Available online at www.sciencedirect.com
SCIENCE DIRECT
PHYSICS LETTERS B
Physics Letters B 603 (2004) 63–73
www.elsevier.com/locate/physletb

# Some simple predictions from $E_{11}$ symmetry

Peter West

*Department of Mathematics King's College, London WC2R 2LS, UK*
*Erwin Schrödinger International Institute for Mathematical Physics, Boltzmanngasse 9, 1090 Wien, Austria*

Received 15 July 2004; accepted 6 October 2004

Available online 18 October 2004

Editor: P.V. Landshoff

**Abstract**

The simplest consequences of the common $E_{11}$ symmetry of the eleven-dimensional, IIA and IIB theories are derived and are shown to imply the known relations between these three theories.


## 1. Introduction

It has been argued that eleven-dimensional supergravity [1] when suitably extended possess a non-linearly realised $E_{11}$ symmetry [2]. Furthermore, both the IIA supergravity [3] and IIB supergravity theories [4,5], when suitably extended, should also possess a non-linearly realised $E_{11}$ symmetry [2,6]. As explained below, the three different theories arise from the same underlying algebra due to the different possible embeddings in $E_{11}$ of the sub-algebras that describe gravity [2,6]. Their common $E_{11}$ symmetry can be exploited to find explicit relations between the eleven-dimensional, IIA and IIB non-linearly realised theories [7]. Indeed, one can find a one-to-one correspondence between the fields that occur in any two of these three theories providing a very concrete idea of what M theory actually is [7]. In this Letter we derive the simplest of these relations which are those that involve fields that are associated with the Cartan sub-algebra of $E_{11}$. We recover the known relations between the eleven-dimensional, IIA and IIB theories, when dimensionally reduced on a suitable torus, in a simple way. Originally these relations were found by using a mixture of string and solitonic properties [8–13], but we will show that they follow from the way the sub-algebra associated with the gravity sectors of the different theories are embedded in $E_{11}$. We also give an example of how the correspondence works for a field not associated with the Cartan sub-algebra and derive the effect of the Weyl transformations of the $E_8^{+++}$ theory for the IIA and IIB theories.

*E-mail address:* pwest@mth.kcl.ac.uk (P. West).


doi:10.1016/j.physletb.2004.10.015

An account of Kac–Moody algebras suitable for this Letter can be found in previous papers of the author, e.g., [14] or in the extended version of this Letter that appears on hep-th.

So little is known about Kac–Moody algebras that it is difficult to calculate the general properties of a non-linear realisation based upon them. However, by setting to zero all the fields of the non-linear realisation, except those associated with the Cartan sub-algebra, the group element takes the very simple form

$$g=\exp\left(\sum_a q_a H_a\right). \tag{1.1}$$

The fields $q_a$ are then the only fields of the theory. Provided one restricts ones attention to operations that preserve the Cartan sub-algebra it is then essentially trivial to examine the consequences. Such is the case for Weyl transformations. Indeed, these were considered in just such a setting for the $E_8^{+++}$ non-linear realisation appropriate to the eleven-dimensional theory and the Weyl transformations were shown [15] to be are none other than the U-duality transformations [18].

The algebra $G^{+++}$ contains a GL(D) sub-algebra, with generators $K^a{}_b$, $a,b=1,\ldots,D$ which leads in the non-linear realisation to the gravity sector of the resulting theory where $D$ is the dimension of the space–time of the theory. The $A_{D-1}$, or SL(D), part of this sub-algebra is obtained by taking $D-1$ dots of the Dynkin diagram of $G^{+++}$ which are connected to the very extended node, i.e., selecting an $A_{D-1}$ sub-Dynkin diagram which contains as an extreme node the very extended node. As we shall see, there is more than one way to do this in general and these lead to different physical theories. The part of the group element of $G^{+++}$ which contains the generators of the preferred sub-algebra is of the form $\exp(\sum_{a\leqslant b} h_a{}^b K^a{}_b)$ and carrying out the non-linear realisation one finds that the vierbein $e_\mu{}^a$ is identified with $e_\mu{}^a=(e^h)_\mu{}^a$, where in this last equation, we treat $h$ as a matrix [2,16]. This Gl(D), or in some cases the SL(D), sub-algebra is referred to as the gravity sub-algebra and the $D-1$ dots of the Dynkin diagram of $G^{+++}$ which belong to the SL(D), or $A_{D-1}$, sub-algebra are referred to as the gravity line.

The eleven-dimensional, IIA and IIB theories all have an underlying $E_8^{+++}$, but they are distinguished by their different gravity sub-algebras. The eleven-dimensional theory must possess an $A_{10}$ gravity algebra and there is only one such algebra. We must choose the gravity algebra to be the $A_{10}$ sub-Dynkin diagram that consist of nodes labeled one to ten. That is it is found by deleting node eleven in the $E_8^{+++}$ Dynkin diagram [2].

The IIA and IIB theories are ten-dimensional and so to find these theories we seek an $A_9$ gravity algebra. Looking at the $E_8^{+++}$ Dynkin diagram there are only two ways to do this. Starting from the very extended node we must choose a $A_9$ sub-Dynkin diagram, but once we get to the junction of $E_8^{+++}$ Dynkin diagram, situated at the node labeled 8, we can continue along the horizontal line with two further nodes taking only the first node to belong to the $A_9$, or we can find the final $A_9$ node by taking it to be the only node in the other choice of direction at the junction. These two ways correspond to the IIA and IIB theories, respectively. Hence, in the IIA theory we take the gravity line to be nodes labeled one to nine inclusive while for the IIB theory the gravity line contains nodes one to eight and in addition node eleven [2,6].

The gravity sub-algebra is such that $K^a{}_a$, $a=1,\ldots,D$ are part of the Cartan sub-algebra of $E_8^{+++}$. For the eleven-dimensional theory, these eleven generators span the Cartan sub-algebra and so one can also write the group element of Eq. (1.6) in the form

$$g=\exp\left(\sum_{a=1}^{11} h_a{}^a K^a{}_a\right)=\exp\left(h^T K\right). \tag{1.2}$$

In the second equation we have used matrix notation whose meaning should be clear. The relationship between the Chevalley generators $H_a$ and the physical generators $K^a{}_a$ can be written in matrix form as $K=\rho H$. It is given by [2]

$$H_a=K^a{}_a-K^{a+1}{}_{a+1},\quad a=1,\ldots,10,\qquad H_{11}=-\frac{1}{3}\left(K^1{}_1+\cdots+K^8{}_8\right)+\frac{2}{3}\left(K^9{}_9+K^{10}{}_{10}+K^{11}{}_{11}\right). \tag{1.3}$$

We also record the relations

$$E_a = K^a{}_{a+1}, \quad a = 1, \ldots, 10, \qquad E_{11} = R^{91011}, \tag{1.4}$$

between the Chevalley generators $E_a$ and the simple root generators of SL(11) and the generator $R^{a_1a_2a_3}$ which is responsible in the non-linear realisation for the introduction of the gauge field $A_{a_1a_2a_3}$ of the eleven-dimensional supergravity theory. Hence, keeping only fields associated with the Cartan sub-algebra implies keeping only the diagonal parts of the metric and, as we will see below for the IIA and IIB theories, also the dilaton field.

The normalisation of the fields in the group element of Eq. (1.7), like all the analogous such group elements in this Letter, is chosen with reference to the normalisation of the generators of $E_8^{+++}$. However, one can carry out the non-linear realisation of $E_8^{+++}$ at low levels, as was done in effect in [16] and in [20], to find the supergravity equations in terms of these fields. By comparison with the formulation of supergravity in terms of ones preferred fields one can find the relationship between the two sets of fields.

The form of the $H_a$ of Eq. (1.8) can essentially be determined given that they must obey Eq. (1.4) with the Cartan matrix of $E_8^{+++}$, together with the knowledge of the simple roots generators of Eq. (1.10) and that tensors, such as $R^{a_1a_2a_3}$, transform in the obvious way under GL(11), i.e., $[K^c{}_d, R^{a_1a_2a_3}] = \delta^{a_1}_d R^{ca_2a_3} + \delta^{a_2}_d R^{ca_3a_1} + \delta^{a_3}_d R^{ca_1a_2}$.

We denote quantities in the IIA and IIB theories with a tilde and hat, respectively. For these theories the Cartan sub-algebra of the gravity sub-algebra, i.e., the $\tilde{K}^a{}_a$, $a = 1, \ldots, 10$ for the IIA theory and the $\hat{K}^a{}_a$, $a = 1, \ldots, 10$ for the IIB theory, account for only ten of the eleven generators of the Cartan sub-algebra of $E_8^{+++}$. The final commuting generator is associated with the dilaton which appears in the IIA and IIB theories. We denote this generator by the symbol $R$ and the dilaton by $A$ with appropriate tildes or hats. As such, for the IIA theory the $E_8^{+++}$ group element of Eq. (1.6) can be written in terms of the physical generators in the form

$$\tilde{g} = \exp\left(\sum_{a=1}^{10} \tilde{h}_a{}^a \tilde{K}^a{}_a\right) \exp(\tilde{A}\tilde{R}) = \exp\bigl(\tilde{h}^T \tilde{K}\bigr). \tag{1.5}$$

In the second equation we have used matrix notation for which $\tilde{h}$ is a column vector whose first ten components are $\tilde{h}_a{}^a$, $a = 1, \ldots, 10$ and whose eleventh component is $\tilde{A}$ and similarly $\tilde{K}$ has its first ten components as $\tilde{K}^a{}_a$ and eleventh component $\tilde{R}$. The Cartan sub-algebra generators $H_a$ of $E_{11}$ and the physical generators $\tilde{K}^a{}_a$, $a = 1, \ldots, 10$ and $\tilde{R}$ are related by $H = \tilde{\rho}\tilde{K}$ which is given by [2]

$$\begin{aligned} &H_a = \tilde{K}^a{}_a - \tilde{K}^{a+1}{}_{a+1}, \quad a = 1, \ldots, 9, \qquad H_{10} = -\frac{1}{8}\bigl(\tilde{K}^1{}_1 + \cdots + \tilde{K}^9{}_9\bigr) + \frac{7}{8}\tilde{K}^{10}{}_{10} - \frac{3}{2}\tilde{R}, \\ &H_{11} = -\frac{1}{4}\bigl(\tilde{K}^1{}_1 + \cdots + \tilde{K}^8{}_8\bigr) + \frac{3}{4}\bigl(\tilde{K}^9{}_9 + \tilde{K}^{10}{}_{10}\bigr) + \tilde{R}. \end{aligned} \tag{1.6}$$

While the $E_a$ Chevalley generators of $E_8^{+++}$ are given in terms of IIA generators by [2]

$$E_a = \tilde{K}^a{}_{a+1}, \quad a = 1, \ldots, 9, \qquad E_{10} = \tilde{R}^{10}, \qquad E_{11} = \tilde{R}^{910}. \tag{1.7}$$

The fields associated with the generators $\tilde{R}^a$ and $R^{ab}$ in the non-linear realisation are the one form and two form fields of the IIA supergravity theory.

Equating the Chevalley generators $H_a$ of Eqs. (1.8) and (1.11) we find that the generators in the physical basis of the eleven-dimensional and IIA theory are related by [2]

$$K^a{}_a = \tilde{K}^a{}_a, \quad a = 1, \ldots, 10, \qquad K^{11}{}_{11} = \frac{1}{8}\sum_{a=1}^{10} \tilde{K}^a{}_a + \frac{3}{2}\tilde{R}. \tag{1.8}$$

For the IIB theory, the generators $\hat{K}^a{}_a$, $a = 1, \ldots, 10$ and $\hat{R}$ span the Cartan sub-algebra of $E_8^{+++}$ and so the group element of Eq. (1.6) can be expressed as

$$\hat{g} = \exp\left(\sum_{a=1}^{10} \hat{h}_a{}^a \hat{K}^a{}_a\right) \exp(\hat{A}\hat{R}) = \exp\bigl(\hat{h}^T \hat{K}\bigr). \tag{1.9}$$

In the second equation we have used matrix notation for which $\hat{h}$ is a column vector whose first ten components are $\hat{h}_a{}^a$, $a = 1, \ldots, 10$ and whose eleventh component is $\hat{A}$ and similarly $\hat{K}$ has its first ten components as $\hat{K}^a{}_a$ and eleventh component $\hat{R}$. The relationship between the Cartan sub-algebra generators $H_a$ of $E_8^{+++}$ and the physical generators $\hat{K}^a{}_a$, $a = 1, \ldots, 10$ and $\hat{R}$ can be written in the form $H = \hat{\rho}\hat{K}$ and it is explicitly given by [6]

$$H_a = \hat{K}^a{}_a - \hat{K}^{a+1}{}_{a+1}, \quad a = 1, \ldots, 8, \qquad H_9 = \hat{K}^9{}_9 + \hat{K}^{10}{}_{10} + \hat{R} - \frac{1}{4}\sum_{a=1}^{10} \hat{K}^a{}_a,$$
$$H_{10} = -2\hat{R}, \qquad H_{11} = \hat{K}^9{}_9 - \hat{K}^{10}{}_{10}. \tag{1.10}$$

The Chevalley generators $E_a$ of $E_8^{+++}$, as they appears in IIB theory are given by [6]

$$E_a = \hat{K}^a{}_{a+1}, \quad a = 1, \ldots 8, \qquad E_9 = \hat{R}_1^{910}, \qquad E_{10} = \hat{R}_2, \qquad E_{11} = \hat{K}^9{}_{10}. \tag{1.11}$$

The fields associated with the generators $\hat{R}_1^{ab}$ and $\hat{R}_2$ are the NS–NS two form and the axion, $\hat{\chi}$ of the IIB theory. The last equation reflects the fact that the node labeled eleven is the last node in the IIB gravity line.

Equating the Chevalley generators $H_a$ of Eqs. (1.8) and (1.11) we find that the generators in the physical basis of the eleven dimensional and IIB theory are related by [7]

$$K^a{}_a = \hat{K}^a{}_a, \quad a = 1, \ldots, 9, \qquad \hat{K}^{10}{}_{10} = \frac{1}{3}\sum_{a=1}^{9} K^a{}_a - \frac{2}{3}\bigl(K^{10}{}_{10} + K^{11}{}_{11}\bigr),$$
$$\hat{R} = -\frac{1}{2}\bigl(K^{10}{}_{10} - K^{11}{}_{11}\bigr). \tag{1.12}$$

For completeness we note the relationship between the IIA and IIB physical generators;

$$\hat{K}^a{}_a = \tilde{K}^a{}_a, \quad a = 1, \ldots, 9, \qquad \hat{K}^{10}{}_{10} = \frac{1}{4}\sum_{a=1}^{9} \tilde{K}^a{}_a - \frac{3}{4}\tilde{K}^{10}{}_{10} - \tilde{R},$$
$$\hat{R} = \frac{1}{16}\sum_{a=1}^{9} \tilde{K}^a{}_a - \frac{7}{16}\tilde{K}^{10}{}_{10} + \frac{3}{4}\tilde{R}, \tag{1.13}$$

We note that the generator corresponding to the node labeled ten in the eleven-dimensional theory is $K^{10}{}_{11}$ and so is associated with the exchange of the ten and eleven space–time coordinates, while in the IIB theory it is $\hat{R}_2$ which is the non-perturbative part of the $\mathrm{SL}(2, \mathbf{Z})$ symmetry of the IIB theory.

## 2. Relations between the eleven-dimensional, IIA and IIB theories

As explained in Ref. [7], the common $E_8^{+++}$ origin of these three theories implies a one-to-one correspondence between the fields of the three theories. In particular, any field in the non-linearly realised IIB theory arises in the group element as the coefficient of a particular generator which is in the Borel sub-algebra of $E_8^{+++}$, however, the generators of $E_8^{+++}$ are essentially unique and so we can identify this generator from the viewpoint of the eleven-dimensional theory. For example, the component graviton field $\hat{h}_9{}^{10}$ of the IIB theory is associated with the generator $\hat{K}^9{}_{10}$ which is equal to the Chevalley generator $E_{11}$ of $E_8^{+++}$. However, from the eleven-dimensional perspective this Chevalley generator is equal to the generator $R^{9\,10\,11}$ that is associated with the field $A_{9\,10\,11}$ which is one component of the third rank anti-symmetric field of the eleven-dimensional supergravity theory. In this section, we will find these correspondences at the simplest possible level.

### 2.1. The correspondence between the eleven-dimensional and IIA theories

To find the correspondence for the Cartan sub-algebra we simply equate the two group elements in the eleven-dimensional and IIA theories of Eqs. (1.2) and (1.5), respectively;

$$g = \tilde{g} \quad \text{or} \quad \exp\left(\sum_{a=1}^{11} h_a{}^a K^a{}_a\right) = \exp\left(\sum_{a=1}^{10} \tilde{h}_a{}^a \tilde{K}^a{}_a\right) \exp(\tilde{A}\tilde{R}). \tag{2.1}$$

Using Eq. (1.8), we conclude that

$$\tilde{h}_a{}^a = h_a{}^a + \frac{1}{8} h_{11}{}^{11}, \quad a = 1, \ldots, 10, \qquad \tilde{A} = \frac{3}{2} h_{11}{}^{11}. \tag{2.2}$$

We expect these relations to hold even if one does not carries out dimensional reduction of the theory on a torus, but then one must also carry out a corresponding exchange of the generalised coordinates [17]. However, if we do dimensionally reduce some of the dimensions on a torus then it is useful to change to the variables

$$h_a{}^a = \ln \frac{R_a}{l_p}, \quad a = 1, \ldots, 11, \tag{2.3}$$

where $l_p$ is the eleven-dimensional Planck scale. We note that in the group elements used to construct the non-linear realisation the fields are dimensionless and so the resulting part of the action in $D$ space–time dimensions that has two space–time derivatives is multiplied by $l_p^{-(D-2)}$. In particular, we will apply the change of variable to the constant background part of the fields. For a rectangular torus, the coordinate and parameterisation invariant length of its cycle in the direction is $l_p \int e_a{}^a \, dx^a = R_a$.

Similarly we introduce the analogous IIA variables by

$$\tilde{h}_a{}^a = \ln \frac{\tilde{R}_a}{\tilde{l}_p}, \quad a = 1, \ldots, 10, \qquad \tilde{A} = \ln \tilde{g}_s, \tag{2.4}$$

where $\tilde{l}_p$ is the ten-dimensional Planck scale of the IIA theory. Comparing the low energy action with that calculated from string scattering allows us to identify the string scale $l_s$ by $(\tilde{l}_p)^8 = \tilde{g}_s^2 (\tilde{l}_s)^8$ and $\tilde{g}_s$ in Eq. (2.4) with the string coupling constant in the usual way.

The last relation in Eq. (2.2) implies that

$$(\tilde{g}_s)^2 = \left(\frac{R_{11}}{l_p}\right)^3. \tag{2.5}$$

Since the eleven-dimensional theory after reduction on a circle coincides with the IIA theory we may take $\tilde{R}_a = R_a$, $a = 1, \ldots, 10$ and then we find that

$$\left(\frac{l_p}{\tilde{l}_p}\right)^{12} = \tilde{g}_s \quad \text{or} \quad l_p^3 = (\tilde{l}_s)^3 \tilde{g}_s. \tag{2.6}$$

The first relation in the above equation together with Eq. (2.5) implies that $\frac{R_{11}}{l_p^9} = \frac{1}{\tilde{l}_p^8}$. Eqs. (2.5) and (2.6) are the known relations between the IIA theory and the so-called eleven-dimensional M theory. They encouraged the idea that eleven-dimensional M theory is the strong coupling limit of the IIA string theory [10,11].

### 2.2. The correspondence between the eleven-dimensional and IIB theories

We now find the analogous relations between the fields, which are associated with their Cartan sub-algebra, of the eleven-dimensional and IIB theories. Equating the eleven-dimensional and IIB group elements of Eq. (1.1) and Eq. (1.9) we find that

$$g = \hat{g} \quad \text{or} \quad \exp\left(\sum_{a=1}^{11} h_a{}^a K^a{}_a\right) = \exp\left(\sum_{a=1}^{10} \hat{h}_a{}^a \hat{K}^a{}_a\right) \exp(\hat{A}\hat{R}) \tag{2.7}$$

which using the identifications of Eqs. (1.12) implies that

$$h_a{}^a = \hat{h}_a{}^a + \frac{1}{3}\hat{h}_{10}{}^{10}, \qquad h_{10}{}^{10} = -\frac{2}{3}\hat{h}_{10}{}^{10} - \frac{1}{2}\hat{A}, \qquad h_{11}{}^{11} = -\frac{2}{3}\hat{h}_{10}{}^{10} + \frac{1}{2}\hat{A}. \tag{2.8}$$

These relations hold without compactifications, but for a torus compactification it is appropriate to adopt the variables

$$\hat{h}_a{}^a = \ln\frac{\hat{R}_a}{\hat{l}_p}, \quad a = 1, \dots, 10, \qquad \hat{A} = \ln \hat{g}_s, \tag{2.9}$$

where $\hat{l}_p$ is the Planck length in the IIB theory and $\hat{g}_s$ its string coupling. Introducing the IIB string scale by $(\hat{l}_p)^8 = \hat{g}_s^2(\hat{l}_s)^8$ the relations given in Eq. (2.8) become

$$\frac{\hat{l}_s^4 \hat{g}_s}{l_p^3} = \hat{R}_{10}, \qquad \frac{R_{10}^6}{l_p^6} = \frac{\hat{l}_s^4}{\hat{R}_{10}^4 \hat{g}_s^2}, \qquad \frac{R_{11}^6}{l_p^6} = \frac{\hat{l}_s^4 \hat{g}_s^4}{\hat{R}_{10}^4}, \tag{2.10}$$

respectively. These are equivalent to the more familiar relations

$$\hat{g}_s = \frac{R_{11}}{R_{10}}, \qquad \hat{l}_s^2 = \frac{l_p^3}{R_{11}}, \qquad \hat{R}_{10} = \frac{l_p^3}{R_{10}R_{11}} \tag{2.11}$$

which relate the eleven-dimensional theory reduced on rectangular torus with radii $R_{10}$ and $R_{11}$ to the IIB theory reduced on a circle of radius $\hat{R}_{10}$ [9,12,13].

As explained in Ref. [7], there is a one-to-one map between all the fields of the IIB and the eleven-dimensional non-linearly realised theories and not just those associated with the Cartan sub-algebra. We close this section by giving a simple illustration of how this map works for a field outside the Cartan sub-algebra. Eqs. (1.4) and (1.11) state that $E_{10} = K^{10}{}_{11} = \hat{R}_2$ and as explained at the beginning of this section this implies that the eleven-dimensional field $h_{10}{}^{11}$ corresponds to the axion field $\hat{\chi}$ of the IIB theory. We now enlarge the fields which are non-zero by including these fields in addition to those associated with the Cartan sub-algebra. As a result, the eleven-dimensional group element takes the form

$$g = \exp\left(\sum_{a=1}^{11} h_a{}^a K^a{}_a\right) \exp\bigl(h_{10}{}^{11} K^{10}{}_{11}\bigr). \tag{2.12}$$

Putting only the Cartan sub-algebra elements in the first exponential will allow us to perform the computation more easily, but it is not quite the form given in the non-linear realisation of Refs. [2,16] and used to find the eleven-dimensional supergravity theory. As a result, we must use the form of the vierbein that follows from the $g$ of Eq. (2.12); its non-vanishing components are given by

$$e_\mu{}^a = \delta_\mu^a, \quad a, \mu = 1, \dots, 9, \qquad e_\mu{}^a = \begin{pmatrix} e^{h_{10}{}^{10}} & e^{h_{10}{}^{10}} h_{10}{}^{11} \\ 0 & e^{h_{11}{}^{11}} \end{pmatrix}_\mu{}^a, \quad a, \mu = 10, 11. \tag{2.13}$$

On the other hand, the IIB group element can be written as

$$\hat{g} = \exp\left(\sum_{a=1}^{10} \hat{h}_a{}^a \hat{K}^a{}_a\right) \exp(\hat{\chi}\hat{R}_2) \exp(\hat{A}\hat{R}) = \exp\left(\sum_{a=1}^{10} \hat{h}_a{}^a \hat{K}^a{}_a\right) \exp(\hat{A}\hat{R}) \exp\bigl(e^{\hat{A}}\hat{\chi}\hat{R}_2\bigr). \tag{2.14}$$

The first form of $\hat{g}$ is the one used to construct the non-linear realisation of IIB supergravity in [6] while the second form is suitable for our comparison with eleven-dimensional group element. To change from one form to the other we used the relation $[\hat{R}, \hat{R}_2] = -\frac{1}{2}[H_{10}, E_{10}] = -E_{10} = -\hat{R}_2$.

Setting $g = \hat{g}$ and using Eqs. (1.12), (1.4) and (1.11) we find the same relations of Eq. (2.8) as well as

$$e^{\hat{A}}\hat{\chi} = h_{10}{}^{11}. \tag{2.15}$$

Let us now suppose that the ten and eleven directions of the eleven-dimensional theory are a torus with lengths $R_{10}$ and $R_{11}$. To discuss the properties of the torus it is simplest to make a rigid coordinate transformation from the coordinates $x^T = (x^{10}, x^{11})$ to the coordinates $y^T = (y^{10}, y^{11})$ that diagonalises the metric in these directions. In particular, we will diagonalise the veirbein in the ten and eleven directions. We denoted the latter by the matrix $e$ which can be read off from the last relation in Eq. (2.13). The transformation $e \to \Lambda e$ given by

$$\Lambda = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \tag{2.16}$$

where $m = -\frac{e_{10}{}^{11}}{e_{11}{}^{11}}$, has the desired result. The new veirbein has the same diagonal components as the old one. Using Eq. (2.13) we find that $m = -h_{10}{}^{11}\exp(h_{10}{}^{10} - h_{11}{}^{11})$. In the diagonal coordinates $y$ we take the cycles of the torus to be given by $y^{11} = u$, $y^{10} = 0$; $0 \leqslant u < 1$ and $y^{10} = v$, $y^{11} = 0$; $0 \leqslant v < 1$. The coordinate and parameterisation invariant length of the first cycle is $\int_0^1 e_{11}{}^{11} \frac{dy^{11}}{du} du = e_{11}{}^{11} = R_{11}$ and similarly with the invariant length of the second cycle is given by $e_{10}{}^{10} = R_{10}$. Hence, we still have the relation $\frac{R_{11}}{R_{10}} = \exp(\hat{A}) = \hat{g}_s$ of Eq. (2.11).

In terms of the original $x$ coordinates which are related by $x = \Lambda^T y$ the cycles of the torus are $x^{10} = 0$, $x^{11} = u$; $0 \leqslant u < 1$ and $x^{10} = v$, $x^{11} = mv$; $0 \leqslant v < 1$. If we define the complex coordinate $z = x^{11} - ix^{10}$ then the periods corresponding to the first and second cycles are $z \to z = 1$ and $z \to z + \tau$, respectively, where $\tau = \tau_1 + i\tau_2$ with $\tau_1 = 1$ and $\tau_2 = -m$. Hence, the modulus parameter of the torus is given by

$$\frac{\tau_1}{\tau_2} = m = \hat{\chi}. \tag{2.17}$$

This agrees with the identification of Refs. [12,13] after one takes into account that one must make the field redefinition $\hat{\chi} \to \exp(-\hat{A})\hat{\chi}$ to find the $\hat{\chi}$ of [6] from that of [12] in order to gain agreement between the field equations of the two references. By a judicious choice of coordinates we can, as in [12], arrange for $\tau_2$ to be $\exp(-\hat{A})$, but the physically relevant quantity $\frac{\tau_1}{\tau_2}$ remains the same.

## 3. Weyl transformations in the IIB and IIA theories

The Weyl reflection $S_a$ corresponding to the simple root $\alpha_a$ on any weight $\beta$ is given by $S_a\beta = \beta - 2\frac{(\beta,\alpha_a)}{(\alpha_a,\alpha_a)}\alpha_a$. For the simple roots this becomes

$$S_a\alpha_b = \alpha_b - 2\frac{(\alpha_b, \alpha_a)}{(\alpha_a, \alpha_a)}\alpha_a = (s_a)_b{}^c\alpha_c. \tag{3.1}$$

The action of the Weyl transformation $S_a$ on the Cartan sub-algebra of a Kac–Moody algebra is given by

$$H_b' = S_a H_b = (s_a)_b{}^c H_c. \tag{3.2}$$

Since the Weyl group acts on Cartan sub-algebra generators to give Cartan sub-algebra generators it makes sense to consider their action on elements restricted to be of the form of Eq. (1.1). Writing the group element in matrix form $g = \exp(q^T H)$, we conclude that the Weyl group acts on the fields $q$ as $q'^T = q^T s$, or $q' = s^T q$ as $s^2 = I$. Clearly, these transformations hold for the eleven-dimensional theory and the IIA and IIB theories.

To find the physical effects of the Weyl transformations we need to find their action on the physical variables ${h_a}^a$ and also the dilaton field, for the cases of the IIA and IIB theories, However, the relationship between the Chevalley generators $H_a$ and the physical generators depends upon which theory we are considering and so the effect of the Weyl transformations on the physical generators and fields is different for each theory. Using matrix notation, in the eleven-dimensional theory we may write $H = \rho K$ and then the effect of the Weyl transformation is $S_a K = K' = \rho^{-1} s_a \rho K = r_a K$ and so the physical fields $h$ transform as $h' = r_a^T h$. However, for the IIB theory, $H = \hat{\rho}\hat{K}$ and so we have the equations

$$S_a \hat{K} = \hat{K}' = \hat{\rho}^{-1} s_a \hat{\rho} \hat{K} = \hat{r}_a \hat{K} \quad \text{and} \quad \hat{h}' = \hat{r}_a^T \hat{h}. \tag{3.3}$$

The equation for IIA being found by replacing ˆ's by ˜'s. Using Eqs. (2.3), (2.4) and (2.9) the effects of the Weyl transformations can then be readily deduced on the radii of any compactified directions and the appropriate length scales and coupling constants.

This calculation was carried out in Ref. [15] for the eleven-dimensional theory and we briefly summarize the result. The Weyl transformations $S_a$, $a = 1, \dots, 10$ implied that $R_a \leftrightarrow R_{a+1}$, $l_p \to l_p$. However, $S_{11}$ induces the transformations ${h_a}'^a = {h_a}^a + \frac{1}{3}({h_9}^9 + {h_{10}}^{10} + {h_{11}}^{11})$, $a = 1, \dots, 8$ and ${h_a}'^a = {h_a}^a - \frac{2}{3}({h_9}^9 + {h_{10}}^{10} + {h_{11}}^{11})$, $a = 9, 10, 11$ which in turn implies that

$$R_9' = \frac{l_p^3}{R_{10} R_{11}}, \qquad R_{10}' = \frac{l_p^3}{R_{11} R_9}, \qquad R_{11}' = \frac{l_p^3}{R_9 R_{10}}, \qquad (l_p')^3 = \frac{l_p^6}{R_9 R_{10} R_{11}}. \tag{3.4}$$

For the IIB theory, the Weyl transformations $S_a$, $a = 1, \dots, 8$ correspond ${\hat{K}^a}_a \leftrightarrow {\hat{K}^{a+1}}_{a+1}$, for $a = 1, \dots, 8$ as well as $\hat{R} \to \hat{R}$. The effect on the variables of Eq. (2.9) is $\hat{R}_a \leftrightarrow \hat{R}_{a+1}$ for $a = 1, \dots, 7$ as well as $\hat{g}_s \to \hat{g}_s$. The Weyl transformation $S_{11}$ leaves $\hat{R}$ and all the ${\hat{K}^a}_a$ inert except for ${\hat{K}^9}_9 \leftrightarrow {\hat{K}^{10}}_{10}$. The effect is to take $\hat{R}_9 \leftrightarrow \hat{R}_{10}$ with all other variables being inert. This is consistent with the node labeled eleven being the last on the gravity line of the IIB theory and one finds that all the Weyl transformations corresponding to all points on the gravity line just exchanges the corresponding radii.

The Weyl transformation $S_{10}$ acts on the Cartan sub-algebra as $H_{10}' = -H_{10}$, $H_9' = H_9 + H_{10}$ all other elements being inert. Using Eq. (1.10) we find that these transformations imply that

$$\hat{R}' = -\hat{R}, \qquad {\hat{K}'^a}_a = {\hat{K}^a}_a. \tag{3.5}$$

Using Eqs. (2.9) and (3.3), the effect on the physical variables is given by

$$\hat{A}' = -\hat{A}, \qquad {h_a'}^a = {h_a}^a \tag{3.6}$$

Which in turn implies that

$$\hat{g}_s' = \frac{\hat{1}}{\hat{g}_s}, \qquad \hat{R}_a' = \hat{R}_a, \quad a = 1, \dots, 10, \qquad \hat{l}_s'^2 = \hat{g}_s \hat{l}_s^2. \tag{3.7}$$

This is just the non-perturbative S-duality transformations of the IIB theory which holds if the theory is compactified or not. This is to be expected as the node labeled ten just leads to an $\mathrm{SL}(2, \mathbf{R})$ transformation of the supergravity theory. We note that in the eleven-dimensional theory, node eleven is the last node in the gravity line of this theory and the corresponding Weyl transformation swops the eleventh and tenth coordinates.

Finally, we consider the Weyl transformation $S_9$ which induces the transformations $H_9' = -H_9$, $H_{10}' = H_9 + H_{10}$, $H_8' = H_8 + H_9$ with all other elements of the Cartan sub-algebra being inert. The transformation on the physical generators is given by

$$\hat{K}'^a{}_a = \hat{K}^a{}_a, \quad a = 1, \ldots, 8,$$
$$\hat{K}'^a{}_a = \hat{K}^a{}_a + \frac{1}{4}\big(\hat{K}^1{}_1 + \cdots + \hat{K}^8{}_8\big) - \frac{3}{4}\big(\hat{K}^9{}_9 + \cdots + \hat{K}^{10}{}_{10}\big) - \hat{R}, \quad a = 9, 10,$$
$$\hat{R}' = \hat{R} + \frac{1}{8}\big(\hat{K}^1{}_1 + \cdots + \hat{K}^8{}_8\big) - \frac{3}{8}\big(\hat{K}^9{}_9 + \cdots + \hat{K}^{10}{}_{10}\big) - \frac{1}{2}\hat{R}. \tag{3.8}$$

The corresponding effect on the fields of the IIB theory is

$$\hat{h}'_a{}^a = \hat{h}_a{}^a + \frac{1}{4}\left(\hat{h}_9{}^9 + \hat{h}_{10}{}^{10} + \frac{1}{2}\hat{A}\right), \quad a = 1, \ldots, 8,$$
$$\hat{h}'_a{}^a = \hat{h}_a{}^a - \frac{3}{4}\left(\hat{h}_9{}^9 + \hat{h}_{10}{}^{10} + \frac{1}{2}\hat{A}\right), \quad a = 9, 10, \qquad \hat{A}' = \hat{A} - \left(\hat{h}_9{}^9 + \hat{h}_{10}{}^{10} + \frac{1}{2}\hat{A}\right). \tag{3.9}$$

As a result the variables of Eq. (2.9) transform as

$$\hat{R}'_a = \hat{R}_a, \quad a = 1, \ldots, 8, \qquad \frac{\hat{R}'_a}{\hat{R}_a} = \frac{\hat{l}_s^2}{\hat{R}_9\hat{R}_{10}}, \quad a = 9, 10, \qquad \frac{\hat{g}'_s}{\hat{g}_s} = \frac{\hat{l}_s^2}{\hat{R}_9\hat{R}_{10}} \tag{3.10}$$

and $\hat{l}'_s = \hat{l}_s$. We recognise this as a double T duality seen from the IIB viewpoint.

We now briefly discuss the effect on the Weyl transformations of $E_8^{+++}$ for the IIA theory. The Weyl transformations $S_a$, $a = 1, \ldots, 9$ takes $K^a{}_a \leftrightarrow K^{a+1}{}_{a+1}$ and so $R_a \leftrightarrow R_{a+1}$. The Weyl transformation $S_{11}$ leads to the double T duality

$$R'_a = R_a, \quad a = 1, \ldots, 8, \qquad \tilde{R}'_9 = \frac{\tilde{l}_s^2}{\tilde{R}_{10}}, \qquad \tilde{R}'_{10} = \frac{\tilde{l}_s^2}{\tilde{R}_9}, \qquad \tilde{g}'_s = \frac{\tilde{g}_s\tilde{l}_s^2}{\tilde{R}_9\tilde{R}_{10}}, \quad \tilde{l}'_s = \tilde{l}_s. \tag{3.11}$$

Finally, the Weyl transformation $S_{10}$ induces the changes

$$\tilde{h}'_a{}^a = \tilde{h}_a{}^a + \frac{1}{8}\tilde{h}_{10}{}^{10} + \frac{3}{32}\tilde{A}, \quad a = 1, \ldots, 9,$$
$$\tilde{h}'_{10}{}^{10} = \tilde{h}_{10}{}^{10} - \frac{7}{8}\tilde{h}_{10}{}^{10} + \frac{21}{32}\tilde{A}, \quad a = 9, 10, \qquad \tilde{A}' = \tilde{A} + \frac{3}{2}\tilde{h}_{10}{}^{10} - \frac{9}{8}\tilde{A} \tag{3.12}$$

which leads to

$$R'_a = R_a, \quad a = 1, \ldots, 9, \qquad \tilde{R}'_{10} = \tilde{l}_s\tilde{g}_s, \qquad \tilde{g}'^2_s = \left(\frac{\tilde{R}_{10}}{\tilde{l}_s}\right)^3\frac{1}{\tilde{g}_s}, \qquad \left(\frac{\tilde{l}'_s}{\tilde{l}_s}\right)^2 = \frac{\tilde{l}_s\tilde{g}_s}{\tilde{R}_{10}}. \tag{3.13}$$

Clearly, this is a non-perturbative relation which is in some sense the IIA analogue of the $\mathrm{SL}(2, \mathbf{Z})$ symmetry of the IIB theory.

## 4. Discussion

One could use the same techniques as used in this Letter to identify the relations between other $G^{+++}$ non-linearly realised theories where there is a choice of gravity sub-algebra.

The eleven-dimensional, IIA and IIB theories are all expected to possess a non-linearly realised $E_8^{+++}$ symmetry [2,6]. Although, their differences arise from the way their gravity sub-algebras are embedded, their common symmetry allows one to establish a one-to-one correspondence between the fields of these theories [7]. In this Letter, we have found the simplest consequences of this correspondence which are those for the fields associated with the Cartan sub-algebra of $E_8^{+++}$. We have recover the known relations [8–11] between the three theories. We also gave one example of the correspondence for a field outside the Cartan sub-algebra and recovered the fact [12,13] that the axion field of the IIB theory dimensionally reduced on a circle can be identified with the modulus of the two-dimensional torus used to dimensionally reduce the eleven-dimensional theory.

The correspondence between the three theories resulting from their common $E_8^{+++}$ symmetry implies many more results, such as the eleven-dimensional origin of the massive IIA theory and the IIB space-filling brane [7]. However, the purpose of this Letter is to demonstrate that the underlying $E_8^{+++}$ symmetry can be used to find results central to string theory in a very simple way.

As we noted above, the identifications of the fields of the three theories should hold even if one does not perform a dimensional reduction. In this case one is the fields which depend on the generalised coordinates [17] of the theory and, as explained in Ref. [7], one must then also swop the generalised coordinates of the theory. However, these includes central charge coordinates as well as the usual coordinates of space–time and their interchange will have far reaching effects on the theory.

We also computed the effect of the Weyl transformations of the IIA and IIB $E_8^{+++}$ theories on the diagonal components of the metric and dilaton to recover the expected U-duality symmetries of these theories. It would be interesting to compare these results with the different perturbative sub-algebras of the $E_8^{+++}$ algebra for the IIA and IIB theories found in [19].

## Acknowledgements

I wish to thank Matthias Gaberdiel and Dominic Clancy for useful discussions. I wish also to thank the Erwin Schrödinger International Institute for Mathematical Physics at Wien and the Department of Physics at Heraklion for their hospitality. This research was supported by a PPARC senior fellowship PPA/Y/S/2002/001/44 and in part by the PPARC grants PPA/G/O/2000/00451, PPA/G/S4/1998/00613 and the EU Marie Curie, research training network grant HPRN-CT-2000-00122.

## References

[1] E. Cremmer, B. Julia, J. Scherk, Phys. Lett. B 76 (1978) 409.
[2] P. West, Class. Quantum Grav. 18 (2001) 4443, hep-th/0104081.
[3] I.C.G. Campbell, P. West, Nucl. Phys. B 243 (1984) 112;
M. Huq, M. Namanzie, Class. Quantum Grav. 2 (1985);
F. Giani, M. Pernici, Phys. Rev. D 30 (1984) 325.
[4] J. Schwarz, P. West, Phys. Lett. B 126 (1983) 301.
[5] J. Schwarz, Nucl. Phys. B 226 (1983) 269;
P. Howe, P. West, Nucl. Phys. B 238 (1984) 181.
[6] I. Schnakenburg, P. West, Phys. Lett. B 517 (2001) 137, hep-th/0107181.
[7] P. West, hep-th/0402140.
[8] J. Dai, R. Leigh, J. Polchinski, Mod. Phys. Lett. A 4 (1989) 2073.
[9] M. Dine, P. Huet, N. Seiberg, Nucl. Phys. B 322 (1989) 301.
[10] P. Townsend, Phys. Lett. B 350 (1995) 184, hep-th/9501068.
[11] E. Witten, Nucl. Phys. B 443 (1995) 85, hep-th/9503124.
[12] J. Schwarz, Phys. Lett. B 367 (1996) 97, hep-th/9510086;
J. Schwarz, Phys. Lett. B 360 (1995) 13;
J. Schwarz, Phys. Lett. B 364 (1995) 252, Erratum, hep-th/9508143.
[13] P. Aspinwall, Nucl. Phys. B (Proc. Suppl.) 46 (1996) 30, hep-th/9508154.
[14] M.R. Gaberdiel, D.I. Olive, P. West, Nucl. Phys. B 645 (2002) 403, hep-th/0205068.
[15] F. Englert, L. Houart, A. Taormina, P. West, JHEP 0309 (2003) 020, hep-th/0304206.

[16] P.C. West, JHEP 0008 (2000) 007, hep-th/0005270.
[17] P. West, Phys. Lett. B 575 (2003) 333, hep-th/0307098.
[18] C.M. Hull, P.K. Townsend, Nucl. Phys. B 438 (1995) 109, hep-th/9410167.
[19] M. Gaberdiel, P. West, JHEP 0208 (2002) 049, hep-th/0207032.
[20] P. West, Class. Quantum Grav. 20 (2003) 2393, hep-th/0212291.