Journal of the Egyptian Mathematical Society (2016) **24**, 175–180

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Some Simpson type integral inequalities for functions whose third derivatives are (α, m)-GA-convex functions

YuJiao Li [a], **TingSong Du** [a,b,*]

[a] *Department of Mathematics, Science College, China Three Gorges University, Yichang 443002, China*
[b] *Hubei Province Key Laboratory of System Science in Metallurgical Process, Wuhan University of Science and Technology, Wuhan 430081, China*

Received 6 April 2015; accepted 6 May 2015
Available online 26 September 2015

Keywords

Convex function;
(α, m)-GA-convex functions;
Simpson type inequality

Abstract By using power-mean integral inequality and Hölder's integral inequality, this paper establishes some new inequalities of Simpson type for functions whose three derivatives in absolute value are the class of (α, m)-geometric-arithmetically-convex functions. Finally, some applications to special means of positive real numbers have also been presented.

2010 MATHEMATICAL SUBJECT CLASSIFICATION: 26D15; 26A51; 26E60; 41A55

1. Introduction

The classical Simpson type inequality has attracted considerable attention since it is very important and remarkable in the area of inequality application. Plenty of new Simpson type inequalities for convex functions have been refined and extended by many mathematicians in a lot of references, such as [1–3], and so on. In recent years, Many studies about Simpson type inequalities can be found by Xi and Qi [4] for logarithmically convex functions, by Sarikaya et al. [5] for s-convex functions, by Chun and Qi [6] for extended s-convex functions, by Hua et al. [7] for strongly s-convex functions, and by Qaisar et al. [8] for (α, m)-convex functions in the published papers.

With the development of inequality researches, the inequalities for generalized convex functions have a rapid blossom in the field of convex analysis. For example, geometric-arithmetically-convex functions is one of the generalized convex functions. Very recently, Shuang et al. [9] established

* Corresponding author.
E-mail addresses: yujiaolictgu@163.com (Y. Li), hbycdts@163.com (T. Du).
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.05.009

Hermite–Hadamard type integral inequalities for s-geometric-arithmetically-convex functions. Hua et al. [10] also studied s-geometric-arithmetically-convex functions, which concerning about two differentiable mappings. In 2013, Park [11] and Ji et al. [12] used the definition of (α, m)-geometric-arithmetically-convex functions to prove some new Hermite-Hadamard type inequalities, while Park introduced twice differentiable (α, m)-geometric-arithmetically convex functions. As mentioned above, these papers are all involved with Hermite–Hadamard type inequalities. However, to our knowledge, Simpson type inequalities for functions whose three derivatives in absolute value are the class of (α, m)-geometric-arithmetically-convex functions have not been reported. So we turn our attention to this new research.

Motivated by [11–13], we are concerned in this paper with some new Simpson type inequalities for functions whose three derivatives in absolute value are the class of (α, m)-geometric-arithmetically-convex functions rather than Hermite-Hadamard type inequalities. Although the inequality for functions whose derivatives in absolute value are all (α, m)-geometric-arithmetically-convex functions in [11,12], a point that should be stressed is that this paper is associated to third derivatives, which has higher derivative than the previous works [11,12]. That is to say, this present paper continues the extension of previous works.

An outline of this paper is as follows. Some preliminaries, including definitions and lemmas are introduced in Section 2. Some new results about Simpson type inequalities for (α, m)-geometric-arithmetically-convex functions are then established in Section 3. Finally some applications to special means of real numbers are given in Section 4 .

2. Preliminaries

Throughout this paper, we consider a real interval $I \subseteq \mathbb{R}$, and we denote that I^0 is the interior of I.

The following inequality is well known in the literature as Simpson type inequality:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4, \tag{2.1}$$

where the mapping $f:[a,\ b]\to\mathbb{R}$ is assumed to be four times continuously differentiable on the interval (a, b) and $\|f^{(4)}\|_\infty = sup_{t\in(a,b)}|f^{(4)}(t)| < \infty$.

Next let us recall concepts of geometric-arithmetically-convex and (α, m)-geometric-arithmetically-convex functions.

Definition 2.1 [11]**.** The function $f:\mathbb{R}_0\to\mathbb{R}$ is said to be geometric-arithmetically convex or GA-convex on I, if

$$f(x^t y^{1-t}) \le t f(x) + (1-t) f(y) \tag{2.2}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$, where $x^t y^{1-t}$ and $tf(x) + (1-t)f(y)$ are respectively called the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

Definition 2.2 [14]**.** The function $f:[0,\ b]\to\mathbb{R}$ and $(\alpha, m)\in[0,1]^2$. If

$$f(x^t y^{m(1-t)}) \le t^\alpha f(x) + m(1-t^\alpha) f(y) \tag{2.3}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$, then $f(x)$ is said to be (α, m)-GA-convex function. If (2.3) is reversed, then $f(x)$ is said to be (α, m)-GA-concave function.

Remark 2.1. It is sure that GA-convexity means just (α, m)-GA-convexity when $\alpha = 1$ and $m = 1$.

To establish some new Simpson type inequalities for (α, m)-GA-convex functions, we need the following lemmas.

Lemma 2.1 ([15, Lemma 2.5])**.** *For $t \in [0, 1]$, a, $b > 0$, we have*

$$\frac{1-t}{2}a + \frac{1+t}{2}b \ge a^{\frac{1-t}{2}} b^{\frac{1+t}{2}}. \tag{2.4}$$

Lemma 2.2 ([6, Lemma 2.1])**.** *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on I^0, and $a, b \in I$ with $a < b$. If $f''' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du = \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left[f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right) - f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt. \tag{2.5}$$

3. Simpson type inequalities for (α, m)-GA-convex functions

In what follows, we are now in a position to present and prove some new Simpson type inequalities for functions whose three derivative absolute values are (α, m)-GA-convex functions.

Theorem 3.1. *Let $f: I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable on I^0, a, $b \in I$ with $0 < a < b < \infty$, $f''' \in L[a, b]$, and $|f'''|$ be decreasing on $[a, b]$. If $|f'''|^q$ is $(\alpha,\ m)-GA$-convex on $[0, max\{a^{\frac{1}{m}}, b\}]$ for $(\alpha, m) \in [0, 1]^2$ and $q \ge 1$, then the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(b-a)^3}{96}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[\left(K_1|f'''(b)|^q + m\left(\frac{1}{12}-K_1\right)\left|f'''\left(a^{\frac{1}{m}}\right)\right|^q\right)^{\frac{1}{q}} + \left(K_2|f'''(b)|^q + m\left(\frac{1}{12}-K_2\right)\left|f'''\left(a^{\frac{1}{m}}\right)\right|^q\right)^{\frac{1}{q}}\right] \tag{3.1}$$

holds, where

$$K_1 = \left(\frac{1}{2}\right)^\alpha \frac{(\alpha-2)2^{\alpha+4}+\alpha^2+11\alpha+34}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \quad \text{and}$$

$$K_2 = \left(\frac{1}{2}\right)^\alpha \frac{1}{(\alpha+3)(\alpha+4)}.$$

Proof. Since $|f'''|$ is decreasing on $[a, b]$, from Lemma 2.1, 2.2 and using power-mean integral inequality, it follows that

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left[\left|f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right.$$

$$+\left|f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\Bigg]dt$$

$$\le \frac{(b-a)^3}{96}\left[\int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}\right)\right|dt + \int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt\right]$$

$$\le \frac{(b-a)^3}{96}\left[\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1 t(1-t)^2\left|f'''\left((a^{\frac{1}{m}})^{\frac{(1-t)m}{2}}b^{\frac{1+t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1 t(1-t)^2\left|f'''\left((a^{\frac{1}{m}})^{\frac{(1+t)m}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Since $|f'''|^q$ is $(\alpha,\ m)-GA$-convex on $[0, max\{a^{\frac{1}{m}},\ b\}]$, the following inequalities

$$\left|f'''\left((a^{\frac{1}{m}})^{\frac{(1-t)m}{2}}b^{\frac{1+t}{2}}\right)\right|^q \le \left(\frac{1+t}{2}\right)^\alpha |f'''(b)|^q + m\left(1-\left(\frac{1+t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q$$

and

$$\left|f'''\left((a^{\frac{1}{m}})^{\frac{(1+t)m}{2}}b^{\frac{1-t}{2}}\right)\right|^q \le \left(\frac{1-t}{2}\right)^\alpha |f'''(b)|^q + m\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q$$

hold, so we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{(b-a)^3}{96}\left[\left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1 t(1-t)^2\left(\left(\frac{1+t}{2}\right)^\alpha|f'''(b)|^q + m\left(1-\left(\frac{1+t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)dt\right)^{\frac{1}{q}} + \left(\int_0^1 t(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)^2\left(\left(\frac{1-t}{2}\right)^\alpha|f'''(b)|^q + m\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)dt\right)^{\frac{1}{q}}\right].$$

Using the fact that

$$\int_0^1 t(1-t)^2\left(\frac{1+t}{2}\right)^\alpha = \left(\frac{1}{2}\right)^\alpha\frac{(\alpha-2)2^{\alpha+4}+\alpha^2+11\alpha+34}{(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)} \triangleq K_1$$

and

$$\int_0^1 t(1-t)^2\left(\frac{1-t}{2}\right)^\alpha = \left(\frac{1}{2}\right)^\alpha\frac{1}{(\alpha+3)(\alpha+4)} \triangleq K_2,$$

we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{(b-a)^3}{96}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[\left(K_1|f'''(b)|^q + m\left(\frac{1}{12}-K_1\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}} + \left(K_2\left|f'''(b)\right|^q + m\left(\frac{1}{12}-K_2\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right],$$

which is the required. □

Remark 3.1. Under the condition of Theorem 3.1,
(1) if we choose $q=1$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(b-a)^3}{96}\left((K_1+K_2)|f'''(b)| + m\left(\frac{1}{6}-K_1-K_2\right)\left|f'''(a^{\frac{1}{m}})\right|\right);$$

(2) if we choose $\alpha=1$ and $m=1$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(b-a)^3}{96}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}\left[\left(\frac{7}{120}|f'''(b)|^q+\frac{3}{120}|f'''(a)|^q\right)^{\frac{1}{q}} + \left(\frac{3}{120}|f'''(b)|^q+\frac{7}{120}|f'''(a)|^q\right)^{\frac{1}{q}}\right];$$

(3) if we choose $q=1$, $\alpha=1$ and $m=1$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(b-a)^3}{1152}(|f'''(a)|+|f'''(b)|).$$

Theorem 3.2. *Let $f: I\subseteq\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be a differentiable on I^0, $a,b\in I$ with $0<a<b<\infty$, $f'''\in L[a,b]$, and $|f'''|$ be decreasing on $[a,b]$. If $|f'''|^q$ is $(\alpha,\ m)-GA$-convex on $[0, max\{a^{\frac{1}{m}},\ b\}]$ for $(\alpha,m)\in[0,1]^2$ and $q>1$, then the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$

$$\le \frac{(b-a)^3}{96}\left(B\left(\frac{2q-1}{q-1},\frac{3q-1}{q-1}\right)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+2}-1}{\alpha+1}|f'''(b)|^q + m\left(1-\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+2}-1}{\alpha+1}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}} + \left(\left(\frac{1}{2}\right)^\alpha\frac{1}{\alpha+1}|f'''(b)|^q + m\left(1-\left(\frac{1}{2}\right)^\alpha\frac{1}{\alpha+1}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right] \tag{3.2}$$

holds, where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

for $x>0$ and $y>0$ is the noted Beta function.

Proof. We proceed similarly as in the proof of Theorem 3.1, but using Hölder's integral inequality instead of the power-mean integral inequality for $(\alpha,\ m)-GA$-convex functions, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left[\left|f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| + \left|f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\le \frac{(b-a)^3}{96}\left[\int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}\right)\right|dt + \int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt\right] \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 (t(1-t)^2)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1\left|f'''\left(\left(a^{\frac{1}{m}}\right)^{\frac{(1-t)m}{2}}b^{\frac{1+t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1\left|f'''\left(\left(a^{\frac{1}{m}}\right)^{\frac{(1+t)m}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t^{\frac{2q-1}{q-1}-1}(1-t)^{\frac{3q-1}{q-1}-1}dt\right)^{1-\frac{1}{q}} \\
&\left[\left(\int_0^1\left(\left(\frac{1+t}{2}\right)^\alpha|f'''(b)|^q + m\left(1-\left(\frac{1+t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left(\left(\frac{1-t}{2}\right)^\alpha\left|f'''(b)\right|^q + m\left(1-\left(\frac{1-t}{2}\right)^\alpha\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By simple integral calculations and the famous Beta function, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(b-a)^3}{96}\left(B\left(\frac{2q-1}{q-1},\frac{3q-1}{q-1}\right)\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+1}-1}{\alpha+1}|f'''(b)|^q + m\left(1-\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+1}-1}{\alpha+1}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left(\frac{1}{2}\right)^\alpha\frac{1}{\alpha+1}|f'''(b)|^q + m\left(1-\left(\frac{1}{2}\right)^\alpha\frac{1}{\alpha+1}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence the statement in Theorem 3.2 is proved. □

Remark 3.2. Under the condition of Theorem 3.2, if we choose $\alpha = 1$ and $m = 1$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(b-a)^3}{96}\left(B\left(\frac{2q-1}{q-1},\frac{3q-1}{q-1}\right)\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\frac{3}{4}|f'''(b)|^q+\frac{1}{4}|f'''(a)|^q\right)^{\frac{1}{q}} + \left(\frac{1}{4}|f'''(b)|^q+\frac{3}{4}|f'''(a)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 3.3. *Let $f : I \subseteq \mathbb{R}_0 = [0,\infty) \to \mathbb{R}$ be a differentiable on I^0, $a,\ b \in I$ with $0 < a < b < \infty$, $f''' \in L[a,b]$, and $|f'''|$ be decreasing on $[a,b]$. If $|f'''|^q$ is (α,m)-GA-convex on $[0, max\{a^{\frac{1}{m}},\ b\}]$ for $(\alpha,m) \in [0,1]^2$ and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(b-a)^3}{96}\left(\frac{(q-1)^2}{(3q-1)(4q-2)}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+1}\alpha+1}{(\alpha+1)(\alpha+2)}|f'''(b)|^q + m\left(\frac{1}{2}-\left(\frac{1}{2}\right)^\alpha\frac{2^{\alpha+1}\alpha+1}{(\alpha+1)(\alpha+2)}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\left(\frac{1}{2}\right)^\alpha\frac{1}{(\alpha+1)(\alpha+2)}|f'''(b)|^q + m\left(\frac{1}{2}-\left(\frac{1}{2}\right)^\alpha\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'''(a^{\frac{1}{m}})\right|^q\right)^{\frac{1}{q}}\right]. \qquad (3.3)
\end{aligned}$$

Proof. Similar as Theorem 3.2, using another version of Hölder's integral inequality, then

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(b-a)^3}{96}\int_0^1 t(1-t)^2\left[\left|f'''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| + \left|f'''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]dt \\
&\le \frac{(b-a)^3}{96}\left[\int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1-t}{2}}b^{\frac{1+t}{2}}\right)\right|dt + \int_0^1 t(1-t)^2\left|f'''\left(a^{\frac{1+t}{2}}b^{\frac{1-t}{2}}\right)\right|dt\right] \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t((1-t)^2)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left(\int_0^1 t\left|f'''\left(\left(a^{\frac{1}{m}}\right)^{\frac{(1-t)m}{2}}b^{\frac{1+t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 t\left|f'''\left(\left(a^{\frac{1}{m}}\right)^{\frac{(1+t)m}{2}}b^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\le \frac{(b-a)^3}{96}\left(\int_0^1 t((1-t)^2)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}} \\
&\left[\left(\int_0^1 t\left(\left(\frac{1+t}{2}\right)^\alpha|f'''(b)|^q\right.\right.\right.
\end{aligned}$$

$$+m\left(1-\left(\frac{1+t}{2}\right)^{\alpha}\right)\left|f'''(a^{\frac{1}{m}})\right|^{q}\right)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t\left(\left(\frac{1-t}{2}\right)^{\alpha}|f'''(b)|^{q}\right.\right.$$
$$\left.\left.+m\left(1-\left(\frac{1-t}{2}\right)^{\alpha}\right)\left|f'''(a^{\frac{1}{m}})\right|^{q}\right)dt\right)^{\frac{1}{q}}\Bigg].$$

Using the fact that

$$\int_0^1 t((1-t)^2)^{\frac{q}{q-1}}dt=\frac{(q-1)^2}{(3q-1)(4q-2)},$$
$$\int_0^1 t\left(\frac{1+t}{2}\right)^{\alpha}dt=\left(\frac{1}{2}\right)^{\alpha}\frac{2^{\alpha+1}\alpha+1}{(\alpha+1)(\alpha+2)}$$

and

$$\int_0^1 t\left(\frac{1-t}{2}\right)^{\alpha}dt=\left(\frac{1}{2}\right)^{\alpha}\frac{1}{(\alpha+1)(\alpha+2)},$$

we get

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(b-a)^3}{96}\left(\frac{(q-1)^2}{(3q-1)(4q-2)}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg[\left(\left(\frac{1}{2}\right)^{\alpha}\frac{2^{\alpha+1}\alpha+1}{(\alpha+1)(\alpha+2)}|f'''(b)|^{q}\right.$$
$$\left.+m\left(\frac{1}{2}-\left(\frac{1}{2}\right)^{\alpha}\frac{2^{\alpha+1}\alpha+1}{(\alpha+1)(\alpha+2)}\right)\left|f'''(a^{\frac{1}{m}})\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{1}{2}\right)^{\alpha}\frac{1}{(\alpha+1)(\alpha+2)}|f'''(b)|^{q}\right.$$
$$\left.+m\left(\frac{1}{2}-\left(\frac{1}{2}\right)^{\alpha}\frac{1}{(\alpha+1)(\alpha+2)}\right)\left|f'''(a^{\frac{1}{m}})\right|^{q}\right)^{\frac{1}{q}}\Bigg],$$

which completes the proof of Theorem 3.3. □

Remark 3.3. Under the condition of Theorem 3.3, if we choose $\alpha=1$ and $m=1$, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)du\right|$$
$$\leq\frac{(b-a)^3}{96}\left(\frac{(q-1)^2}{(3q-1)(4q-2)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{5}{12}|f'''(b)|^{q}+\frac{1}{12}|f'''(a)|^{q}\right)^{\frac{1}{q}}+\left(\frac{1}{12}|f'''(b)|^{q}+\frac{5}{12}|f'''(a)|^{q}\right)^{\frac{1}{q}}\right].$$

4. Applications to special means

For positive numbers, $\beta>\alpha>0$ and $n\in\mathbb{Z}\setminus\{0,-1\}$, define

$$A(\alpha,\ \beta)=\frac{\alpha+\beta}{2}\quad\text{and}\quad L_n(\alpha,\ \beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}.$$

These quantities are respectively called the arithmetic and generalized logarithmic means of two positive number α and β.

Let $f(x)=\frac{x^{r+2}}{r(r+1)(r+2)}$ for $x>0$ and for some fixed $r\in(0,1)$. It is obvious that $|f'''(x)|=x^{(r-1)}$ is decreasing on $\mathbb{R}^+$.

We note that

$$\begin{aligned}|f'''(x^t y^{(1-t)})|^q&=[x^{q(r-1)}]^t[y^{q(r-1)}]^{1-t}\\&\leq tx^{q(r-1)}+(1-t)y^{q(r-1)}\\&=t|f'''(x)|^q+(1-t)|f'''(y)|^q\end{aligned}$$

holds, for $t\in[0,1]$, $x,y>0$ and $q\geq1$. This implies that the function $|f'''(x)|^q=x^{(r-1)q}$ is GA-convex on $\mathbb{R}^+$.

Considering the above function *f*(*x*), we have the following means:

$$\frac{f(a)+f(b)}{2}=\frac{1}{r(r+1)(r+2)}A(a^{r+2},b^{r+2}),$$
$$f\left(\frac{a+b}{2}\right)=\frac{1}{r(r+1)(r+2)}A^{r+2}(a,b)$$

and

$$\frac{1}{b-a}\int_a^b f(u)du=\frac{1}{r(r+1)(r+2)}L_{r+2}^{r+2}(a,b).$$

Now applying Remark 3.1 (2), Remark 3.2 and Remark 3.3 respectively to the function $\left|f'''(x)\right|^q=x^{(r-1)q}$ yield the following propositions.

Proposition 4.1. *Let $f:[a,b]\to\mathbb{R}$, $0<a<b$, $0<r<1$ and $q\geq1$, then we have the following inequality:*

$$\left|\frac{1}{3}A(a^{r+2},b^{r+2})+\frac{2}{3}A^{r+2}(a,b)-L_{r+2}^{r+2}(a,b)\right|$$
$$\leq\frac{r(r+1)(r+2)(b-a)^3}{96}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{7}{120}|b|^{(r-1)q}+\frac{3}{120}|a|^{(r-1)q}\right)^{\frac{1}{q}}+\left(\frac{3}{120}|b|^{(r-1)q}+\frac{7}{120}|a|^{(r-1)q}\right)^{\frac{1}{q}}\right].\tag{4.1}$$

Proposition 4.2. *Let $f:[a,b]\to\mathbb{R}$, $0<a<b$, $0<r<1$ and $q>1$, then we get the following inequality:*

$$\left|\frac{1}{3}A(a^{r+2},b^{r+2})+\frac{2}{3}A^{r+2}(a,b)-L_{r+2}^{r+2}(a,b)\right|$$
$$\leq\frac{r(r+1)(r+2)(b-a)^3}{96}\left(B\left(\frac{2q-1}{q-1},\frac{3q-1}{q-1}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{3}{4}|b|^{(r-1)q}+\frac{1}{4}|a|^{(r-1)q}\right)^{\frac{1}{q}}+\left(\frac{1}{4}|b|^{(r-1)q}+\frac{3}{4}|a|^{(r-1)q}\right)^{\frac{1}{q}}\right].\tag{4.2}$$

Proposition 4.3. *Let $f:[a,b]\to\mathbb{R}$, $0<a<b$, $0<r<1$ and $q>1$, then we obtain the following inequality:*

$$\left|\frac{1}{3}A(a^{r+2},b^{r+2})+\frac{2}{3}A^{r+2}(a,b)-L_{r+2}^{r+2}(a,b)\right|$$
$$\leq\frac{r(r+1)(r+2)(b-a)^3}{96}\left(\frac{(q-1)^2}{(3q-1)(4q-2)}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{5}{12}|b|^{(r-1)q}+\frac{1}{12}|a|^{(r-1)q}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{12}|b|^{(r-1)q}+\frac{5}{12}|a|^{(r-1)q}\right)^{\frac{1}{q}}\right]. \quad (4.3)$$

Acknowledgments

The research was supported by the National Natural Science foundation of China (No. 61374028), Hubei Province Key Laboratory of Systems Science in Metallurgical Process of China (No. Z201402), and the Natural Science Foundation of Hubei Province, China under Grant 2013CFA131. Finally we thank the referees for their time and comments.

References

[1] M. Alomari, M. Darus, S.S. Dragomir, New inequalities of Simpson's type for s-convex functions with applications, RGMIA Res. Rep. Coll. 12 (4) (2009). Article 9. http://ajmaa.org/RGMIA/v12n4.php.

[2] S.S. Dragomir, R.P. Agarwal, P. Cerone, On Simpson's inequality and applications, J. Inequal. Appl. 5 (2000) 533–579.

[3] B.Z. Liu, An inequality of Simpson type, Proc. R. Soc. 461 (4) (2005) 2155–2158.

[4] B.Y. Xi, F. Qi, Integral inequalities of Simpson type for logarithmically convex functions, Adv. Stud. Contemp. Math. (Kyungshang) 23 (4) (2013) 559–566. http://dx.doi.org/10.1515/gmj-2013-0043.

[5] M.Z. Sarikaya, E. Set, M.E. Ozdemir, On new inequalities of Simpson's type for s-convex functions, Comput. Math. Appl. 60 (2010) 2191–2199.

[6] L. Chun, F. Qi, Inequalities of Simpson type for functions whose third derivatives are extended s-convex functions and applications to means, J. Comp. Anal. Appl. 19 (3) (2015) 555–569.

[7] J. Hua, B.Y. Xi, F. Qi, Some new inequalities of Simpson type for strongly s-convex functions, Afr. Mat. (2014). 10.1007/s13370-014-0242-2.

[8] S. Qaisar, C.J. He, S. Hussain, A generalizations of Simpsons type inequality for differentiable functions using (α, m)-convex functions and applications, J. Inequal. Appl. 158 (2013). http://www.journalofinequalitiesandapplications.com/content/2013/1/-158.

[9] Y. Shuang, H.P. Yin, F. Qi, Hermite–Hadamard type integral inequalities for geometric-arithmetically s-convex functions, Analysis 33 (2013) 197–208.

[10] J. Hua, B.Y. Xi, F. Qi, Hermite–Hadamard type inequalities for geometric-arithmetically s-convex functions, Commun. Korean Math. Soc. 29 (1) (2014) 51–63.

[11] J. Park, New Hermite–Hadamard-like type inequalities for twice differentiable (α, m)-ga-convex functions, Int. J. Math. Anal. 7 (51) (2013) 2503–2515.

[12] A.P. Ji, T.Y. Zhang, F. Qi, Integral inequalities of Hermite–Hadamard type for (α, m)-ga-convex functions, J. Funct. Spaces Appl. (2013) 8. Article ID 823856.

[13] F. Qi, B.Y. Xi, Some integral inequalities of Simpson type for GA-ε-convex functions, Georgian Math. J. 20 (4) (2014) 775–788. http://dx.doi.org/10.1515/gmj-2013-0043.

[14] M.K. Bakula, J. Pečarić, Note on some Hadamard-type inequalities, J. Inequal. Pure Appl. Math. 5 (3) (2004) 9. Article 74.

[15] Y.M. Liao, J.P.H. Deng, J.P.R. Wang, Riemann–Liouville fractional Hermite–Hadamard inequalities. Part I: for once differentiable geometric-arithmetically s-convex functions, J. Inequal. Appl. 443 (2013). http://www.journalofinequalitiesandapplications.com/content/2013/1/443.