0.5410836525z + 0.7327229666\bigr).$$

For $n = 9$, we take $\beta = 0.7145672829$ and

$$P'(z) = (z + 0.2157115753)^6\bigl(z^2 - 0.8021671918z + 0.9280147829\bigr).$$

For $n = 12$, we take $\beta = 0.8403619619$ and

$$P'(z) = (z + 0.1155828545)^9\bigl(z^2 - 0.4090272613z + 0.5513532168\bigr).$$

For $n = 13$, we take $\beta = 0.8275325585$ and

$$P'(z) = (z + 0.1246203379)^{10}\bigl(z^2 - 0.5415308686z + 0.6699194279\bigr).$$

For $n=14$, we take $\beta=0.8158105092$ and

$$P'(z)=(z+0.1304708647)^{11}\left(z^2-0.6885970233z+0.7916663399\right).$$

For $n=15$, we take $\beta=0.7999767588$ and

$$P'(z)=(z+0.1400336168)^{12}\left(z^2-0.8389864647z+0.9148263642\right).$$

For $n=19$, we take $\beta=0.8684432238$ and

$$P'(z)=(z+0.0923361850)^{16}\left(z^2-0.6503807257z+0.7337221736\right).$$

For $n=20$, we take $\beta=0.8570396874$ and

$$P'(z)=(z+0.0982636528)^{17}\left(z^2-0.7563752823z+0.8263310816\right).$$

For $n=26$, we take $\beta=0.8817716692$ and

$$P'(z)=(z+0.0797127446)^{23}\left(z^2-0.7969496845z+0.8496586550\right).$$

These verifications complete the proof of Theorem 1.

Supplementary material

The online version of this article contains additional supplementary material.
Please visit DOI: 10.1016/j.jmaa.2008.07.049.

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