Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 321 (2006) 10–23
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Strong convergence theorems for multi-step Noor iterations with errors in Banach spaces ☆

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Received 30 April 2005
Available online 16 September 2005
Submitted by M.A. Noor

Abstract

In this paper, we established two strong convergence theorems for a multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense(asymptotically quasi-nonexpansive, respectively) in Banach spaces. Our results extend and improve the recent ones announced by Xu and Noor [B.L. Xu, M.A. Noor, Fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 267 (2002) 444–453], Cho, Zhou and Guo [Y.J. Cho, H. Zhou, G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl. 47 (2004) 707–717], and many others.

Keywords: Asymptotically nonexpansive in the intermediate sense; Asymptotically quasi-nonexpansive mappings; Completely continuous; Uniformly convex; Uniformly L-Lipschitzian

1. Introduction

Let C be a subset of real normed linear space X. A mapping $T: C \to C$ is said to be *asymptotically nonexpansive* on C if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for each $x, y \in C$,

$$\|T^n x - T^n y\| \leqslant (1 + r_n)\|x - y\|, \quad \forall n \geqslant 1.$$

☆ The authors thank the Thailand Research Fund for their financial support.
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1 Supported by The Thailand Research Fund.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.08.029

If $r_n \equiv 0$, then T is known as a *nonexpansive mapping*. T is called *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\left\|T^n x - T^n y\right\| - \|x-y\|\right) \leqslant 0.$$

T is said to be *asymptotically quasi-nonexpansive mapping*, if there exists a sequence $\{r_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} r_n = 0$ such that for all $x \in C$, $p \in F(T)$,

$$\left\|T^n x - p\right\| \leqslant (1+r_n)\|x-p\|,$$

for all $n \geqslant 1$, where $F(T)$ denotes the set of fixed points of T, i.e., $F(T) = \{x \in C\colon\ Tx = x\}$. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\left\|T^n x - T^n y\right\| \leqslant L\|x-y\|,$$

for all $n \geqslant 1$ and $x, y \in C$.

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive in the intermediate sense, asymptotically quasi-nonexpansive mapping and L-Lipschitzian mapping. But the convergence does not hold such as in the following example:

Example 1.1. (See [9]) Let $X = \mathbb{R}$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is asymptotically nonexpansive in the intermediate sense. *It is well known* [8] *that $T^n x \to 0$ uniformly, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping*.

Fixed-point iterations process for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iterations process have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequations; see [6–18]. In 2000, Noor [13] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. Glowinski and Le Tallec [3] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [3] that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. In 1998, Haubruge, Nguyen and Strodiot [5] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [3] and applied these schemes to obtain new splitting-type algorithms for solving variation inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. Thus we conclude that three-step scheme plays an important and significant part in solving various problems, which arise in pure and applied sciences.

Recently, Xu and Noor [19] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. In 2004, Cho, Zhou and Guo [2] extended the work of Xu and Noor to the three-step iterative scheme with errors and gave weak and strong convergence theorems for asymptotically nonexpansive mappings in a Banach space. Moreover, Suantai [18] gave weak and strong convergence theorems for a new three-step iterative scheme of asymptotically nonexpansive mappings. Inspired and motivated by these facts, we introduce and study a multi-step scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense and asymptotically quasi-nonexpansive, respectively. Our results include the Ishikawa, Mann and Noor iterative schemes for solving variational inclusions (inequalities) as special case. The scheme is defined as follows.

Let C be a nonempty subset of normed space X and let $T:C\to C$ be a mapping. For a given $x_1\in C$, and a fixed $m\in\mathbb{N}$ ($\mathbb{N}$ denote the set of all positive integers), compute the iterative sequences $\{x_n^{(1)}\},\dots,\{x_n^{(m)}\}$ defined by

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)}T^n x_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)},\\
x_n^{(2)} &= \alpha_n^{(2)}T^n x_n^{(1)}+\beta_n^{(2)}x_n+\gamma_n^{(2)}u_n^{(2)},\\
x_n^{(3)} &= \alpha_n^{(3)}T^n x_n^{(2)}+\beta_n^{(3)}x_n+\gamma_n^{(3)}u_n^{(3)},\\
&\ \vdots\\
x_n^{(m-1)} &= \alpha_n^{(m-1)}T^n x_n^{(m-2)}+\beta_n^{(m-1)}x_n+\gamma_n^{(m-1)}u_n^{(m-1)},\\
x_{n+1}=x_n^{(m)} &= \alpha_n^{(m)}T^n x_n^{(m-1)}+\beta_n^{(m)}x_n+\gamma_n^{(m)}u_n^{(m)},\quad n\geqslant 1,
\end{aligned}\tag{1.1}$$

where, $\{u_n^{(1)}\},\dots,\{u_n^{(m)}\}$ are bounded sequences in C and $\{\alpha_n^{(i)}\},\{\beta_n^{(i)}\},\{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)}+\beta_n^{(i)}+\gamma_n^{(i)}=1$ for each $i\in\{1,2,\dots,m\}$.

The iterative schemes (1.1) are called the multi-step Noor iterations with errors. These iterations include the Mann–Ishikawa–Noor iterations as special case. If $m=3$ and $\beta_n^{(i)}=1-\alpha_n^{(i)}-\gamma_n^{(i)}$ for all $i=1,2,3$, then (1.1) reduces to Noor iterations with errors defined by Cho et al. [2]:

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)}T^n x_n+\big(1-\alpha_n^{(1)}-\gamma_n^{(1)}\big)x_n+\gamma_n^{(1)}u_n^{(1)},\\
x_n^{(2)} &= \alpha_n^{(2)}T^n x_n^{(1)}+\big(1-\alpha_n^{(2)}-\gamma_n^{(2)}\big)x_n+\gamma_n^{(2)}u_n^{(2)},\\
x_{n+1}=x_n^{(3)} &= \alpha_n^{(3)}T^n x_n^{(2)}+\big(1-\alpha_n^{(3)}-\gamma_n^{(3)}\big)x_n+\gamma_n^{(3)}u_n^{(3)},
\end{aligned}\tag{1.2}$$

where $\{\alpha_n^{(i)}\},\{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ for all $i\in\{1,2,3\}$.

For $m=3$ and $\gamma_n^{(1)}=\gamma_n^{(2)}=\gamma_n^{(3)}\equiv 0$, then (1.1) reduces to Noor iterations defined by Xu and Noor [19]:

$$\begin{aligned}
x_n^{(1)} &= \alpha_n^{(1)}T^n x_n+\big(1-\alpha_n^{(1)}\big)x_n,\\
x_n^{(2)} &= \alpha_n^{(2)}T^n x_n^{(1)}+\big(1-\alpha_n^{(2)}\big)x_n,\\
x_{n+1}=x_n^{(3)} &= \alpha_n^{(3)}T^n x_n^{(2)}+\big(1-\alpha_n^{(3)}\big)x_n,\quad n\geqslant 1,
\end{aligned}\tag{1.3}$$

where $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$ are appropriate real sequences in $[0,1]$.

The purpose of this paper is to establish several strong convergence theorems of the multi-step Noor iterative scheme with errors for mappings of asymptotically nonexpansive in the intermediate sense (asymptotically quasi-nonexpansive mappings, respectively) in a uniformly convex Banach space. These results presented in this paper extend and improve the corresponding ones announced by Xu and Noor [19], Cho et al. [2], and many others.

2. Preliminaries

In this section, we recall the well-known concepts and results.

Definition 2.1. (See [4]) A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2}\colon \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\} > 0$$

for all $0 < \epsilon \leqslant 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0,2] \to (0,1)$).

It is known [12] that if X is a uniformly convex Banach space and T is a self-mapping of bounded closed convex subset C of X which is an asymptotically nonexpansive in the intermediate sense, then $F(T) \neq \emptyset$.

Lemma 2.2. (See [10]) *Let $\{a_n\}$, $\{b_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1+\gamma_n)a_n + b_n, \quad \forall n = 1, 2, \ldots.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists*;
(ii) $\lim_{n\to\infty} a_n = 0$, *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 2.3. [17, J. Schu's Lemma] *Let X be a uniformly convex Banach space, $0 < \alpha \leqslant t_n \leqslant \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \leqslant a$, $\limsup_{n\to\infty} \|y_n\| \leqslant a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$, for some $a \geqslant 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Non-Lipschitzian mappings

Our first result is the strong convergence theorem for asymptotically nonexpansive in the intermediate sense mappings. Note that the proof given below is different from that of Xu and Noor. In order to prove our main result, the following lemmas are needed.

Lemma 3.1. *Let X be a uniformly convex Banach space with $x_n, y_n \in X$, real numbers $a \geqslant 0$, $\alpha, \beta \in (0,1)$ and $\{\alpha_n\}$ be a real sequence number which satisfies*

(i) $0 < \alpha \leqslant \alpha_n \leqslant \beta < 1$, $\forall n \geqslant n_0$ *and for some* $n_0 \in \mathbb{N}$;
(ii) $\limsup_{n\to\infty} \|x_n\| \leqslant a$ *and* $\limsup_{n\to\infty} \|y_n\| \leqslant a$;
(iii) $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = a$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. The proof is clear by Lemma 2.3. □

Lemma 3.2. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T : C \to C$ be asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y\in C}\left(\left\|T^n x - T^n y\right\| - \|x-y\|\right) \vee 0, \quad \forall n \geqslant 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\{x_n\}$ be defined by (1.1) *with the following restrictions*:

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all* $i \in \{1, 2, \dots, m\}$ *and for all* $n \geqslant 1$;
(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i \in \{1, 2, \dots, m\}$.

If $p \in F(T)$*, then* $\lim_{n\to\infty} \|x_n - p\|$ *exists.*

Proof. By [12], we have $F(T) \neq \emptyset$. Let $p \in F(T)$. For each $n \geqslant 1$, we note that

$$\begin{aligned}
\|x_n^{(1)} - p\| &= \|\alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\| \\
&\leqslant \alpha_n^{(1)} \|T^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\
&\leqslant \alpha_n^{(1)} \|x_n - p\| + \alpha_n^{(1)} G_n + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\
&= (\alpha_n^{(1)} + \beta_n^{(1)}) \|x_n - p\| + \alpha_n^{(1)} G_n + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\
&\leqslant \|x_n - p\| + d_n^{(1)},
\end{aligned} \tag{3.1}$$

where $d_n^{(1)} = \alpha_n^{(1)} G_n + \gamma_n^{(1)} \|u_n^{(1)} - p\|$. Since $\sum_{n=1}^{\infty} G_n < \infty$, we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (3.1) that

$$\begin{aligned}
\|x_n^{(2)} - p\| &\leqslant \alpha_n^{(2)} \|x_n^{(1)} - p\| + \alpha_n^{(2)} G_n + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&\leqslant \alpha_n^{(2)} (\|x_n - p\| + d_n^{(1)}) + \alpha_n^{(2)} G_n + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&= (\alpha_n^{(2)} + \beta_n^{(2)}) \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\
&\leqslant \|x_n - p\| + d_n^{(2)}
\end{aligned} \tag{3.2}$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} + \alpha_n^{(2)} G_n + \gamma_n^{(2)} \|u_n^{(2)} - p\|$. Since $\sum_{n=1}^{\infty} G_n < \infty$ and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned}
\|x_n^{(3)} - p\| &\leqslant \alpha_n^{(3)} \|x_n^{(2)} - p\| + \alpha_n^{(3)} G_n + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&\leqslant \alpha_n^{(3)} (\|x_n - p\| + d_n^{(2)}) + \alpha_n^{(3)} G_n + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&= (\alpha_n^{(3)} + \beta_n^{(3)}) \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\
&\leqslant \|x_n - p\| + d_n^{(3)},
\end{aligned} \tag{3.3}$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)} + \alpha_n^{(3)} G_n + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there are nonnegative real sequences $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} < \infty$ and

$$\|x_n^{(k)} - p\| \leqslant \|x_n - p\| + d_n^{(k)} \quad \text{for all } k = 1, 2, \dots, m.$$

This together with Lemma 2.2 gives that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.3. *Let* X *be a real uniformly convex Banach space,* C *a nonempty closed convex subset of* X *and* $T : C \to C$ *be asymptotically nonexpansive in the intermediate sense. Put*

$$G_n = \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \quad \forall n \geqslant 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$*. Let the sequence* $\{x_n\}$ *be defined by* (1.1) *whenever* $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ *satisfy the same assumptions as in Lemma* 3.2 *for each* $i \in \{1, 2, \dots, m\}$ *and the additional assumption that* $0 < \alpha \leqslant \alpha_n^{(m-1)}, \alpha_n^{(m)} \leqslant \beta < 1$ *for all* $n \geqslant n_0$*, for some* $n_0 \in \mathbb{N}$*. Then*

(a) $\lim_{n\to\infty}\|T^n x_n^{(m-1)} - x_n\| = 0$;
(b) $\lim_{n\to\infty}\|T^n x_n^{(m-2)} - x_n\| = 0$.

Proof. (a) For any $p \in F(T)$, it follows from Lemma 3.2 that $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = a$ for some $a \geqslant 0$. We note that

$$\left\|x_n^{(m-1)} - p\right\| \leqslant \|x_n - p\| + d_n^{(m-1)}, \quad \forall n \geqslant 1,$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n\to\infty}\left\|x_n^{(m-1)} - p\right\| \leqslant \limsup_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|x_n - p\| = a,$$

from which we have

$$\limsup_{n\to\infty}\left\|T^n x_n^{(m-1)} - p\right\| \leqslant \limsup_{n\to\infty}\left(\left\|x_n^{(m-1)} - p\right\| + G_n\right) = \limsup_{n\to\infty}\left\|x_n^{(m-1)} - p\right\| \leqslant a.$$

Next, we observe that

$$\left\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right\| \leqslant \left\|T^n x_n^{(m-1)} - p\right\| + \gamma_n^{(m)}\left\|u_n^{(m)} - x_n\right\|.$$

Thus we have

$$\limsup_{n\to\infty}\left\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right\| \leqslant a. \tag{3.4}$$

Also,

$$\left\|x_n - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right\| \leqslant \|x_n - p\| + \gamma_n^{(m)}\left\|u_n^{(m)} - x_n\right\|,$$

gives that

$$\limsup_{n\to\infty}\left\|x_n - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right\| \leqslant a, \tag{3.5}$$

and note that

$$\begin{aligned}
a &= \lim_{n\to\infty}\left\|x_n^{(m)} - p\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T^n x_n^{(m-1)} + \left(1 - \alpha_n^{(m)}\right)x_n - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \left(1 - \alpha_n^{(m)}\right)p - \alpha_n^{(m)} p\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} x_n + \left(1 - \alpha_n^{(m)}\right)x_n\right. \\
&\quad \left. - \left(1 - \alpha_n^{(m)}\right)p - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)} u_n^{(m)} + \alpha_n^{(m)}\gamma_n^{(m)} x_n\right\| \\
&= \lim_{n\to\infty}\left\|\alpha_n^{(m)}\left(T^n x_n^{(m-1)} - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right)\right. \\
&\quad \left. + \left(1 - \alpha_n^{(m)}\right)\left(x_n - p + \gamma_n^{(m)}\left(u_n^{(m)} - x_n\right)\right)\right\|.
\end{aligned}$$

This together with (3.4), (3.5) and Lemma 3.1, gives

$$\lim_{n\to\infty}\left\|T^n x_n^{(m-1)} - x_n\right\| = 0. \tag{3.6}$$

This completes the proof of (a).

Proof of (b). For each $n \geqslant 1$,

$$\begin{aligned}\|x_n - p\| &\leqslant \|x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - p\| \\ &\leqslant \|x_n - T^n x_n^{(m-1)}\| + \|x_n^{(m-1)} - p\| + G_n.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n\to\infty} G_n$, we obtain that

$$a = \lim_{n\to\infty} \|x_n - p\| \leqslant \liminf_{n\to\infty} \|x_n^{(m-1)} - p\|.$$

It follows that

$$a \leqslant \liminf_{n\to\infty} \|x_n^{(m-1)} - p\| \leqslant \limsup_{n\to\infty} \|x_n^{(m-1)} - p\| \leqslant a.$$

This implies that

$$\lim_{n\to\infty} \|x_n^{(m-1)} - p\| = a.$$

On the other hand, we note that

$$\|x_n^{(m-2)} - p\| \leqslant \|x_n - p\| + d_n^{(m-2)}, \quad \forall n \geqslant 1,$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. So that

$$\limsup_{n\to\infty} \|x_n^{(m-2)} - p\| \leqslant \limsup_{n\to\infty} \|x_n - p\| = a,$$

and hence

$$\limsup_{n\to\infty} \|T^n x_n^{(m-2)} - p\| \leqslant \limsup_{n\to\infty} \left(\|x_n^{(m-2)} - p\| + G_n\right) \leqslant a.$$

Next we observe that

$$\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leqslant \|T^n x_n^{(m-2)} - p\| + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\|.$$

Thus,

$$\limsup_{n\to\infty} \|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leqslant a. \tag{3.7}$$

Also,

$$\|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leqslant \|x_n - p\| + \gamma_n^{(m-1)}\|u_n^{(m-1)} - x_n\|,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)\| \leqslant a, \tag{3.8}$$

and note that

$$\begin{aligned}a &= \lim_{n\to\infty} \|x_n^{(m-1)} - p\| = \lim_{n\to\infty} \|\alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m-1)}(T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n)) \\ &\quad + (1 - \alpha_n^{(m-1)})(x_n - p + \gamma_n^{(m-1)}(u_n^{(m-1)} - x_n))\|.\end{aligned} \tag{3.9}$$

It follows from (3.7)–(3.9) and Lemma 3.1 that

$$\lim_{n\to\infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). □

We now state and prove the first main result of this paper and this is the main motivation of our next result.

Theorem 3.4. *Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T:C\to C$ be completely continuous asymptotically nonexpansive in the intermediate sense. Put*

$$G_n=\sup_{x,y\in C}\left(\left\|T^nx-T^ny\right\|-\|x-y\|\right)\vee 0,\quad \forall n\geqslant 1,$$

so that $\sum_{n=1}^{\infty}G_n<\infty$. Let the sequence $\{x_n\}$ be defined by (1.1) whenever $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma 3.2 for each $i\in\{1,2,\dots,m\}$ and the additional assumption that $0<\alpha\leqslant\alpha_n^{(m-1)},\alpha_n^{(m)}\leqslant\beta<1$ for all $n\geqslant n_0$, for some $n_0\in\mathbb{N}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T for each $k=1,2,3,\dots,m$.

Proof. It follows from Lemma 3.3 that

$$\lim_{n\to\infty}\left\|T^nx_n^{(m-1)}-x_n\right\|=0=\lim_{n\to\infty}\left\|T^nx_n^{(m-2)}-x_n\right\|$$

and this implies that,

$$\|x_{n+1}-x_n\|=\left\|x_n^{(m)}-x_n\right\|\leqslant\alpha_n^{(m)}\left\|T^nx_n^{(m-1)}-x_n\right\|+\gamma_n^{(m)}\left\|u_n^{(m)}-x_n\right\|\to 0 \quad \text{as } n\to\infty. \tag{3.10}$$

It follows from (3.10) that

$$\begin{aligned}\left\|T^nx_n-x_n\right\|&\leqslant\left\|T^nx_n-T^nx_n^{(m-1)}\right\|+\left\|T^nx_n^{(m-1)}-x_n\right\|\\&\leqslant\left\|x_n-x_n^{(m-1)}\right\|+G_n+\left\|T^nx_n^{(m-1)}-x_n\right\|\\&\leqslant\alpha_n^{(m-1)}\left\|x_n-T^nx_n^{(m-2)}\right\|+G_n+\gamma_n^{(m-1)}\left\|u_n^{(m-1)}-x_n\right\|\\&\quad+\left\|T^nx_n^{(m-1)}-x_n\right\|\to 0\quad\text{as } n\to\infty.\end{aligned}\tag{3.11}$$

Since

$$\begin{aligned}\|x_n-Tx_n\|\leqslant{}&\|x_{n+1}-x_n\|+\left\|x_{n+1}-T^{n+1}x_{n+1}\right\|+\left\|T^{n+1}x_{n+1}-T^{n+1}x_n\right\|\\&+\left\|T^{n+1}x_n-Tx_n\right\|,\end{aligned}$$

it follows from (3.10), (3.11) and uniformly continuity of T that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.12}$$

Since $\{x_n\}$ is a bounded and T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k}\to p\in C$ as $k\to\infty$. Moreover, by (3.12), we have $\|Tx_{n_k}-x_{n_k}\|\to 0$ which implies that $x_{n_k}\to p$ as $k\to\infty$. By (3.12) again, we have

$$\|p-Tp\|=\lim_{k\to\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

It shows that $p\in F(T)$. Since $\lim_{n\to\infty}\|x_n-p\|$ exists, we have $\lim_{n\to\infty}\|x_n-p\|=0$; that is $\lim_{n\to\infty}x_n^{(m)}=\lim_{n\to\infty}x_n=p$. Moreover, we observe that $\|x_n^{(k)}-p\|\leqslant\|x_n-p\|+d_n^{(k)}$ for all $k=1,2,3,\dots,m-1$ and each $\lim_{n\to\infty}d_n^{(k)}=0$. Therefore $\lim_{n\to\infty}x_n^{(k)}=p$ for all $k=1,2,3,\dots,m-1$. The proof is completed. □

4. Asymptotically quasi-nonexpansive mappings

In the next result, we prove strong convergence theorem for the multi-step Noor iterations (1.1) for asymptotically quasi-nonexpansive mapping in a uniformly convex Banach space. To do this, we need the following lemmas.

Lemma 4.1. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and T be asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by* (1.1) *with the following restrictions*:

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ *for all $i \in \{1, 2, \dots, m\}$ and for all $n \geqslant 1$*;
(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all $i \in \{1, 2, \dots, m\}$.*

If $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. For each $n \geqslant 1$, we note that

$$\begin{aligned}\|x_n^{(1)} - p\| &= \|\alpha_n^{(1)} T^n x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)} - p\| \\ &\leqslant \alpha_n^{(1)} \|T^n x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leqslant \alpha_n^{(1)} (1 + r_n)\|x_n - p\| + \beta_n^{(1)} \|x_n - p\| + \gamma_n^{(1)} \|u_n^{(1)} - p\| \\ &\leqslant (1 + r_n)\|x_n - p\| + d_n^{(1)}, \end{aligned} \tag{4.1}$$

where $d_n^{(1)} = \gamma_n^{(1)} \|u_n^{(1)} - p\|$. Since $\{u_n^{(1)}\}$ is bounded and $\sum_{n=1}^{\infty} \gamma_n^{(1)} < \infty$, we see that $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$. It follows from (4.1) that

$$\begin{aligned}\|x_n^{(2)} - p\| &\leqslant \alpha_n^{(2)} (1 + r_n)\|x_n^{(1)} - p\| + \beta_n^{(2)} \|x_n - p\| + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leqslant \alpha_n^{(2)} (1 + r_n)\big((1 + r_n)\|x_n - p\| + d_n^{(1)}\big) + \beta_n^{(2)} (1 + r_n)^2 \|x_n - p\| \\ &\quad + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &= \big(\alpha_n^{(2)} + \beta_n^{(2)}\big)(1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &\leqslant (1 + r_n)^2 \|x_n - p\| + \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\| \\ &= (1 + r_n)^2 \|x_n - p\| + d_n^{(2)}, \end{aligned} \tag{4.2}$$

where $d_n^{(2)} = \alpha_n^{(2)} d_n^{(1)} (1 + r_n) + \gamma_n^{(2)} \|u_n^{(2)} - p\|$. Since $\{u_n^{(2)}\}$ is bounded and $\sum_{n=1}^{\infty} d_n^{(1)} < \infty$, it follows that $\sum_{n=1}^{\infty} d_n^{(2)} < \infty$. Moreover, we see that

$$\begin{aligned}\|x_n^{(3)} - p\| &\leqslant \alpha_n^{(3)} (1 + r_n)\|x_n^{(2)} - p\| + \beta_n^{(3)} \|x_n - p\| + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leqslant \alpha_n^{(3)} (1 + r_n)\big((1 + r_n)^2\|x_n - p\| + d_n^{(2)}\big) + \beta_n^{(3)} (1 + r_n)^3 \|x_n - p\| \\ &\quad + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leqslant \big(\alpha_n^{(3)} + \beta_n^{(3)}\big)(1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &\leqslant (1 + r_n)^3 \|x_n - p\| + \alpha_n^{(3)} d_n^{(2)} (1 + r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\| \\ &= (1 + r_n)^3 \|x_n - p\| + d_n^{(3)}, \end{aligned} \tag{4.3}$$

where $d_n^{(3)} = \alpha_n^{(3)} d_n^{(2)}(1+r_n) + \gamma_n^{(3)} \|u_n^{(3)} - p\|$. So that $\sum_{n=1}^{\infty} d_n^{(3)} < \infty$. By continuing the above method, there are nonnegative real sequence $\{d_n^{(k)}\}$ such that $\sum_{n=1}^{\infty} d_n^{(k)} < \infty$ and

$$\|x_n^{(k)} - p\| \leqslant (1+r_n)^k \|x_n - p\| + d_n^{(k)} \quad \text{for all } k = 1, 2, \ldots, m.$$

By Lemma 2.2 , we have $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 4.2. *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to C$ be asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by* (1.1) *whenever $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma* 4.1 *for each $i \in \{1, 2, \ldots, m\}$ and the additional assumption that $0 < \alpha \leqslant \alpha_n^{(m-1)}, \alpha_n^{(m)} \leqslant \beta < 1$ for all $n \geqslant n_0$, for some $n_0 \in \mathbb{N}$. Then*

(a) $\lim_{n\to\infty} \|T^n x_n^{(m-1)} - x_n\| = 0$;
(b) $\lim_{n\to\infty} \|T^n x_n^{(m-2)} - x_n\| = 0$.

Proof. (a) For any $p \in F(T)$, it follows from Lemma 4.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geqslant 0$. We note that

$$\|x_n^{(m-1)} - p\| \leqslant (1+r_n)^{m-1} \|x_n - p\| + d_n^{(m-1)}, \quad \forall n \geqslant 1,$$

where $\{d_n^{(m-1)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-1)} < \infty$. It follows that

$$\limsup_{n\to\infty} \|x_n^{(m-1)} - p\| \leqslant \limsup_{n\to\infty} \big((1+r_n)^{m-1} \|x_n - p\| + d_n^{(m-1)}\big) = \lim_{n\to\infty} \|x_n - p\| = a$$

and so

$$\limsup_{n\to\infty} \|T^n x_n^{(m-1)} - p\| \leqslant \limsup_{n\to\infty} (1+r_n) \|x_n^{(m-1)} - p\| = \limsup_{n\to\infty} \|x_n^{(m-1)} - p\| \leqslant a.$$

Next, consider

$$\|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leqslant \|T^n x_n^{(m-1)} - p\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\|.$$

Thus,

$$\limsup_{n\to\infty} \|T^n x_n^{(m-1)} - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leqslant a. \tag{4.4}$$

Also,

$$\|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leqslant \|x_n - p\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\|,$$

gives that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n^{(m)}(u_n^{(m)} - x_n)\| \leqslant a, \tag{4.5}$$

and we observe that

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(m)} - p\| = \lim_{n\to\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + \beta_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} + (1-\alpha_n^{(m)}) x_n - \gamma_n^{(m)} x_n + \gamma_n^{(m)} u_n^{(m)} - (1-\alpha_n^{(m)}) p - \alpha_n^{(m)} p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m)} T^n x_n^{(m-1)} - \alpha_n^{(m)} p + \alpha_n^{(m)} \gamma_n^{(m)} u_n^{(m)} - \alpha_n^{(m)} \gamma_n^{(m)} x_n + (1-\alpha_n^{(m)}) x_n \end{aligned}$$

$$- \big(1-\alpha_n^{(m)}\big)p - \gamma_n^{(m)}x_n + \gamma_n^{(m)}u_n^{(m)} - \alpha_n^{(m)}\gamma_n^{(m)}u_n^{(m)} + \alpha_n^{(m)}\gamma_n^{(m)}x_n\big\|$$
$$= \lim_{n\to\infty}\big\|\alpha_n^{(m)}\big(T^n x_n^{(m-1)} - p + \gamma_n^{(m)}\big(u_n^{(m)} - x_n\big)\big)$$
$$+ \big(1-\alpha_n^{(m)}\big)\big(x_n - p + \gamma_n^{(m)}\big(u_n^{(m)} - x_n\big)\big)\big\|.$$

It follows from (4.4), (4.5) and Lemma 3.1 that

$$\lim_{n\to\infty}\big\|T^n x_n^{(m-1)} - x_n\big\| = 0.$$

This completes the proof of (a).

Proof of (b). For each $n \geqslant 1$, we have

$$\|x_n - p\| \leqslant \big\|x_n - T^n x_n^{(m-1)}\big\| + \big\|T^n x_n^{(m-1)} - p\big\|$$
$$\leqslant \big\|x_n - T^n x_n^{(m-1)}\big\| + (1+r_n)\big\|x_n^{(m-1)} - p\big\|.$$

Since $\lim_{n\to\infty}\|x_n - T^n x_n^{(m-1)}\| = 0 = \lim_{n\to\infty} r_n$, we obtain that

$$a = \lim_{n\to\infty}\|x_n - p\| \leqslant \liminf_{n\to\infty}\big\|x_n^{(m-1)} - p\big\|.$$

It follows that

$$a \leqslant \liminf_{n\to\infty}\big\|x_n^{(m-1)} - p\big\| \leqslant \limsup_{n\to\infty}\big\|x_n^{(m-1)} - p\big\| \leqslant a,$$

which implies that

$$\lim_{n\to\infty}\big\|x_n^{(m-1)} - p\big\| = a.$$

On the other hand, we note that

$$\big\|x_n^{(m-2)} - p\big\| \leqslant (1+r_n)^{m-2}\|x_n - p\| + d_n^{(m-2)}, \quad \forall n \geqslant 1,$$

where $\{d_n^{(m-2)}\}$ is a nonnegative real sequence such that $\sum_{n=1}^{\infty} d_n^{(m-2)} < \infty$. Thus

$$\limsup_{n\to\infty}\big\|x_n^{(m-2)} - p\big\| \leqslant \limsup_{n\to\infty}(1+r_n)^{m-2}\|x_n - p\| = a,$$

and hence

$$\limsup_{n\to\infty}\big\|T^n x_n^{(m-2)} - p\big\| \leqslant \limsup_{n\to\infty}(1+r_n)\big\|x_n^{(m-2)} - p\big\| \leqslant a.$$

Next, consider

$$\big\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big\| \leqslant \big\|T^n x_n^{(m-2)} - p\big\| + \gamma_n^{(m-2)}\big\|u_n^{(m-1)} - x_n\big\|.$$

Thus,

$$\limsup_{n\to\infty}\big\|T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big\| \leqslant a. \tag{4.6}$$

Also,

$$\big\|x_n - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big\| \leqslant \|x_n - p\| + \gamma_n^{(m-1)}\big\|u_n^{(m-1)} - x_n\big\|,$$

gives that

$$\limsup_{n\to\infty}\big\|x_n - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big\| \leqslant a, \tag{4.7}$$

and note that

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(m-1)} - p\| = \lim_{n\to\infty} \|\alpha_n^{(m-1)} T^n x_n^{(m-2)} + \beta_n^{(m-1)} x_n + \gamma_n^{(m-1)} u_n^{(m-1)} - p\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(m-1)}\big(T^n x_n^{(m-2)} - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big) \\ &\quad + \big(1 - \alpha_n^{(m-1)}\big)\big(x_n - p + \gamma_n^{(m-1)}\big(u_n^{(m-1)} - x_n\big)\big)\|. \end{aligned}$$

It follows from (4.6), (4.7) and Lemma 3.1 that

$$\lim_{n\to\infty} \|T^n x_n^{(m-2)} - x_n\| = 0.$$

This completes the proof of (b). □

Theorem 4.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X and $T : C \to C$ be uniformly L-Lipschitzian, completely continuous asymptotically quasi-nonexpansive with the sequence $\{r_n\}_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} r_n < \infty$ and $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by* (1.1) *whenever $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$, $\{\gamma_n^{(i)}\}$ satisfy the same assumptions as in Lemma* 4.1 *for each $i \in \{1, 2, \ldots, m\}$ and the additional assumption that $0 < \alpha \leqslant \alpha_n^{(i)} \leqslant \beta < 1$ for all $i \in \{m-1, m\}$. Then $\{x_n^{(k)}\}$ converges strongly to a fixed point of T, for each $k = 1, 2, 3, \ldots, m$.*

Proof. It follows from Lemma 4.2 that

$$\lim_{n\to\infty} \|T^n x_n^{(m-1)} - x_n\| = 0 = \lim_{n\to\infty} \|T^n x_n^{(m-2)} - x_n\|.$$

This implies that,

$$\|x_{n+1} - x_n\| = \|x_n^{(m)} - x_n\| \leqslant \alpha_n^{(m)} \|T^n x_n^{(m-1)} - x_n\| + \gamma_n^{(m)} \|u_n^{(m)} - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

Thus, we have

$$\begin{aligned} \|T^n x_n - x_n\| &\leqslant \|T^n x_n - T^n x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leqslant L\|x_n - x_n^{(m-1)}\| + \|T^n x_n^{(m-1)} - x_n\| \\ &\leqslant \alpha_n^{(m-1)} L \|x_n - T^n x_n^{(m-2)}\| + \gamma_n^{(m-1)} L \|u_n^{(m-1)} - x_n\| \\ &\quad + \|T^n x_n^{(m-1)} - x_n\| \to 0 \quad \text{as } n \to \infty \end{aligned} \tag{4.9}$$

and we note that

$$\begin{aligned} \|x_n - Tx_n\| &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\ &\leqslant \|x_{n+1} - x_n\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|. \end{aligned}$$

This together with (4.8) and (4.9) gives

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{4.10}$$

By the boundedness of $\{x_n\}$ and our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to p \in C$ as $k \to \infty$. Moreover, by (4.10), we have $\|Tx_{n_k} - x_{n_k}\| \to 0$ which implies that $x_{n_k} \to p$ as $k \to \infty$. By (4.10) again, we have

$$\|p - Tp\| = \lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0.$$

It show that $p \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - p\|$ exists, we obtain $\lim_{n\to\infty} \|x_n - p\| = 0$, that is $\lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} x_n = p$. Moreover, we observe that $\|x_n^{(k)} - p\| \leqslant \|x_n - p\| + d_n^{(k)}$ for all $k = 1, 2, 3, \dots, m-1$ and each $\lim_{n\to\infty} d_n^{(k)} = 0$. Therefore $\lim_{n\to\infty} x_n^{(k)} = p$ for all $k = 1, 2, 3, \dots, m-1$. The proof is completed. □

For $m = 3$ and $\beta_n^{(i)} = 1 - \alpha_n^{(i)} - \gamma_n^{(i)}$ for all $i = 1, 2, 3$ in Theorem 3.4 or Theorem 4.3, we obtain the following result.

Theorem 4.4. (See [2]) *Let X be uniformly convex Banach space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with the nonempty fixed-point set $F(T)$ and a sequence $\{r_n\}$ in $[0, \infty)$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let a sequence $\{x_n\}$ be defined by (1.2) with the following restrictions:*

(i) $0 < a \leqslant \alpha_n^{(3)} < b < 1$;
(ii) $\limsup_{n\to\infty} (1 + r_n)\alpha_n^{(2)} < 1$;
(iii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$ *for all* $i = 1, 2, 3$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point p of T.

When $m = 3$ and $\gamma_n^{(1)} = \gamma_n^{(2)} = \gamma_n^{(3)} \equiv 0$ in Theorem 3.4 or Theorem 4.3, we obtain strong convergence theorem for Noor iteration as follows:

Theorem 4.5. [19, Theorem 2.1] *Let X be a real uniformly convex Banach space, C be a nonempty closed, bounded convex subset of X. Let $T : C \to C$ be a completely continuous asymptotically nonexpansive self-mapping with sequence $\{r_n\}$ satisfying $r_n \geqslant 0$ and $\sum_{n=1}^{\infty} r_n < \infty$. Let $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\alpha_n^{(3)}\}$ be real sequences in $[0, 1]$ satisfying:*

(i) $0 < \liminf_{n\to\infty} \alpha_n^{(3)} \leqslant \limsup_{n\to\infty} \alpha_n^{(3)} < 1$, *and*
(ii) $0 < \liminf_{n\to\infty} \alpha_n^{(2)} \leqslant \limsup_{n\to\infty} \alpha_n^{(2)} < 1$.

For a given $x_1 \in C$, the sequence $\{x_n\}$, $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$ defined by (1.3) converges strongly to a fixed point of T.

Proof. It follows from the conditions (i) and (ii) that there are $\alpha, \beta \in (0, 1)$ and $n_0 \in \mathbb{N}$ such that

$$0 < \alpha \leqslant \alpha_n^{(2)}, \qquad \alpha_n^{(3)} \leqslant \beta < 1$$

for all $n \geqslant n_0$. So that the conclusion of the theorem follows from Theorem 3.4 or Theorem 4.3.

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