Opial Type Inequalities Involving Fractional Derivatives of Two Functions and Applications

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Abstract—A large variety of very general, but basic $L_p$ ($1 \leq p \leq \infty$) form, Opial type inequalities [1] is established involving generalized fractional derivatives [2,3] of two functions in different orders and powers.

The above rely on a generalization of Taylor's formula for generalized fractional derivatives [2]. From the developed results derive several other concrete results of special interest. The sharpness of inequalities is established there. Finally, applications of some of these special inequalities are given in establishing uniqueness of solution and in giving upper bounds to solutions of initial value problems involving a very general system of two fractional differential equations. Also, upper bounds to various fractional derivatives of the solutions that are involved in the above systems are presented. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Opial type inequality, Fractional derivative, System of fractional differential equations, Uniqueness of solution, Upper bound of solution.

1. INTRODUCTION

Opial inequalities appeared for the first time in [1] and then, many authors dealt with them in different directions and for various cases. For a complete recent account on the activity of this field, see [4], and still there remains a very active area of research. One of their main attractions to these inequalities is their applications, especially to establishing uniqueness and upper bounds of solution of initial value problems in differential equations. The author was the first to present Opial inequalities involving fractional derivatives of functions [2,5] with applications to fractional differential equations.

Fractional derivatives come up naturally in a number of fields, especially in physics, see the recent book [6]. To name a few topics such as, fractional kinetics of Hamiltonian chaotic systems, polymer physics and rheology, regular variation in thermodynamics, biophysics, fractional time evolution, fractal time series, etc. One there deals also with stochastic fractional-difference equations and fractional diffusion equations. Great applications of these can be found in the study of DNA sequences. Other fractional differential equations arise in the study of suspensions, coming from the fluid dynamical modeling of certain blood flow phenomena. An excellent account in the
study of fractional differential equations is in the recent book [7]. One can also have applications of fractional calculus to viscoelasticity, Bode's analysis of feedback amplifiers, Capacitor theory, electrical circuits, electronanalytical chemistry, biology, control theory, fitting of experimental data, fractional-order physics. The study of fractional differential equations ranges from the very theoretical topics of existence and uniqueness of solution to finding numerical solutions.

The study of fractional calculus started from 1695 by L'Hospital and Leibniz, also continued later by Fourier in 1822 and Abel in 1823, and continues to our time in an increased fashion due to its many applications and necessity to deal with fractional phenomena and structures. So, this field is keeping a lot of people active and interested.

In this article, the author is greatly motivated and inspired by the very important papers [8,9]. Of course, there, the authors are dealing with other kinds of derivative. So, here, the author continues his study of fractional Opial inequalities now involving two different functions and produces a wide variety of corresponding results with important applications to systems of two fractional differential equations. Dealing with two functions makes the study more complicated and involved.

We start in Section 2 with preliminaries, we continue in Section 3 with the main results, and we finish in Section 4 with the applications.

To give a flavor to the reader of the kind of inequalities we are dealing with, we briefly mention

\[ \int_a^b q(w) \left[ \left| (D^\alpha_{a+} f_1) (w) \right|^\lambda + \left| (D^\beta_{a+} f_2) (w) \right|^\mu \right] dw \]

\[ \leq C (a, b, q(w), \gamma_1, \nu, \lambda_\alpha, \lambda_\beta, p(w), p) \]

\[ \cdot \left[ \int_a^b p(w) \left[ \left| (D^\alpha_{a+} f_1) (w) \right|^p + \left| (D^\beta_{a+} f_2) (w) \right|^p \right] dw \right]^{(\lambda_\alpha + \lambda_\beta)/p} \]  

for certain continuous functions \( f_1, f_2, p(w), q(w) \) on \([a, b] \), all exponents and orders are fractional, etc. Furthermore, one system of fractional differential equations we are working on, briefly looks like

\[ (D^\alpha_{a} f_j) (t) = F_j (t, \{ (D^\alpha_{a} f_1) (t) \}_{i=1}^{r}, \{ (D^\alpha_{a} f_2) (t) \}_{i=1}^{r}) \]

for \( j = 1, 2 \), and with

\[ f_j^{(i)} (a) = a_{ij} \in \mathbb{R}, \quad i = 0, 1, \ldots, n - 1, \]

where \( n := \lfloor \nu \rfloor, \nu \geq 2, \text{etc.} \)

In literature, there are many different definitions of fractional derivatives, some of them being equivalent, see [6,7]. In this paper, we use one of the most recent, attributed to Canavati [3], generalized in [2,5].

One of the advantages of Canavati fractional derivative is that in applications to fractional initial value problems, we need only \( n \) initial conditions, like with the ordinary derivative case, while with other definitions of fractional derivatives, we need \( n + 1 \) or more conditions, see [7].

To the best of our knowledge, the presented results here are totally new to the literature of fractional analysis.

2. PRELIMINARIES

In the sequel, we follow [3]. Let \( g \in C([0, 1]) \). Let \( \nu \) be a positive number, \( n := \lfloor \nu \rfloor \) and \( \alpha := \nu - n \) (\( 0 < \alpha < 1 \)). Define

\[ (J_\nu g) (x) := \frac{1}{\Gamma (\nu)} \int_0^x (x - t)^{\nu - 1} g(t) \, dt, \quad 0 \leq x \leq 1, \]
the Riemann-Liouville integral, where $\Gamma$ is the gamma function. We define the subspace $C^\nu([0,1])$ of $C^n([0,1])$ as follows:

$$C^\nu([0,1]) := \{ g \in C^n([0,1]) : J_{1-a} D^n g \in C^1([0,1]) \},$$

where $D := \frac{d}{dt}$. So, for $g \in C^\nu([0,1])$, we define the $\nu$-fractional derivative of $g$ as

$$D^\nu g := D J_{1-a} D^n g. \quad (2)$$

When $\nu \geq 1$, we have Taylor's formula,

$$g(t) = g(0) + g'(0) t + g''(0) \frac{t^2}{2!} + \cdots + g^{(n-1)}(0) \frac{t^{n-1}}{(n-1)!} + (J_0 D^\nu g)(t), \quad \text{for all } t \in [0,1]. \quad (3)$$

When $0 < \nu < 1$, we find

$$g(t) = (J_0 D^\nu g)(t), \quad \text{for all } t \in [0,1]. \quad (4)$$

Next, we carry above notions over to arbitrary $[a,b] \subseteq \mathbb{R}$ (see [2]). Let $x, x_0 \in [a,b]$, such that $x \geq x_0$, where $x_0$ is fixed. Let $f \in C([a,b])$ and define

$$(J_0^\nu f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x - t)^{\nu - 1} f(t) \, dt, \quad x_0 \leq x \leq b, \quad (5)$$

the generalized Riemann-Liouville integral. We define the subgroup $C^\nu_0([a,b])$ of $C^\nu([a,b])$,

$$C^\nu_0([a,b]) := \{ f \in C^\nu([a,b]) : J_{1-a}^\nu f \in C^1([x_0,b]) \}. \quad (6)$$

For $f \in C^\nu_0([a,b])$, we define the generalized $\nu$-fractional derivative of $f$ over $[x_0,b]$ as

$$D^\nu x_0 f := D J_{1-a}^\nu f^{(n)} \quad (f^{(n)} := D^n f). \quad (7)$$

Notice that,

$$(J_{1-a}^\nu f^{(n)})(x) = \frac{1}{\Gamma(1 - \alpha)} \int_{x_0}^x (x - t)^{-\alpha} f^{(n)}(t) \, dt$$

exists for $f \in C^\nu_0([a,b])$.

We recall the following generalization of Taylor's formula (see [2,3]).

**Theorem 1.** Let $f \in C^\nu_0([a,b])$, $x_0 \in [a,b]$ be fixed.

(i) If $\nu \geq 1$, then,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2} + \cdots + f^{(n-1)}(x_0) \frac{(x - x_0)^{n-1}}{(n-1)!} + (J_{1-a}^\nu D^\nu x_0 f)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \quad (8)$$

(ii) If $0 < \nu < 1$, then,

$$f(x) = (J_{1-a}^\nu D^\nu x_0 f)(x), \quad \text{for all } x \in [a,b] : x \geq x_0. \quad (9)$$

We make the following remark.

**Remark 1.**

(1) $(D^\nu x_0 f) = f^{(n)}$, $n \in \mathbb{N}$.

(2) Let $f \in C^\nu_0([a,b])$, $\nu \geq 1$ and $f^{(i)}(x_0) = 0$, $i = 0,1,\ldots,n - 1$, $n := [\nu]$. Then, by (7),

$$f(x) = (J_{1-a}^\nu D^\nu x_0 f)(x),$$

i.e.,

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x - t)^{\nu - 1} (D^\nu x_0 f)(t) \, dt, \quad (9)$$

for all $x \in [a,b]$ with $x \geq x_0$. Notice that (9) is true, also, when $0 < \nu < 1$. 

We need the following lemma from [5].

**Lemma 1.** Let $f \in C([a, b])$, $\mu, \nu > 0$. Then,

$$J_{\mu}^{\nu} (J_{\mu}^{\nu} f) = J_{\mu + \nu}^{\nu} (f).$$

(10)

We also make the following remark.

**Remark 2.** Let $\nu, \gamma \geq 1$, such that $\nu - \gamma \geq 1$, so that $\gamma < \nu$. Call $n := [\nu]$, $\alpha := \nu - n$; $m := [\gamma]$, $\rho := \gamma - m$. Note that, $\nu - m \geq 1$ and $n - m \geq 1$. Let $f \in C_{\alpha}^{m}([a, b])$ be, such that

$$f^{(i)}(x_0) = 0, \quad i = 0, 1, \ldots, n - 1.$$  

Hence, by (7),

$$f(x) = (J_{\nu}^{\nu} D_{\nu}^{\nu} f)(x), \quad \text{for all } x \in [a, b] : x \geq x_0.$$  

Therefore, by Leibnitz's formula and $\Gamma(p + 1) = p\Gamma(p)$, $p > 0$, we get that

$$f^{(m)}(x) = (J_{\nu}^{\nu} D_{\nu}^{\nu} f)(x), \quad \text{for all } x \geq x_0.$$  

(11)

It follows that $f \in C_{\alpha}^{m}([a, b])$, and thus,

$$(D_{\nu}^{\nu} f)(x) := (DJ_{1-\rho}^{\nu} f^{(m)})(x) \text{ exists, for all } x \geq x_0.$$  

(12)

Really, by the use of (11), we have on $[x_0, b]

$$J_{1-\rho}^{\nu} \left( f^{(m)} \right) = J_{1-\rho}^{\nu} \left( J_{\nu}^{\nu} D_{\nu}^{\nu} f \right) = (J_{1-\rho}^{\nu} \circ J_{\nu}^{\nu} - m) \left( D_{\nu}^{\nu} f \right)$$

$$= J_{\nu}^{\nu} - m \left( D_{\nu}^{\nu} f \right) = J_{\nu}^{\nu} - m + 1 \left( D_{\nu}^{\nu} f \right),$$

by (10). That is,

$$J_{1-\rho}^{\nu} f^{(m)}(x) = \frac{1}{\Gamma(\nu - \gamma + 1)} \int_{x_0}^{x} (x - t)^{\nu - \gamma - 1} (D_{\nu}^{\nu} f)(t) \, dt.$$  

Therefore,

$$(D_{\nu}^{\nu} f)(x) = D \left( J_{1-\rho}^{\nu} f^{(m)} \right)(x) = \frac{1}{\Gamma(\nu - \gamma)} \cdot \int_{x_0}^{x} (x - t)^{\nu - \gamma - 1} (D_{\nu}^{\nu} f)(t) \, dt,$$

hence,

$$(D_{\nu}^{\nu} f)(x) = (J_{\nu}^{\nu} D_{\nu}^{\nu} f)(x) \text{ and is continuous in } x \text{ on } [x_0, b].$$

In particular, when $\nu \geq 2$, we have

$$(D_{\nu}^{\nu - 1} f)(x) = \int_{x_0}^{x} (D_{\nu}^{\nu} f)(t) \, dt, \quad x \geq x_0.$$  

(14)

That is,

$$(D_{\nu}^{\nu - 1} f)' = D_{\nu}^{\nu} f, \quad (D_{\nu}^{\nu - 1} f)(x_0) = 0.$$  

3. MAIN RESULTS

We present our first main result.

**Theorem 2.** Let $\nu, \gamma_1, \gamma_2 \geq 1$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$ and $f_1, f_2 \in C_{\alpha}^{\nu}([a, b])$ with

$$f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, \quad i = 0, 1, \ldots, n - 1, \quad n := [\nu].$$

Here, $x, x_0 \in [a, b] : x \geq x_0$. Consider also $p(t) > 0$, and $q(t) \geq 0$ continuous functions on $[x_0, b]$. 

Let $\lambda_\nu > 0$ and $\lambda_\alpha, \lambda_\beta \geq 0$, such that $\lambda_\nu < p$, where $p > 1$. Set

$$P_k(w) := \int_{x_0}^{w} (w - t)^{(\nu - \gamma_k - 1)p/(p-1)} (p(t))^{-1/(p-1)} \, dt, \quad k = 1, 2, \quad x_0 \leq x \leq b,$$

$$A(w) := \frac{q(w) \cdot (P_1(w))^{\lambda_\alpha((p-1)/p)} \cdot (P_2(w))^{\lambda_\beta((p-1)/p)} (p(w))^{-\lambda_\nu/p}}{\Gamma(\nu - \gamma_1)^{\lambda_\alpha} \cdot \Gamma(\nu - \gamma_2)^{\lambda_\beta}},$$

$$A_0(x) := \left( \int_{x_0}^{x} A(w)^{p/(p-\lambda_\nu)} \, dw \right)^{(p-\lambda_\nu)/p},$$

and

$$\delta_1 := \begin{cases} 2^{1-(\lambda_\alpha + \lambda_\nu)/p}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 1, & \text{if } \lambda_\alpha + \lambda_\nu \geq p. \end{cases}$$

If $\lambda_\beta = 0$, we obtain that,

$$\int_{x_0}^{x} q(w) \left[ \left( \int_{x_0}^{w} A(w)^{\lambda_\alpha} \cdot \left| (D_{\nu}^\alpha f_1)(w) \right|^{\lambda_\nu} + \left| (D_{\nu}^\alpha f_2)(w) \right|^{\lambda_\nu} \right) \, dw \right]$$

$$\leq \left( A_0(x)\right)_{\lambda_\beta=0} \cdot \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} \cdot \delta_1 \cdot \left( \int_{x_0}^{x} p(w) \left[ \left| (D_{\nu}^\alpha f_1)(w) \right|^p + \left| (D_{\nu}^\alpha f_2)(w) \right|^p \right] \, dw \right)^{(\lambda_\alpha + \lambda_\nu)/p}.\tag{19}$$

**Proof.** From (13), we have

$$(D_{\nu}^\alpha f_j)(w) = \frac{1}{\Gamma(\nu - \gamma_k)} \int_{x_0}^{w} (w - t)^{\nu - \gamma_k - 1} (D_{\nu}^\alpha f_j)(t) \, dt,$$

for $k = 1, 2, j = 1, 2$, and, for all $x_0 \leq w \leq b$.

Next, applying Hölder’s inequality with indices $p, p/(p-1)$, we get

$$\left| (D_{\nu}^\alpha f_j)(w) \right| \leq \frac{1}{\Gamma(\nu - \gamma_k)} \int_{x_0}^{w} (w - t)^{\nu - \gamma_k - 1} (p(t))^{-1/p} (p(t))^{1/p} \left| (D_{\nu}^\alpha f_j)(t) \right| \, dt$$

$$\leq \frac{1}{\Gamma(\nu - \gamma_k)} \left( \int_{x_0}^{w} (w - t)^{\nu - \gamma_k - 1} (p(t))^{-1/p} \, dt \right)^{p/(p-1)} \left( \int_{x_0}^{w} p(t) \left| (D_{\nu}^\alpha f_j)(t) \right|^p \, dt \right)^{1/p}$$

$$= \frac{1}{\Gamma(\nu - \gamma_k)} (P_k(w))^{(p-1)/p} \left( \int_{x_0}^{w} p(t) \left| (D_{\nu}^\alpha f_j)(t) \right|^p \, dt \right)^{1/p},$$

i.e., it holds

$$\left| (D_{\nu}^\alpha f_j)(w) \right| \leq \frac{1}{\Gamma(\nu - \gamma_k)} (P_k(w))^{(p-1)/p} \left( \int_{x_0}^{w} p(t) \left| (D_{\nu}^\alpha f_j)(t) \right|^p \, dt \right)^{1/p}.\tag{20}$$

Put

$$z_j(w) := \int_{x_0}^{w} p(t) \left| (D_{\nu}^\alpha f_j)(t) \right|^p \, dt,$$

thus,

$$z_j'(w) = p(w) \left| (D_{\nu}^\alpha f_j)(w) \right|^p, \quad z_j(x_0) = 0, \quad j = 1, 2.$$
Hence, we have
\[ |(D_{z_0}^\alpha f_j)(w)| \leq \frac{1}{\Gamma(\nu - \gamma)} (P_k(w))^{(p-1)/p} (z_j(w))^{1/p}, \]
and
\[ |(D_{z_0}^\alpha f_j)(w)|^{\lambda_j} = p(w)^{-\lambda_j/p} (z_j'(w))^{\lambda_j/p}, \quad j = 1, 2. \]

Therefore, we obtain
\[
q(w) |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} |(D_{z_0}^\alpha f_2)(w)|^{\lambda_2} |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} \\
\leq q(w) \left( \frac{1}{(\Gamma(\nu - \gamma)^{\lambda_1}} (P_1(w))^{\lambda_1/(p-1)} (z_1(w))^{\lambda_1/p} \frac{1}{(\Gamma(\nu - \gamma)^{\lambda_2}} (P_2(w))^{\lambda_2/(p-1)} \right) \\
\cdot (z_2(w))^{\lambda_2/p} (p(w))^{-\lambda_2/p} (z_2'(w))^{\lambda_2/p} \\
= A(w) (z_1(w))^{\lambda_1/p} (z_2(w))^{\lambda_2/p} (z_2'(w))^{\lambda_2/p}.
\]

Consequently, by another Hölder’s inequality application, we find (by \(p/\lambda_j > 1\))
\[
\int_{x_0}^{x} q(w) |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} |(D_{z_0}^\alpha f_2)(w)|^{\lambda_2} |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} dw
\]
\[
\leq A_0(x) \left[ \int_{x_0}^{x} (z_1(w))^{\lambda_1/\lambda_2} (z_2(w))^{\lambda_2/\lambda_1} z_1'(w) dw \right]^{\lambda_1/p}. \tag{21}
\]

Similarly, one finds
\[
\int_{x_0}^{x} q(w) |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} |(D_{z_0}^\alpha f_2)(w)|^{\lambda_2} |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} dw
\]
\[
\leq A_0(x) \left[ \int_{x_0}^{x} (z_1(w))^{\lambda_1/\lambda_2} (z_2(w))^{\lambda_2/\lambda_1} z_2'(w) dw \right]^{\lambda_2/p}. \tag{22}
\]

Taking \(\lambda_2 = 0\) and adding (21) and (22), we obtain
\[
\int_{x_0}^{x} q(w) \left[ |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} + |(D_{z_0}^\alpha f_2)(w)|^{\lambda_2} + |(D_{z_0}^\alpha f_1)(w)|^{\lambda_1} \right] dw
\]
\[
\leq \left( A_0(x)|_{\lambda_2 = 0} \right) \left\{ \left[ \int_{x_0}^{x} (z_1(w))^{\lambda_1/\lambda_1} z_1'(w) dw \right]^{\lambda_1/p} + \left[ \int_{x_0}^{x} (z_2(w))^{\lambda_1/\lambda_1} z_2'(w) dw \right]^{\lambda_2/p} \right\}
\]
\[
= \left( A_0(x)|_{\lambda_2 = 0} \right) \left\{ \left( z_1(x) \right)^{(\lambda_1 + \lambda_1)/p} + \left( z_2(x) \right)^{(\lambda_1 + \lambda_1)/p} \right\} \left( \frac{\lambda_1}{\lambda_1 + \lambda_1} \right)^{\lambda_1/p}
\]
\[
= \left( A_0(x)|_{\lambda_2 = 0} \right) \left( \frac{\lambda_1}{\lambda_1 + \lambda_1} \right)^{\lambda_1/p} \left\{ \left( \int_{x_0}^{x} p(t) |(D_{z_0}^\alpha f_1)(t)|^p dt \right)^{(\lambda_1 + \lambda_1)/p} \right. \]
\[
\left. + \left( \int_{x_0}^{x} p(t) |(D_{z_0}^\alpha f_2)(t)|^p dt \right)^{(\lambda_1 + \lambda_1)/p} \right\} =: (*). \tag{23}
\]

In this article, we are using frequently, the basic inequalities,
\[
2^{r-1} (a^r + b^r) \leq (a + b)^r \leq 2^r (a^r + b^r), \quad a, b \geq 0, \quad 0 \leq r \leq 1, \tag{23}
\]
\[
a^r + b^r \leq (a + b)^r \leq 2^{r-1} (a^r + b^r), \quad a, b \geq 0, \quad r \geq 1. \tag{24}
\]

Finally, using (23), (24), and (18), we get
\[
(*) \leq \left( A_0(x)|_{\lambda_2 = 0} \right) \left( \frac{\lambda_1}{\lambda_1 + \lambda_1} \right)^{\lambda_1/p} \cdot \delta_1
\]
\[
\cdot \left\{ \left( \int_{x_0}^{x} p(t) \left[ |(D_{z_0}^\alpha f_1)(t)|^p + |(D_{z_0}^\alpha f_2)(t)|^p \right] dt \right)^{(\lambda_1 + \lambda_1)/p} \right\}.
\]

Inequality (19) has been established.

It follows the counterpart of the last theorem.
THEOREM 3. All here, as in Theorem 2. Denote
\[
\delta_3 := \begin{cases} 
2^{\lambda_\beta/\lambda_\nu} - 1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\
1, & \text{if } \lambda_\beta \leq \lambda_\nu.
\end{cases}
\] (25)

If \( \lambda_\alpha = 0 \), then it holds
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^2 f_2) (w) \right|^{\lambda_\beta} \cdot \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\nu} + \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\beta} \cdot \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_\nu} \right] dw \\
\leq \left( A_0 (x) \right|_{\lambda_\alpha=0} \right) 2^{p-\lambda_\nu/p} \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right)^{\lambda_\nu/p} \delta_3^{\lambda_\nu/p} \\
\cdot \left( \int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\nu} f_1) (w) \right|^{p} + \left| (D_{x_0}^{\nu} f_2) (w) \right|^{p} \right] dw \right)^{(\lambda_\nu + \lambda_\beta)/p}, \quad \text{all } x_0 \leq x \leq b.
\] (26)

PROOF. When \( \lambda_\alpha = 0 \) from (21) and (22), we obtain
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^2 f_2) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\nu} \right] dw \\
\leq \left( A_0 (x) \right|_{\lambda_\alpha=0} \right) \left[ \int_{x_0}^{x} \left( z_2 (w) \right)^{\lambda_\beta/\lambda_\nu} z_1' (w) dw \right]^{\lambda_\nu/p},
\] (27)

and
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_\nu} \right] dw \\
\leq \left( A_0 (x) \right|_{\lambda_\alpha=0} \right) \left[ \int_{x_0}^{x} \left( z_1 (w) \right)^{\lambda_\beta/\lambda_\nu} z_2' (w) dw \right]^{\lambda_\nu/p},
\] (28)
all \( x_0 \leq x \leq b \). Adding (27) and (28), we get
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^2 f_2) (w) \right|^{\lambda_\beta} \cdot \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\nu} + \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\beta} \cdot \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_\nu} \right] dw \\
\leq \left( A_0 (x) \right|_{\lambda_\alpha=0} \right) \left\{ \left[ \int_{x_0}^{x} \left( z_2 (w) \right)^{\lambda_\beta/\lambda_\nu} z_1' (w) dw \right]^{\lambda_\nu/p} \\
+ \left[ \int_{x_0}^{x} \left( z_1 (w) \right)^{\lambda_\beta/\lambda_\nu} z_2' (w) dw \right]^{\lambda_\nu/p} \right\} \\
\leq \left( A_0 (x) \right|_{\lambda_\alpha=0} \right) \cdot 2^{p-\lambda_\nu/p} \cdot (M (x))^{\lambda_\nu/p} =: (\ast),
\] (29)
by
\[
0 < \lambda_\nu/p < 1
\]
and (23), where
\[
M (x) := \int_{x_0}^{x} \left( z_2 (w) \right)^{\lambda_\beta/\lambda_\nu} z_1' (w) + \left( z_1 (w) \right)^{\lambda_\beta/\lambda_\nu} \left( z_2' (w) \right) dw.
\] (30)

Call
\[
\delta_2 := \begin{cases} 
1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\
2^{1-(\lambda_\beta/\lambda_\nu)}, & \text{if } \lambda_\beta \leq \lambda_\nu.
\end{cases}
\]
Next, we work on \( M(x) \). We have that
\[
M (x) = \int_{x_0}^{x} \left( z_1 (w) \right)^{\lambda_\beta/\lambda_\nu} + \left( z_2 (w) \right)^{\lambda_\beta/\lambda_\nu} \left( z_1' (w) + z_2' (w) \right) dw
\]
\[
- \int_{x_0}^{x} \left[ (z_1(w))^{\theta/\lambda_\nu} \, z_1'(w) + (z_2(w))^{\theta/\lambda_\nu} \, z_2'(w) \right] \, dw
\]

(by (23),(24))

\[
\delta_2 \int_{x_0}^{x} (z_1(w) + z_2(w))^{\theta/\lambda_\nu} \, (z_1(w) + z_2(w))' \, dw
\]

\[
= \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \left[ (z_1(x))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} + (z_2(w))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} \right]
\]

\[
= \delta_2 \left( (z_1(x) + z_2(x))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} \right) \left( \frac{\lambda_\nu}{\lambda_\nu + \lambda_\beta} \right) - \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right)
\]

\[
\times \left[ (z_1(x))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} + (z_2(w))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} \right]
\]

\[
= \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \delta_2 \left( z_1(x) + z_2(x) \right)^{\lambda_\nu/\lambda_\beta - 1} \left( (z_1(x))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} + (z_2(x))^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu} \right)
\]

\[
\text{(by (23),(24))}
\]

\[
\leq \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \delta_3 \left( z_1(x) + z_2(x) \right)^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu},
\]

i.e., we present that

\[
M(x) \leq \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \delta_3 \left( z_1(x) + z_2(x) \right)^{(\lambda_\nu + \lambda_\beta)/\lambda_\nu}.
\]

Consequently, by (29) and (31), we get

\[
(*) \leq \left( A_0(x) |_{\lambda_\nu=0} \right) \delta^{\theta-p/\lambda_\nu} \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \delta_3^{\lambda_\nu/p} \left( z_1(x) + z_2(x) \right)^{(\lambda_\nu + \lambda_\beta)/p}
\]

\[
= \left( A_0(x) |_{\lambda_\nu=0} \right) \delta^{\theta-p/\lambda_\nu} \left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right) \delta_3^{\lambda_\nu/p}
\]

\[
\cdot \left( \int_{x_0}^{x} p(t) \left[ \left( (D_{x_0}^\nu f_1)(t) \right)^p + \left( (D_{x_0}^\nu f_2)(t) \right)^p \right] \, dt \right)^{(\lambda_\nu + \lambda_\beta)/p}.
\]

We have established (26).

The full case, when \( \lambda_\alpha, \lambda_\beta \neq 0 \), follows.

**Theorem 4.** All here, as in Theorem 2. Denote

\[
\tilde{\gamma}_1 := \begin{cases} 2^{(\lambda_\alpha + \lambda_\beta)/\lambda_\nu - 1}, & \text{if } \lambda_\alpha + \lambda_\beta \geq \lambda_\nu, \\ 1, & \text{if } \lambda_\alpha + \lambda_\beta \leq \lambda_\nu, \end{cases}
\]

and

\[
\tilde{\gamma}_2 := \begin{cases} 1, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \geq p, \\ 2^{1 - (\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p}, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \leq p. \end{cases}
\]
Then, it holds

\[
\int_{x_0}^{x} q(w) \left[ \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \left| (D^\beta_{\mu} f_2)(w) \right|^\lambda_\mu \right] \, dw
\]

\[+ \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \left| (D^\beta_{\mu} f_2)(w) \right|^\lambda_\mu \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \right] \, dw
\]

\[\leq A_0(x) \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)} \right)^{\lambda_\nu/p} \left[ \frac{\lambda_\alpha^p}{(\lambda_\alpha + \lambda_\beta)} + 2^{(p-\lambda_\nu)/p} (\frac{\lambda_\beta}{\lambda_\nu})^{q/p} \right]
\]

\[
\cdot \left( \int_{x_0}^{x} p(w) \left[ \left| (D^\nu_{\nu} f_1)(w) \right|^p + \left| (D^\nu_{\nu} f_2)(w) \right|^p \right] \, dw \right)^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p},
\]

all \( x_0 \leq x \leq b \).

**Proof.** Here, we use the basic inequality,

\[ a^p b^q \leq \frac{a^p}{p} + \frac{b^q}{q}, \]

where \( a, b \geq 0 \) and \( p, q > 1 \), such that \( p + q = 1 \). From (21), we obtain

\[
\int_{x_0}^{x} q(w) \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \left| (D^\beta_{\mu} f_2)(w) \right|^\lambda_\mu \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \, dw
\]

\[\leq A_0(x) \left[ \int_{x_0}^{x} \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)} \right) (z_1(w))^{(\lambda_\alpha + \lambda_\beta)/\lambda_\nu} \right. \]

\[+ \left. \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\beta} \right) (z_2(w))^{(\lambda_\alpha + \lambda_\beta)/\lambda_\nu} z_1'(w) \, dw \right]^{\lambda_\nu/p},
\]

\[\leq A_0(x) \left[ \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\beta} \right) (z_1(w))^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/\lambda_\nu} \right.
\]

\[+ \left. \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\beta} \right) (z_2(w))^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/\lambda_\nu} z_1'(w) \, dw \right]^{\lambda_\nu/p},
\]

Therefore,

\[
\int_{x_0}^{x} q(w) \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \left| (D^\beta_{\mu} f_2)(w) \right|^\lambda_\mu \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \, dw
\]

\[\leq A_0(x) \left\{ \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)} \right)^{\lambda_\nu/p} \left( z_1(w) \right)^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p} \right.
\]

\[+ \left. \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\beta} \right)^{\lambda_\nu/p} \left( \int_{x_0}^{x} (z_2(w))^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/\lambda_\nu} z_1'(w) \, dw \right)^{\lambda_\nu/p} \right\}.
\]

Similarly, using (22), we find

\[
\int_{x_0}^{x} q(w) \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \left| (D^\beta_{\mu} f_2)(w) \right|^\lambda_\mu \left| (D^\alpha_{\nu} f_1)(w) \right|^\lambda_\nu \, dw
\]

\[\leq A_0(x) \left\{ \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)} \right)^{\lambda_\nu/p} \left( z_2(w) \right)^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p} \right.
\]

\[+ \left. \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\beta} \right)^{\lambda_\nu/p} \left( \int_{x_0}^{x} (z_1(w))^{(\lambda_\alpha + \lambda_\beta + \lambda_\nu)/\lambda_\nu} z_2'(w) \, dw \right)^{\lambda_\nu/p} \right\}.
\]
Next, adding (36) and (37), we observe

\[
\Omega := \int_{x_0}^{x} q(w) \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\nu} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\omega} \, dw
\]

\[
+ \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\nu} \, dw
\]

\[
\leq A_0(x) \left\{ \left( \frac{\lambda_\nu \lambda_\alpha}{(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\beta + \lambda_\nu)} \right)^{\lambda_\nu/p} \times \left[ (z_1(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} + (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right] \\
+ \left( \frac{\lambda_\beta}{\lambda_\alpha + \lambda_\beta} \right)^{\lambda_\nu/p} 2^{(p-\lambda_\nu)/p} \left( \frac{\tilde{\gamma}_1 \lambda_\nu}{\lambda_\alpha + \lambda_\beta + \lambda_\nu} \right)^{\lambda_\nu/p} (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right\}
\]

\[
\leq A_0(x) \left\{ \left( \frac{\lambda_\nu \lambda_\alpha}{(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\beta + \lambda_\nu)} \right)^{\lambda_\nu/p} \times \left[ (z_1(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} + (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right] \\
+ \left( \frac{\lambda_\beta}{\lambda_\alpha + \lambda_\beta} \right)^{\lambda_\nu/p} 2^{(p-\lambda_\nu)/p} \left( \frac{\tilde{\gamma}_1 \lambda_\nu}{\lambda_\alpha + \lambda_\beta + \lambda_\nu} \right)^{\lambda_\nu/p} (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right\}
\]

where

\[
\Gamma(x) := \int_{x_0}^{x} \left( (z_1(w))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} z_2'(w) + (z_2(w))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} z_1'(w) \right) \, dw.
\]

Again, see (30) and (31), we get

\[
\Gamma(x) \leq \left( \frac{\tilde{\gamma}_1 \lambda_\nu}{\lambda_\alpha + \lambda_\beta + \lambda_\nu} \right) (z_1(x) + z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)}.
\]

Hence, by (38), we obtain

\[
\Omega \leq A_0(x) \left\{ \left( \frac{\lambda_\nu \lambda_\alpha}{(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\beta + \lambda_\nu)} \right)^{\lambda_\nu/p} \times \left[ (z_1(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} + (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right] \\
+ \left( \frac{\lambda_\beta}{\lambda_\alpha + \lambda_\beta} \right)^{\lambda_\nu/p} 2^{(p-\lambda_\nu)/p} \left( \frac{\tilde{\gamma}_1 \lambda_\nu}{\lambda_\alpha + \lambda_\beta + \lambda_\nu} \right)^{\lambda_\nu/p} (z_2(x))^{((\lambda_\alpha + \lambda_\beta + \lambda_\nu)/p)} \right\}
\]
We have proved

\[
\Omega \leq A_0(x) \left( \frac{\lambda_u}{(\lambda_\alpha + \lambda_\beta)} \left( \frac{\lambda_\alpha}{\lambda_\alpha + \lambda_\beta + \lambda_u} \right) \right)^{\lambda_u/p} \left[ \frac{\lambda_\alpha^{\lambda_u/p}}{\gamma_2 + 2(\gamma_\beta - \lambda_\gamma)/p} \right] \\
\cdot \left[ \int_{x_0}^x p(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^p + \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^p \right] \, dw \right]^{(\lambda_\alpha + \lambda_\beta + \lambda_u)/p}.
\]

We have established (34).

A special important case follows.

**THEOREM 5.** Let \( \nu \geq 3 \) and \( \gamma_1 \geq 1 \), such that \( \nu - \gamma_1 \geq 2 \). Let \( f_1, f_2 \in C^\nu_0([a, b]) \) with

\[
f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, \quad i = 0, 1, \ldots, n - 1,
\]

\( n := [\nu] \). Here, \( x, x_0 \in [a, b] : x \geq x_0 \). Consider also, \( p(t) > 0 \), and \( q(t) \geq 0 \) continuous functions on \([x_0, b] \). Let

\[
\lambda_\alpha \geq 0, \quad 0 < \lambda_\alpha + 1 < 1,
\]

and \( p > 1 \). Denote

\[
\theta_3 := \begin{cases} 
2^{\lambda_\alpha/(\lambda_\alpha + 1)} - 1, & \text{if } \lambda_\alpha \geq \lambda_\alpha + 1, \\
1, & \text{if } \lambda_\alpha \leq \lambda_\alpha + 1,
\end{cases}
\]

\( L(x) := \left( 2 \int_{x_0}^x \left( q(w) \right)^{(1/1 - (\lambda_\alpha + 1))} \, dw \right)^{(1-\lambda_\alpha + 1)/(\lambda_\alpha + \lambda_\alpha + 1)}, \)

and

\[
P_1(x) := \int_{x_0}^x (x - t)^{(\nu - \gamma_1)p/(p-1)} (p(t))^{-1/(p-1)} \, dt,
\]

\( T(x) := L(x) \cdot \left( \frac{P_1(x)^{(\nu-1)/p}}{\Gamma(\nu - \gamma_1)} \right)^{(\lambda_\alpha + \lambda_\alpha + 1)}, \)

and

\[
\omega_1 := \begin{cases} 
2^{1-(\lambda_\alpha + \lambda_\alpha + 1)/p}, & \text{if } \lambda_\alpha + \lambda_\alpha + 1 \leq p, \\
1, & \text{if } \lambda_\alpha + \lambda_\alpha + 1 > p,
\end{cases}
\]

\( \Phi(x) := T(x) \omega_1. \)

Then, it holds

\[
\int_{x_0}^x q(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^\lambda_\alpha \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^{\lambda_\alpha + 1} \\
+ \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^\lambda_\alpha \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha + 1} \right] \, dw \\
\leq \Phi(x) \left[ \int_{x_0}^x p(w) \left[ \left| (D_{x_0}^{\gamma_2} f_1)(w) \right|^p + \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^p \right] \, dw \right]^{(\lambda_\alpha + \lambda_\alpha + 1)/p}, \quad \text{all } x_0 \leq x \leq b.
\]

**PROOF.** For convenience, we set

\( \gamma_2 := \gamma_1 + 1. \)
From (13), we obtain
\[ \left| (D_{x_0}^\gamma f_j) (w) \right| \leq \frac{1}{\Gamma (\nu - \gamma_k)} \int_{x_0}^w \left( w - t \right)^{\nu - \gamma_k - 1} \left| (D_{x_0}^\nu f_j) (t) \right| \, dt =: g_{j, \gamma_k} (w), \]
where \( j = 1, 2, k = 1, 2, \) all \( x_0 \leq w \leq b. \) We observe that
\[ (D_{x_0}^\gamma f_j (x))' = (D_{x_0}^{\gamma+1} f_j) (x) = (D_{x_0}^\nu f_j) (x), \]
all \( x_0 \leq x \leq b. \) And also,
\[ (g_{j, \gamma_1} (w))' = g_{j, \gamma_2} (w); \quad g_{j, \gamma_k} (x_0) = 0. \]

Notice that, if \( \nu - \gamma_k = 1, \) then,
\[ g_{j, \gamma_k} (w) = \int_{x_0}^w \left| (D_{x_0}^\nu f_j) (t) \right| \, dt. \]

Next, we apply Hölder’s inequality with indices \( 1/\lambda_{\alpha+1}, 1/(1 - \lambda_{\alpha+1}), \) we obtain
\[ \int_{x_0}^x q (w) \left| (D_{x_0}^\gamma f_1) (w) \right|^{\lambda_{\alpha}} \left| (D_{x_0}^{\gamma+1} f_2) (w) \right|^{\lambda_{\alpha+1}} \, dw \]
\[ \leq \int_{x_0}^x q (w) (g_{1, \gamma_1} (w))^{\lambda_{\alpha}} (g_{2, \gamma_1} (w))^{\lambda_{\alpha+1}} \, dw \]
\[ \leq \left( \int_{x_0}^x (q (w))^{1/(1-\lambda_{\alpha+1})} \, dw \right)^{(1-\lambda_{\alpha+1})} \left( \int_{x_0}^x (g_{1, \gamma_1} (w))^{\lambda_{\alpha}/\lambda_{\alpha+1}} (g_{2, \gamma_1} (w))' \, dw \right)^{\lambda_{\alpha+1}} . \]

Similarly, we get
\[ \int_{x_0}^x q (w) \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_{\alpha}} \left| (D_{x_0}^{\gamma+1} f_1) (w) \right|^{\lambda_{\alpha+1}} \, dw \]
\[ \leq \left( \int_{x_0}^x (q (w))^{1/(1-\lambda_{\alpha+1})} \, dw \right)^{(1-\lambda_{\alpha+1})} \left( \int_{x_0}^x (g_{2, \gamma_1} (w))^{\lambda_{\alpha}/\lambda_{\alpha+1}} (g_{1, \gamma_1} (w))' \, dw \right)^{\lambda_{\alpha+1}} . \]

Adding (49) and (50), we observe
\[ \int_{x_0}^x q (w) \left[ \left| (D_{x_0}^\gamma f_1) (w) \right|^{\lambda_{\alpha}} \left| (D_{x_0}^{\gamma+1} f_2) (w) \right|^{\lambda_{\alpha+1}} + \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_{\alpha}} \left| (D_{x_0}^{\gamma+1} f_1) (w) \right|^{\lambda_{\alpha+1}} \right] \, dw \]
\[ \leq \left( \int_{x_0}^x (q (w))^{1/(1-\lambda_{\alpha+1})} \, dw \right)^{(1-\lambda_{\alpha+1})} \left[ \left( \int_{x_0}^x (g_{1, \gamma_1} (w))^{\lambda_{\alpha}/\lambda_{\alpha+1}} (g_{2, \gamma_1} (w))' \, dw \right)^{\lambda_{\alpha+1}} \right. \]
\[ \leq \left( \int_{x_0}^x (q (w))^{1/(1-\lambda_{\alpha+1})} \, dw \right)^{(1-\lambda_{\alpha+1})} \left[ \left( \int_{x_0}^x (g_{2, \gamma_1} (w))^{\lambda_{\alpha}/\lambda_{\alpha+1}} (g_{1, \gamma_1} (w))' \, dw \right)^{\lambda_{\alpha+1}} \right] \]
\[ \leq \frac{L (x)}{\Gamma (\nu - \gamma_k)^{(\lambda_{\alpha+\lambda_{\alpha+1}})}} \left\{ \left( \int_{x_0}^x (x - t)^{\nu - \gamma_k - 1} (p (t))^{1/p} (p (t))^{1/p} \left| (D_{x_0}^\nu f_1) (t) \right| \, dt \right)^{(\lambda_{\alpha+\lambda_{\alpha+1}})} \right\} \]
\[ = \frac{L (x)}{\Gamma (\nu - \gamma_k)^{(\lambda_{\alpha+\lambda_{\alpha+1}})}} \left\{ \left( \int_{x_0}^x (x - t)^{\nu - \gamma_k - 1} (p (t))^{1/p} (p (t))^{1/p} \left| (D_{x_0}^\nu f_1) (t) \right| \, dt \right)^{(\lambda_{\alpha+\lambda_{\alpha+1}})} \right\} \]
(applying Hölder’s inequality twice, with indices \( p/(p-1) \) and \( p \), we find)

\[
\begin{align*}
L(x) & \leq \frac{L(x)}{(\Gamma(\nu - \gamma_1))^{\lambda_\alpha + \lambda_\beta + 1}} \cdot \left( \int_{x_0}^{x} (x-t)^{(\nu - \gamma_1 -1) p/(p-1)} (p(t))^{-1/(p-1)} dt \right)^{(p-1)/p} \\
& \quad \cdot \left[ \left( \int_{x_0}^{x} p(t) \left| (D^\nu_{x_0} f_1)(t) \right|^p dt \right)^{\lambda_\alpha + \lambda_\beta + 1/p} \right] \\
& \quad \cdot \left( \int_{x_0}^{x} p(t) \left| (D^\nu_{x_0} f_2)(t) \right|^p dt \right)^{(\lambda_\alpha + \lambda_\beta + 1)/p} \\
& \quad \cdot \left( \int_{x_0}^{x} p(t) \left| (D^\nu_{x_0} f_1)(t) \right|^p dt \right)^{\lambda_\alpha + \lambda_\beta + 1/p} \\
& = \Phi(x) \cdot \left[ \int_{x_0}^{x} p(t) \left( (D^\nu_{x_0} f_1)(t) \right)^p + \left( (D^\nu_{x_0} f_2)(t) \right)^p \right]^{(\lambda_\alpha + \lambda_\beta + 1)/p}.
\end{align*}
\]

We have proved (45).

Next, we treat the case of exponents \( \lambda_\beta = \lambda_\alpha + \lambda_\nu \).

**Theorem 6.** All here, as in Theorem 2. Consider the special case \( \lambda_\beta = \lambda_\alpha + \lambda_\nu \). Denote

\[
\hat{T}(x) := A_0(x) \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} 2^{(p-2\lambda_\alpha - 3\lambda_\nu)/p}. \tag{51}
\]

Then, it holds

\[
\begin{align*}
\int_{x_0}^{x} q(w) \left[ \left| (D^\nu_{x_0} f_1)(w) \right|^\lambda + \left| (D^\nu_{x_0} f_2)(w) \right|^\lambda \right] \left( (D^\nu_{x_0} f_1)(w) \right)^{\lambda_\alpha + \lambda_\nu} + \left( (D^\nu_{x_0} f_2)(w) \right)^{\lambda_\alpha + \lambda_\nu} \right] dw \\
& \leq \hat{T}(x) \left( \int_{x_0}^{x} p(w) \left( (D^\nu_{x_0} f_1)(w) \right)^p + \left( (D^\nu_{x_0} f_2)(w) \right)^p \right) \left( 2^{(\lambda_\alpha + \lambda_\nu)/p} \right),
\end{align*}
\]

all \( x_0 \leq x \leq b \).

**Proof.** We apply (21) and (22) for \( \lambda_\beta = \lambda_\alpha + \lambda_\nu \), and add to get

\[
\begin{align*}
\int_{x_0}^{x} q(w) \left[ \left| (D^\nu_{x_0} f_1)(w) \right|^\lambda + \left| (D^\nu_{x_0} f_2)(w) \right|^\lambda \right] \left( (D^\nu_{x_0} f_1)(w) \right)^{\lambda_\alpha + \lambda_\nu} + \left( (D^\nu_{x_0} f_2)(w) \right)^{\lambda_\alpha + \lambda_\nu} \right] dw \\
& \leq A_0(x) \left\{ \left( \int_{x_0}^{x} (z_1(w))^{\lambda_\alpha/\lambda_\nu} \left( z_2(w) \right)^{\lambda_\alpha/\lambda_\nu} z_1'(w) dw \right)^{\lambda_\nu/p} \right\} \\
& \quad + \left( \int_{x_0}^{x} (z_1(w))^{\lambda_\alpha/\lambda_\nu} \left( z_2(w) \right)^{\lambda_\alpha/\lambda_\nu} z_1'(w) dw \right)^{\lambda_\nu/p} \\
& \leq A_0(x) 2^{-1 - (\lambda_\nu/p)} \left\{ \left( \int_{x_0}^{x} (z_1(w))^{\lambda_\alpha/\lambda_\nu} \left( z_2(w) \right)^{\lambda_\alpha/\lambda_\nu} z_1'(w) dw \right)^{\lambda_\nu/p} \right\} \\
& \quad + \left( \int_{x_0}^{x} (z_1(w))^{\lambda_\alpha/\lambda_\nu} \left( z_2(w) \right)^{\lambda_\alpha/\lambda_\nu} z_1'(w) dw \right)^{\lambda_\nu/p} \\
& = A_0(x) 2^{-1 - (\lambda_\nu/p)} \left\{ \left( \int_{x_0}^{x} (z_1(w))^{\lambda_\alpha/\lambda_\nu} \left( z_2(w) \right)^{\lambda_\alpha/\lambda_\nu} z_1'(w) dw \right)^{\lambda_\nu/p} \right\}.
\[ A_0(x) 2^{1-(\lambda_\nu/p)} \cdot \left\{ \int_{x_0}^x (z_1(w)z_2(w))^{\lambda_\alpha/\lambda_\nu} (z_1(w)z_2(w))' dw \right\}^{\lambda_\nu/p} \]
\[ = A_0(x) 2^{1-(\lambda_\nu/p)} \left( \frac{(z_1(x)z_2(x))^{(\lambda_\alpha/\lambda_\nu)+1}}{(\lambda_\alpha/\lambda_\nu) + 1} \right)^{\lambda_\nu/p} \]
\[ = A_0(x) 2^{p-(\lambda_\nu/p)} \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} (z_1(x)z_2(x))^{(\lambda_\alpha + \lambda_\nu)/p} \]
\[ \leq A_0(x) 2^{p-(\lambda_\nu/p)} \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} \left( \frac{(z_1(x) + z_2(x))^2}{4} \right)^{(\lambda_\alpha + \lambda_\nu)/p} \]
\[ = \hat{T}(x) (z_1(x) + z_2(x))^{2(\lambda_\alpha + \lambda_\nu)/p} \]
\[ = \hat{T}(x) \left( \int_{x_0}^x p(w) \left( |(D_{z_0}^{\nu} f_1)(w)|^p + |(D_{z_0}^{\nu} f_2)(w)|^p \right) dw \right) 2^{((\lambda_\alpha + \lambda_\nu)/p)} . \]

We have established (52).

Next, follow special cases of the above theorems.

**Corollary 1.** (In relationship to Theorem 2, \( \lambda_\beta = 0 \), \( p(t) = q(t) = 1 \).) It holds
\[
\int_{x_0}^x \left[ \left| (D_{z_0}^{\nu} f_1)(w) \right|^\lambda_\alpha \left| (D_{z_0}^{\nu} f_1)(w) \right|^\lambda_\nu \right] dw \\
\leq C_1(x) \cdot \left( \int_{x_0}^x \left( |(D_{z_0}^{\nu} f_1)(w)|^p + |(D_{z_0}^{\nu} f_2)(w)|^p \right) dw \right)^{(\lambda_\alpha + \lambda_\nu)/p} ,
\]
all \( x_0 \leq x \leq b \), where
\[
C_1(x) := \left( A_0(x)|_{\lambda_\beta=0} \right) \cdot \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} \cdot \delta_1,
\]
\[
\delta_1 : = \begin{cases} 2^{1-((\lambda_\alpha + \lambda_\nu)/p)}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 1, & \text{if } \lambda_\alpha + \lambda_\nu > p. \end{cases}
\]
We find that
\[
\left( A_0(x)|_{\lambda_\beta=0} \right) = \left\{ \left( \frac{(p-1)^{((\lambda_\alpha - p)/(\lambda_\alpha - \lambda_\beta))}}{((\nu - \gamma_1)^{\lambda_\alpha}(\nu p - \gamma_1 p - 1)^{((\lambda_\alpha - p)/(\lambda_\alpha - \lambda_\beta))}} \right) \cdot \left( \frac{p-\lambda_\nu}{(p-\lambda_\nu)/p} \right) \right\} \cdot \left( \frac{\lambda_\alpha^{\nu p - \lambda_\alpha \gamma_1 p - \lambda_\alpha + p - \lambda_\nu}}{(p-\lambda_\nu)/p} \right) \cdot \left( x - x_0 \right)^{(\lambda_\alpha^{\nu p - \lambda_\alpha \gamma_1 p - \lambda_\alpha + p - \lambda_\nu})/p}. \]

**Proof.** Here, we need to calculate \( A_0(x)|_{\lambda_\beta=0} \). From (17), we have
\[
A_0(x)|_{\lambda_\beta=0} = \left( \int_{x_0}^x \left( A(w)|_{\lambda_\beta=0} \right)^{p/(p-\lambda_\nu)} dw \right)^{(p-\lambda_\nu)/p},
\]
where from (16), we get
\[
A(w)|_{\lambda_\beta=0} = \frac{P_1(w)^{\lambda_\alpha/(p-1)}}{((\nu - \gamma_1)^{\lambda_\alpha}},
\]
and here, from (15), it is
\[
P_1(w) = \int_{x_0}^w (w-t)^{((\nu - \gamma_1 - 1)/(p-1))} dt.
Therefore, we obtain

\[ P_1(w) = \left( \frac{p-1}{\nu p - \gamma_1 p - 1} \right) (w - x_0)^{(\nu p - \gamma_1 p - 1)/(p-1)}, \]

and

\[ A(w) |_{\lambda_0 = 0} = \left( \frac{p-1}{\nu p - \gamma_1 p - 1} \right)^{\left( \frac{\lambda_0 p - \lambda_\nu}{\nu p - \gamma_1 p - 1} \right)/p} \cdot \left( \frac{1}{\Gamma (\nu - \gamma_1)} \right)^{\lambda_\nu} \cdot (w - x_0)^{(\lambda_0 \nu p - \lambda_\nu \gamma_1 p - \lambda_\nu)/p}. \]

That is,

\[
A_0(x) |_{\lambda_0 = 0} = \left( \frac{p-1}{\nu p - \gamma_1 p - 1} \right)^{\left( \frac{\lambda_0 p - \lambda_\nu}{\nu p - \gamma_1 p - 1} \right)/p} \cdot \left( \frac{1}{\Gamma (\nu - \gamma_1)} \right)^{\lambda_\nu} \cdot (x - x_0)^{(\lambda_0 \nu p - \lambda_\nu \gamma_1 p - \lambda_\nu + p - \lambda_\nu)/(p-\lambda_\nu)/p). \]

**Corollary 2.** (In relationship to Theorem 2, \( \lambda_0 = 0, \nu(t) = q(t) = 1, \lambda = \lambda_\nu = 1, p = 2. \))

In detail, let \( \nu, \gamma_1, \gamma_2 \geq 1 \), such that \( \nu - \gamma_1 \geq 1, \nu - \gamma_2 \geq 1 \) and \( f_1, f_2 \in C^\nu \) \([a, b]\) with \( f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0 \), \( i = 0, 1, \ldots, n - 1, n := [\nu] \). Here, \( x, x_0 \in [a, b] : x \neq x_0 \). Then, it holds

\[
\int_{x_0}^x \left[ \left| (D^\nu_{x_0} f_1)(w) \right| + \left| (D^\nu_{x_0} f_2)(w) \right| \right] \, dw \leq \left( \frac{x - x_0}{\Gamma (\nu - \gamma_1) \sqrt{\nu - \gamma_1} \sqrt{2\nu - 2\gamma_1 - 1}} \right) \left( \int_{x_0}^x \left[ \left( (D^\nu_{x_0} f_1)(w) \right)^2 + \left( (D^\nu_{x_0} f_2)(w) \right)^2 \right] \, dw \right),
\]

all \( x_0 \leq x \leq b \).

**Proof.** We apply Corollary 1. Here, \( \delta_1 = 1 \) and

\[
\left( \frac{\lambda_\nu}{\lambda_0 + \lambda_\nu} \right)^{\lambda_\nu/p} = \frac{1}{\sqrt{2}}.
\]

Furthermore, we have

\[
(A_0(x) |_{\lambda_0 = 0}) = \left( \frac{1}{\Gamma (\nu - \gamma_1) \sqrt{2\nu - 2\gamma_1 - 1}} \right) \cdot \left( \frac{1}{\sqrt{2\nu - 2\gamma_1 - 1}} \right) \cdot (x - x_0)^{(\nu - \gamma_1)}.
\]

Finally, we get

\[
C_1(x) = \frac{(x - x_0)^{(\nu - \gamma_1)}}{2\Gamma (\nu - \gamma_1) \sqrt{\nu - \gamma_1} \sqrt{2\nu - 2\gamma_1 - 1}}.
\]
COROLLARY 3. (In relationship to Theorem 3, \( \lambda_0 = 0 \), \( p(t) = q(t) = 1 \).) It holds

\[
\int_{x_0}^{x} \left[ |(D_{x_0}^{\gamma_1} f_2) (w)|^\lambda_0 + |(D_{x_0}^{\gamma_1} f_1) (w)|^\lambda_0 + |(D_{x_0}^{\gamma_1} f_3) (w)|^\lambda_0 \right] dw
\]

\[
\leq C_2(x) \left( \int_{x_0}^{x} \left[ |(D_{x_0}^{\gamma_1} f_1) (w)|^p + |(D_{x_0}^{\gamma_1} f_2) (w)|^p \right] dw \right)^{(\lambda_0 + \lambda_0)/(p)},
\]

all \( x_0 \leq x \leq b \), where

\[
C_2(x) := (A_0(x)|_{\lambda_0=0}) 2^{(p-\lambda_0)/p} \left( \frac{\lambda_0}{\lambda_0 + \lambda_0} \right)^{\lambda_0/p} \delta_0^{\lambda_0/p},
\]

\[
\delta_0 := \begin{cases} 
2^{\lambda_0/\lambda_0 - 1}, & \text{if } \lambda_0 \geq \lambda_0, \\
1, & \text{if } \lambda_0 \leq \lambda_0.
\end{cases}
\]

We find that

\[
\left( A_0(x)|_{\lambda_0=0} \right) = \left\{ \left( \frac{(p-1)^{\left(\lambda_0 p - \lambda_0 \right)/(p)} \left( \Gamma(\nu - \gamma_2) \right)^{\lambda_0 p - \lambda_0}}{\left( \lambda_0 \nu - \lambda_0 \gamma_2 p - \lambda_0 + \nu \right)^{(p-\lambda_0)/(p)}} \right) \right\}
\cdot \left( \frac{(p-1)^{\left(\lambda_0 p - \lambda_0 \right)/(p)} \left( \Gamma(\nu - \gamma_2) \right)^{\lambda_0 p - \lambda_0}}{\left( \lambda_0 \nu - \lambda_0 \gamma_2 p - \lambda_0 + \nu \right)^{(p-\lambda_0)/(p)}} \right)
\cdot (x - x_0)^{\left(\lambda_0 p - \lambda_0 \gamma_2 p - \lambda_0 + \nu \right)/(p)}. \tag{59}
\]

PROOF. Here, we need to calculate

\[
A_0(x)|_{\lambda_0=0}.
\]

From (17), we have

\[
A_0(x)|_{\lambda_0=0} = \left( \int_{x_0}^{x} \left( A(w)|_{\lambda_0=0} \right)^{p/(p-\lambda_0)} dw \right)^{(p-\lambda_0)/p},
\]

where from (16), we get

\[
A(w)|_{\lambda_0=0} = \frac{P_2(w)^{\lambda_0((p-1)/p)}}{\left( \Gamma(\nu - \gamma_2) \right)^{\lambda_0}},
\]

and here, from (15), it is

\[
P_2(w) = \int_{x_0}^{x_0} (w - t)^{(\nu - \gamma_2 - 1)p/(p-1)} dt.
\]

Therefore, we obtain

\[
P_2 (w) = \left( \frac{p-1}{\nu p - \gamma_2 p - 1} \right) \left( w - x_0 \right)^{(\nu - \gamma_2 - 1)p/(p-1)},
\]

and

\[
A(w)|_{\lambda_0=0} = \frac{(p-1)^{\left(\lambda_0 p - \lambda_0 \right)/(p)} \left( w - x_0 \right)^{\lambda_0 p - \lambda_0 \gamma_2 p - \lambda_0 + \nu}}{\left( \nu p - \gamma_2 p - 1 \right)^{((\lambda_0 p - \lambda_0 \gamma_2 p - \lambda_0 + \nu)/(p-1)} \left( \Gamma(\nu - \gamma_2) \right)^{\lambda_0}}.
\]
That is,
\[
A_0(x)|_{\lambda_0=0} = \left( \frac{(p-1)((\lambda_\beta p - \lambda_\beta)/p)}{(\nu p - \gamma_2 p - 1) ((\lambda_\alpha p - \lambda_\alpha)/p) \Gamma(\nu - \gamma_2) \lambda_\beta} \right) \\
\cdot \left( \int_{x_0}^{x} (w - x_0) \left( (\lambda_\alpha p - \lambda_\beta p - \lambda_\alpha)/(p - \lambda_\beta) \right) dw \right)^{(p - \lambda_\beta)/p} \\
= \left\{ \left( \frac{(p-1)((\lambda_\alpha p - \lambda_\beta p - \lambda_\alpha)/(p - \lambda_\beta)) \Gamma(\nu - \gamma_2) \lambda_\beta}{(\nu p - \gamma_2 p - 1) ((\lambda_\alpha p - \lambda_\beta p - \lambda_\alpha)/(p - \lambda_\beta)) \Gamma(\nu - \gamma_2) \lambda_\beta} \right) \\
\cdot \left( \frac{(p - \lambda_\beta)/(p - \lambda_\beta)}{(\nu p - \gamma_2 p - 1) ((\lambda_\alpha p - \lambda_\beta p - \lambda_\alpha)/(p - \lambda_\beta)) \Gamma(\nu - \gamma_2) \lambda_\beta} \right) \right\}^{(p - \lambda_\beta)/p} \\
\cdot (x - x_0)^{(\lambda_\alpha p - \lambda_\beta p - \lambda_\alpha)/(p - \lambda_\beta)}.
\]

**Corollary 4.** (In relationship to Theorem 3, \(\lambda_0 = 0\), \(p(t) = q(t) = 1\), \(\lambda_\beta = \lambda_\nu = 1\), \(p = 2\).) In detail, let \(\nu, \gamma_1, \gamma_2 \geq 1\), such that \(\nu - \gamma_1 \geq 1\), \(\nu - \gamma_2 \geq 1\) and \(f_1, f_2 \in C_{\nu_0}^n ([u, b])\) with \(f_i^{(i)}(x_0) = 0\), \(i = 0, 1, \ldots, n - 1\), \(n := [\nu]\). Here, \(x, x_0 \in [u, b] : x \geq x_0\). Then, it holds
\[
\int_{x_0}^{x} \left[ |(D_{x_0}^\nu f_2)(w)| \right] dw \\
\leq \left( \frac{(x - x_0)^{(\nu - \gamma_2)}}{\sqrt{2\Gamma(\nu - \gamma_2) \sqrt{\nu - \gamma_2} \sqrt{2\nu - 2\gamma_2 - 1}}} \right) \\
\cdot \left( \int_{x_0}^{x} \left( (D_{x_0}^\nu f_1)(w) \right)^2 + \left( (D_{x_0}^\nu f_2)(w) \right)^2 \right) dw,
\]
all \(x_0 \leq x \leq b\).

**Proof.** We apply Corollary 3. Here, \(\delta_0^{\nu / p} = 1, 2^{(p - \lambda_\beta)/p} = \sqrt{2}\),
\[
\left( \frac{\lambda_\nu}{\lambda_\beta + \lambda_\nu} \right)^{\nu / p} = \frac{1}{\sqrt{2}}.
\]
Furthermore, we have
\[
(A_0(x)|_{\lambda_0=0}) = \left( \frac{(x - x_0)^{(\nu - \gamma_2)}}{\sqrt{2\Gamma(\nu - \gamma_2) \sqrt{\nu - \gamma_2} \sqrt{2\nu - 2\gamma_2 - 1}}} \right).
\]
Finally, we get
\[
C_2(x) = \left( \frac{(x - x_0)^{(\nu - \gamma_2)}}{\sqrt{2\Gamma(\nu - \gamma_2) \sqrt{\nu - \gamma_2} \sqrt{2\nu - 2\gamma_2 - 1}}} \right).
\]

**Corollary 5.** (In relationship to Theorem 4, \(\lambda_\alpha = \lambda_\beta = \lambda_\nu = 1\), \(p = 3\), \(p(t) = q(t) = 1\).) It holds
\[
\int_{x_0}^{x} \left[ \left( |(D_{x_0}^\nu f_1)(w)| \right)^3 + \left( |(D_{x_0}^\nu f_2)(w)| \right)^3 \right] dw \\
\leq A_0(x) \left( \sqrt{3} + \frac{1}{3^{3/2}} \right) \cdot \left( \int_{x_0}^{x} \left( |(D_{x_0}^\nu f_1)(w)|^3 + |(D_{x_0}^\nu f_2)(w)|^3 \right) dw \right),
\]
all \(x_0 \leq x \leq b\). Here,
\[
A_0(x) = \frac{4(x - x_0)^{(2\nu - \gamma_1 - \gamma_2)}}{\Gamma(\nu - \gamma_1) \Gamma(\nu - \gamma_2) [3(3\nu - 3\gamma_1 - 1)(3\nu - 3\gamma_2 - 1)(2\nu - \gamma_1 - \gamma_2)]^{2/3}}.
\]
Proof. We apply inequality (34). Here, $\tilde{\gamma}_1 = 3$, $\tilde{\gamma}_2 = 1$, and

$$
\left( \frac{\lambda_{\nu}}{\left( \lambda_{\alpha} + \lambda_{\beta} \right) \left( \lambda_{\alpha} + \lambda_{\beta} + \lambda_{\nu} \right)} \right)^{\nu/p} = \frac{1}{\sqrt[6]{6}}.
$$

Furthermore,

$$
\left( \frac{\lambda_{\nu}/p}{\gamma_2 + 2(\nu-\lambda_{\nu})/p} \right) = 1 + \sqrt[6]{2}.
$$

The product of the last two quantities is $(\sqrt[6]{2} + 1/\sqrt[6]{6})$. It still remains to find $A_0(x)$ for this case.

Here, by (17),

$$
A_0(x) = \left( \int_{x_0}^{x} A(w)^{3/2} dw \right)^{2/3},
$$

and, by (16),

$$
A(w) = \left( \frac{P_1(w)^{2/3} (P_2(w)^{2/3})}{\Gamma(\nu-\gamma_1) \Gamma(\nu-\gamma_2)} \right).
$$

Also, by (15), we have for $k = 1, 2$,

$$
P_k(w) = \int_{x_0}^{w} (w-t)^{\left(3\nu-3\gamma_k-3\right)/2} dt.
$$

That is,

$$
P_k(w) = \frac{2(w-x_0)^{\left(3\nu-3\gamma_k-1 \right)/2}}{(3\nu-3\gamma_k-1)}, \quad k = 1, 2,
$$

and

$$
A(w) = \frac{24/3 (w-x_0)^{\left(6\nu-3\gamma_1-3\gamma_2-2/3\right)}}{\Gamma(\nu-\gamma_1) \Gamma(\nu-\gamma_2) (3\nu-3\gamma_1-1)^{2/3} (3\nu-3\gamma_2-1)^{2/3}}.
$$

Finally, we get

$$
A_0(x) = \left( \frac{2^{4/3}}{\Gamma(\nu-\gamma_1) \Gamma(\nu-\gamma_2) (3\nu-3\gamma_1-1)^{2/3} (3\nu-3\gamma_2-1)^{2/3}} \right)
$$

$$
\cdot \left( \int_{x_0}^{x} \left( (w-x_0)^{\left(6\nu-3\gamma_1-3\gamma_2-2/3\right)} dw \right)^{2/3} \right)
$$

$$
= \frac{4 \cdot (x-x_0)^{2(\nu-\gamma_1-\gamma_2)}}{\Gamma(\nu-\gamma_1) \Gamma(\nu-\gamma_2) (3\nu-3\gamma_1-1)^{2/3} (3\nu-3\gamma_2-1)^{2/3}}.
$$

Corollary 6. (In relationship to Theorem 5, $\lambda_\alpha = 1$, $\lambda_{\alpha+1} = 1/2$, $p = 3/2$, $p(t) = q(t) = 1$.) In detail, let $\nu \geq 3$ and $\gamma_1 \geq 1$, such that $\nu - \gamma_1 \geq 2$. Let $f_1, f_2 \in C_{x_0}^{\nu} ([a,b])$ with $f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n-1$, $n = [\nu]$. Here, $x, x_0 \in [a,b] : x \geq x_0$. Then, it holds

$$
\int_{x_0}^{x} \left[ |D_{x_0}^{\nu} f_1(w)| \sqrt{\left| D_{x_0}^{\nu+1} f_2(w) \right|} + \left| D_{x_0}^{\nu} f_2(w) \right| \sqrt{\left| D_{x_0}^{\nu+1} f_1(w) \right|} \right] dw
$$

$$
\leq \Phi(x) \cdot \left[ \int_{x_0}^{x} \left( \left| D_{x_0}^{\nu} f_1(w) \right|^{3/2} + \left| D_{x_0}^{\nu} f_2(w) \right|^{3/2} \right) dw \right],
$$

all $x_0 \leq x \leq b$. Here, we find

$$
\Phi(x) = \left( \sqrt{\frac{2}{3\nu-3\gamma_1-2}} \right) \left( \frac{x-x_0)^{\left(3\nu-3\gamma_1-1 \right)/2}}{(\Gamma(\nu-\gamma_1))^{3/2}} \right).
$$
Opial Type Inequalities

PROOF. We apply inequality (45). Here, \( \omega_1 = 1 \), that is, \( \Phi(x) = T(x) \), and \( \theta_3 = 3 \). Furthermore, we find \( L(x) = \sqrt{2}(x - x_0) \), and

\[
P_1(x) = \frac{(x - x_0)^{3\nu - 3\gamma_1 - 2}}{(3\nu - 3\gamma_1 - 2)}.
\]

Thus,

\[
T(x) = \sqrt{2}(x - x_0) \left( \frac{(x - x_0)^{(3\nu - 3\gamma_1 - 2)/3}}{\sqrt{3\nu - 3\gamma_1 - 2}} \right)^{3/2} = \frac{\sqrt{2} \cdot (x - x_0)^{(3\nu - 3\gamma_1 - 1)/2}}{(\Gamma(\nu - \gamma_1))^{3/2} \cdot \sqrt{3\nu - 3\gamma_1 - 2}}.
\]

COROLLARY 7. (In relationship to Theorem 6, \( p = 2(\lambda_\alpha + \lambda_\nu) > 1 \), \( p(t) = q(t) = 1 \). It holds

\[
\int_{x_0}^{x} \left( [(D_{x_0}^\nu f_1)(w)]^{\lambda_\alpha} [(D_{x_0}^\nu f_2)(w)]^{\lambda_\nu} + [(D_{x_0}^\nu f_1)(w)]^{\lambda_\nu} [(D_{x_0}^\nu f_2)(w)]^{\lambda_\alpha} \right) dw \\
\leq \tilde{T}(x) \left( \int_{x_0}^{x} \left( [(D_{x_0}^\nu f_1)(w)]^{2(\lambda_\alpha + \lambda_\nu)} + [(D_{x_0}^\nu f_2)(w)]^{2(\lambda_\lambda + \lambda_\nu)} \right) dw \right),
\]

\( x_0 \leq x \leq b \). Here,

\[
\tilde{T}(x) = A_0(x) \left( \frac{\lambda_\nu}{(2(\lambda_\alpha + \lambda_\nu))^{\lambda_\nu/2(\lambda_\alpha + \lambda_\nu)}} \right),
\]

and

\[
A_0(x) = \sigma^* (x - x_0)^\theta,
\]

where

\[
\sigma := \frac{1}{(\Gamma(\nu - \gamma_1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2))^{\lambda_\nu}},
\]

\[
\frac{2\lambda_\alpha + 2\lambda_\nu - 1}{(2\lambda_\alpha + 2\lambda_\nu - 2\lambda_\alpha\gamma_1 - 2\lambda_\nu\gamma_1 - 1) (2\lambda_\nu + 2\lambda_\nu - 1) ((2\lambda_\alpha + 2\lambda_\nu - 1)/2)} (2\lambda_\alpha + 2\lambda_\nu - 1),
\]

\[
\sigma^* := \frac{2\lambda_\alpha + \lambda_\nu}{(4\lambda_\alpha^2 + 6\lambda_\alpha\lambda_\nu - 2\lambda_\alpha^2\gamma_1 - 2\lambda_\alpha\lambda_\nu^2 - 2\lambda_\alpha\lambda_\nu\gamma_1 - 2\lambda_\alpha^2\gamma_2 - 4\lambda_\alpha\lambda_\nu^2\gamma_1 + 2\lambda_\alpha^2\nu - 2\lambda_\nu^2\gamma_2)} (2\lambda_\alpha + 2\lambda_\nu),
\]

and

\[
\theta := \frac{4\lambda_\alpha^2 + 6\lambda_\alpha\lambda_\nu - 2\lambda_\alpha^2\gamma_1 - 2\lambda_\alpha\lambda_\nu\gamma_1 - 2\lambda_\alpha^2\gamma_2 - 4\lambda_\alpha\lambda_\nu^2\gamma_1 + 2\lambda_\alpha^2\nu - 2\lambda_\nu^2\gamma_2}{2\lambda_\alpha + 2\lambda_\nu}.
\]

PROOF. We apply inequality (52). Constant \( \tilde{T}(x) \) comes from (51) and assumption on \( p \) here. Still, we need to determine \( A_0(x) \). Here, from (17), we have

\[
A_0(x) = \left( \int_{x_0}^{x} (A(w))^{(2(\lambda_\alpha + \lambda_\nu)/2(\lambda_\alpha + \lambda_\nu)} dw \right)^{(2(\lambda_\alpha + \lambda_\nu)/2(\lambda_\alpha + \lambda_\nu)}},
\]

where, from (16), we get

\[
A(w) = \frac{(P_1(w))^{(\lambda_\alpha(2\lambda_\alpha + 2\lambda_\nu - 1)/(2(\lambda_\alpha + \lambda_\nu))} (P_2(w))((2\lambda_\alpha + 2\lambda_\nu - 1))}{(\Gamma(\nu - \gamma_1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2))^{\lambda_\nu}},
\]
and, from (15), we find for \( k = 1, 2, \)

\[
P_k(w) = \int_{w_0}^{w} (w - t)^{(\nu - \gamma_k - 1)/(2\lambda_\nu + 2\lambda_\nu - 1)} dt.
\]

Hence, for \( k = 1, 2, \)

\[
P_k(w) = \frac{(2\lambda_\alpha + 2\lambda_\nu - 1) (w - x_0)^{(2\lambda_\alpha + 2\lambda_\nu - 2\lambda_\alpha \gamma_k - 2\lambda_\nu \gamma_k - 1)/(2\lambda_\alpha + 2\lambda_\nu - 1)}}{(2\lambda_\alpha + 2\lambda_\nu - 2\lambda_\alpha \gamma_k - 2\lambda_\nu \gamma_k - 1)},
\]

and

\[
P_k(w) = \frac{(4\lambda_\alpha^2 \nu + 6\lambda_\alpha \lambda_\nu \nu - 2\lambda_\alpha^2 \gamma_1 - 2\lambda_\nu^2 \gamma_2 - 2\lambda_\alpha \lambda_\nu \gamma_1 - 2\lambda_\nu \gamma_2 - 2\lambda_\alpha \lambda_\nu \gamma_1 - 2\lambda_\nu \gamma_2 - 2\lambda_\alpha \lambda_\nu \gamma_1 - 2\lambda_\nu \gamma_2)}{(2\lambda_\alpha + 2\lambda_\nu - 2\lambda_\alpha \gamma_k - 2\lambda_\nu \gamma_k - 1)}.
\]

Finally, we obtain

\[
A_0(x) = \sigma (w - x_0) \frac{(2\lambda_\alpha + \lambda_\nu)/(2\lambda_\alpha + \lambda_\nu)}{\int_{x_0}^{w} (w - t)^{(2\lambda_\alpha + \lambda_\nu)/2(\lambda_\alpha + \lambda_\nu)} dt}.
\]

**Corollary 8.** (In relationship to Theorem 6, \( p = 4, \lambda_\alpha = \lambda_\nu = 1, p(t) = q(t) = 1. \)) It holds

\[
\int_{x_0}^{w} \left| \left( [D_{x_0}^{\nu} f_1](w) \right) \left( [D_{x_0}^{\nu} f_2](w) \right)^2 \left( [D_{x_0}^{\nu} f_1](w) \right) \right| \right| (D_{x_0}^{\nu} f_1)(w) \right| (D_{x_0}^{\nu} f_2)(w) \right| \right| dw \leq T^*(x) \left( \int_{x_0}^{w} \left( (D_{x_0}^{\nu} f_1)(w) \right)^4 + (D_{x_0}^{\nu} f_2)(w) \right)^4 \right) \right| \right| dw,
\]

all \( x_0 \leq x \leq b. \) Here,

\[
T^*(x) = \frac{\tilde{A}_0(x)}{\sqrt{2}},
\]

and

\[
\tilde{A}_0(x) = \tilde{\sigma} \tilde{\sigma}^* (x - x_0)^{\tilde{\theta}},
\]

where

\[
\tilde{\sigma} := \frac{1}{\Gamma(\nu - \gamma_1) \Gamma(\nu - \gamma_2)} \left( \frac{3}{4\nu - 4\gamma_1 - 1} \right)^{3/4} \left( \frac{3}{4\nu - 4\gamma_2 - 1} \right)^{3/2},
\]

\[
\tilde{\sigma}^* := \left( \frac{3}{12\nu - 4\gamma_1 - 8\gamma_2} \right)^{3/4},
\]

and

\[
\tilde{\theta} := 3\nu - \gamma_1 - 2\gamma_2.
\]

**Proof.** We prove Corollary 8 by Corollary 7.

We continue with related results regarding the sup-norm \( \| \cdot \|_\infty. \)
THEOREM 7. Let $\nu, \gamma_1, \gamma_2 \geq 1$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$ and $f_1, f_2 \in C_{x_0}^\nu ([a, b])$ with $f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0$, $i = 0, 1, \ldots, n - 1$, $n := [\nu]$. Here, $x, x_0 \in [a, b] : x \geq x_0$. Consider $p(x) \geq 0$ continuous function on $[x_0, b]$. Let $\lambda_\alpha, \lambda_\beta, \lambda_\gamma \geq 0$. Set
\[
\rho (x) := \frac{(x - x_0)^{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1)}}{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1) (\Gamma (\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma (\nu - \gamma_2 + 1))^{\lambda_\beta}} \| p (x) \|_\infty. \tag{77}
\]

Then, it holds
\[
\int_{x_0}^{x} p(w) \left( \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\gamma} \right. \nonumber \\
+ \left. \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\gamma} \right) \, dw \leq \rho (x) \frac{1}{2} \left( \| D_{x_0}^{\nu} f_1 \|_{\infty}^{2(\lambda_\alpha + \lambda_\gamma)} + \| D_{x_0}^{\nu} f_1 \|_{\infty}^{2\lambda_\alpha} + \| D_{x_0}^{\nu} f_2 \|_{\infty}^{2\lambda_\beta} + \| D_{x_0}^{\nu} f_2 \|_{\infty}^{2(\lambda_\alpha + \lambda_\gamma)} \right), \tag{78}
\]
all $x_0 \leq x \leq b$.

PROOF. From (13), we get for $j = 1, 2; k = 1, 2$, that
\[
\left| (D_{x_0}^{\nu} f_j) (w) \right| \leq \frac{1}{\Gamma (\nu - \gamma_k)} \left( \int_{w - \gamma_k}^{w} (w - t)^{\nu - \gamma_k - 1} \, dt \right) \| D_{x_0}^{\nu} f_j \|_\infty \\
= \frac{1}{\Gamma (\nu - \gamma_k)} \frac{(w - x_0)^{\nu - \gamma_k}}{(\nu - \gamma_k)} \| D_{x_0}^{\nu} f_j \|_\infty.
\]
That is,
\[
\left| (D_{x_0}^{\nu} f_j) (w) \right| \leq \frac{(w - x_0)^{\nu - \gamma_k}}{\Gamma (\nu - \gamma_k + 1)} \| D_{x_0}^{\nu} f_j \|_\infty, \tag{79}
\]
all $x_0 \leq w \leq b$. Then, we have
\[
\left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\alpha} \leq \frac{(w - x_0)^{(\nu - \gamma_1) \lambda_\alpha}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda_\alpha}} \| D_{x_0}^{\nu} f_1 \|_{\infty}^{\lambda_\alpha}, \tag{80}
\]
\[
\left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\beta} \leq \frac{(w - x_0)^{(\nu - \gamma_2) \lambda_\beta}}{(\Gamma (\nu - \gamma_2 + 1))^{\lambda_\beta}} \| D_{x_0}^{\nu} f_2 \|_{\infty}^{\lambda_\beta}, \tag{81}
\]
\[
\left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\gamma} \leq \frac{(w - x_0)^{(\nu - \gamma_1) \lambda_\gamma}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda_\gamma}} \| D_{x_0}^{\nu} f_1 \|_{\infty}^{\lambda_\gamma}. \tag{82}
\]
Multiplying (80)-(82), we obtain
\[
\left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\gamma} \leq \frac{(w - x_0)^{\nu - \gamma_1 + \nu - \gamma_2 + \gamma_1 \lambda_\alpha + \gamma_2 \lambda_\beta}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma (\nu - \gamma_2 + 1))^{\lambda_\beta}} \| D_{x_0}^{\nu} f_1 \|_{\infty}^{\lambda_\alpha + \lambda_\gamma + \lambda_\gamma} \| D_{x_0}^{\nu} f_2 \|_{\infty}^{\lambda_\beta}. \tag{83}
\]
Similarly, we observe
\[
\left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\beta} \leq \frac{(w - x_0)^{(\nu - \gamma_1) \lambda_\beta}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda_\beta}} \| D_{x_0}^{\nu} f_1 \|_{\infty}^{\lambda_\beta}, \tag{84}
\]
\[
\left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\alpha} \leq \frac{(w - x_0)^{(\nu - \gamma_1) \lambda_\alpha}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda_\alpha}} \| D_{x_0}^{\nu} f_2 \|_{\infty}^{\lambda_\alpha}, \tag{85}
\]
\[
\left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\gamma} \leq \| D_{x_0}^{\nu} f_2 \|_{\infty}^{\lambda_\gamma}. \tag{86}
\]
Multiplying \((84)-(86)\), we get
\[
\left| \left(D^\alpha f_1 \right)(w) \right|^\lambda \left| \left(D^\beta f_2 \right)(w) \right|^\mu \leq \frac{(w - x_0)^{\nu (\lambda \beta - \gamma_1 \lambda_\alpha + \nu_\beta \lambda_\alpha - \gamma_2 \lambda_\beta)}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda \beta} (\Gamma (\nu - \gamma_2 + 1))^{\lambda \beta}} \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \left(\lambda_\alpha + \lambda_\beta \right).
\]
(87)

Adding (83) and (87), we have
\[
\left| \left(D^\alpha f_1 \right)(w) \right|^\lambda \left| \left(D^\beta f_2 \right)(w) \right|^\mu \leq \frac{(w - x_0)^{\nu (\lambda \beta - \gamma_1 \lambda_\alpha + \nu_\beta \lambda_\alpha - \gamma_2 \lambda_\beta)}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda \beta} (\Gamma (\nu - \gamma_2 + 1))^{\lambda \beta}} \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \lambda_\alpha + \lambda_\beta \right).
\]
(88)

It follows that
\[
\int_{x_0}^{x} p(w) \left| \left(D^\alpha f_1 \right)(w) \right|^\lambda \left| \left(D^\beta f_2 \right)(w) \right|^\mu \leq \frac{(w - x_0)^{\nu (\lambda \beta - \gamma_1 \lambda_\alpha + \nu_\beta \lambda_\alpha - \gamma_2 \lambda_\beta)}}{(\Gamma (\nu - \gamma_1 + 1))^{\lambda \beta} (\Gamma (\nu - \gamma_2 + 1))^{\lambda \beta}} \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \lambda_\alpha + \lambda_\beta \right) \int_{x_0}^{x} dw.
\]
(89)

We have established (78).

Some special cases to the last theorem follow.

**Theorem 8.** (Assume, as in Theorem 7, \(\lambda_\beta = 0\).) It holds
\[
\int_{x_0}^{x} p(w) \left( \left| \left(D^\alpha f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D^\beta f_2 \right)(w) \right|^{\lambda_\beta} \left| \left(D^\alpha f_1 \right)(w) \right|^{\lambda_\mu} \right) dw \leq \frac{(x - x_0)^{\nu (\lambda \beta - \gamma_1 \lambda_\alpha + 1)}}{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + 1)} \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \right) \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \right) \lambda_\alpha + \lambda_\beta \]
(90)

all \(x_0 \leq x \leq b\).

**Proof.** The proof of Theorem 8 is similar to the proof for Theorem 7.

**Theorem 9.** (In relationship to Theorem 7, \(\lambda_\beta = \lambda_\alpha + \mu_\beta \).) It holds
\[
\int_{x_0}^{x} p(w) \left( \left| \left(D^\alpha f_1 \right)(w) \right|^{\lambda_\alpha} \left| \left(D^\beta f_2 \right)(w) \right|^{\lambda_\beta + \lambda_\mu} \left| \left(D^\alpha f_1 \right)(w) \right|^{\lambda_\mu} \right) dw \leq \frac{(x - x_0)^{2(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + 1)}}{(2 \nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\alpha + 1)} \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \right) \left( \left\| D^\alpha f_1 \right\|_\infty \left\| D^\alpha f_2 \right\|_\infty \right) \lambda_\alpha + \lambda_\beta \]
(90)

all \(x_0 \leq x \leq b\).

**Proof.** The proof of Theorem 9 is similar to the proof for Theorem 7.
THEOREM 10. (In relationship to Theorem 7, \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta. \)) It holds

\[
\int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^{\lambda_\beta} + \left| (D_{x_0}^{\gamma_2} f_1)(w) \right|^{\lambda_\alpha} \right] \, dw \\
\leq \rho^*(x) \left[ \left\| D_{x_0}^{\nu} f_1 \right\|_{\infty}^{2\lambda_\alpha} + \left\| D_{x_0}^{\nu} f_2 \right\|_{\infty}^{2\lambda_\beta} \right], \quad \text{all } x_0 \leq x \leq b.
\]

Here,

\[
\rho^*(x) := \frac{(x - x_0)^{2(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha - \gamma_2 \lambda_\beta + 1)} \cdot \| p(x) \|_{\infty}}{(2\nu \lambda_\alpha - \gamma_1 \lambda_\alpha - \gamma_2 \lambda_\beta + 1) (\Gamma (\nu - \gamma_1 + 1))^\lambda_\alpha (\Gamma (\nu - \gamma_2 + 1))^\lambda_\beta}.
\]

PROOF. The proof of Theorem 10 is similar to the proof for Theorem 7, when \( \lambda_\nu = 0 \), we follow the proof and we obtain

\[
\int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^{\lambda_\beta} + \left| (D_{x_0}^{\gamma_2} f_1)(w) \right|^{\lambda_\alpha} \right] \, dw \\
\leq \rho(x) \left[ \left\| D_{x_0}^{\nu} f_1 \right\|_{\infty}^{\lambda_\alpha} \left\| D_{x_0}^{\nu} f_2 \right\|_{\infty}^{\lambda_\beta} \right],
\]

all \( x_0 \leq x \leq b \), which inequality is, by itself, of interest. Then, setting \( \lambda_\alpha = \lambda_\beta \) into (93) and (77), we derive (91).

THEOREM 11. (In relationship to Theorem 7, \( \lambda_\alpha = 0, \lambda_\beta = \lambda_\nu \).) It holds

\[
\int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\beta} + \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \right] \, dw \\
\leq \left( \frac{(x - x_0)^{(\nu \lambda_\beta - \gamma_1 \lambda_\beta - \gamma_2 \lambda_\alpha + 1)} \cdot \| p(x) \|_{\infty}}{(\nu \lambda_\beta - \gamma_2 \lambda_\alpha + 1) (\Gamma (\nu - \gamma_2 + 1))^\lambda_\beta} \right) \cdot \left[ \left\| (D_{x_0}^{\nu} f_1) \right\|_{\infty}^{\lambda_\alpha} + \left\| (D_{x_0}^{\nu} f_2) \right\|_{\infty}^{2\lambda_\beta} \right],
\]

all \( x_0 \leq x \leq b \).

PROOF. Again, we follow the proof of Theorem 7.

COROLLARY 9. (In relationship to Theorem 10 and, all as in Theorem 7, \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta, \gamma_2 = \gamma_1 + 1 \).) It holds

\[
\int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1+1} f_2)(w) \right|^{\lambda_\beta} + \left| (D_{x_0}^{\gamma_1+1} f_1)(w) \right|^{\lambda_\alpha} \right] \, dw \\
\leq \left( \frac{(x - x_0)^{(2\nu \lambda_\nu - 2\gamma_1 \lambda_\nu - \lambda_\alpha + 1)} \cdot \| p(x) \|_{\infty}}{(2\nu \lambda_\nu - 2\gamma_1 \lambda_\nu - \lambda_\alpha + 1) (\Gamma (\nu - \gamma_1))^\lambda_\alpha} \right) \cdot \left[ \left\| D_{x_0}^{\nu} f_1 \right\|_{\infty}^{\lambda_\alpha} + \left\| D_{x_0}^{\nu} f_2 \right\|_{\infty}^{2\lambda_\beta} \right], \quad \text{all } x_0 \leq x \leq b.
\]

PROOF. The proof of this corollary is obvious.

COROLLARY 10. (In relationship to Corollary 9.) In detail, let \( \nu, \gamma_1 \geq 1 \), such that \( \nu - \gamma_1 \geq 2 \) and \( f_1, f_2 \in \mathbb{C}_{\nu}^\nu ([a, b]) \) with \( f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, i = 0, 1, \ldots, n - 1 \), \( n := [\nu] \). Here, \( x, x_0 \in [a, b] \) : \( x \geq x_0 \). Then,

\[
\int_{x_0}^{x} \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right| \left| (D_{x_0}^{\gamma_1+1} f_2)(w) \right| + \left| (D_{x_0}^{\gamma_1+1} f_1)(w) \right| \left| (D_{x_0}^{\gamma_2} f_2)(w) \right| \right] \, dw \\
\leq \left( \frac{(x - x_0)^{(2\nu - \gamma_1)}}{(2 (\nu - \gamma_1))^2 (\Gamma (\nu - \gamma_1))^2} \right) \cdot \left[ \left\| D_{x_0}^{\nu} f_1 \right\|_{\infty}^2 + \left\| D_{x_0}^{\nu} f_2 \right\|_{\infty}^2 \right], \quad \text{all } x_0 \leq x \leq b.
\]

PROOF. The proof of this corollary is obvious.
PROPOSITION 1. Inequality (96) is sharp, in fact, it is attained when $f_1 = f_2$.

PROOF. Clearly, (96) collapses to

$$\int_{0}^{x} |(D_{a+}^\gamma f_1)(w)| \left| (D_{a+}^{\gamma+1} f_1)(w) \right| dw \leq \left( \frac{(x - x_0)^{2(\nu - \gamma)}}{2(\nu - \gamma)(\Gamma(\nu - \gamma))^2} \right) \|D_{a+}^\gamma f_1\|_\infty^2,$$

for all $x_0 \leq x \leq b$.

In [2], see Propositions 9 and 10. We proved that (97) is sharp, in fact, it is attained.

4. APPLICATIONS

We present a uniqueness of solution result for a system of fractional differential equations.

THEOREM 12. Let $\nu, \gamma_i \geq 1, \nu - \gamma_i \geq 1, i = 1, \ldots, r \in \mathbb{N}$, $n := [\nu]$, $f_j \in C_n^\nu([a, b])$, $j = 1, 2$; $f_j^{(i)}(a) = a_{ij} \in \mathbb{R}, i = 0, 1, \ldots, n - 1$. Furthermore, we have, for $j = 1, 2$ that

$$\langle D_{a+}^\nu f_j \rangle(t) = F_j(t, \{(D_{a}^\nu f_1)(t)\}_{i=1}^r, \{(D_{a}^\nu f_2)(t)\}_{i=1}^r), \quad \text{all } t \in [a, b].$$

Here, $F_j$ are continuous functions on $[a, b] \times \mathbb{R}^r \times \mathbb{R}^r$, and satisfy the Lipschitz condition,

$$|F_j(t, z_1, \ldots, z_r, y_1, \ldots, y_r) - F_j(t, z_1', \ldots, z_r', y_1, \ldots, y_r')| \leq \sum_{i=1}^{r} \left[ q_{1,i,j}(t) |z_i - z_i'| + q_{2,i,j}(t) |y_i - y_i'| \right],$$

where $q_{1,i,j}(t), q_{2,i,j}(t) \geq 0, 1 \leq i \leq r$ are continuous functions over $[a, b]$. Call

$$M_{1,i} := \max \left( \|q_{1,i,1}\|_\infty, \|q_{2,i,2}\|_\infty \right) \quad \text{and} \quad M_{2,i} := \max \left( \|q_{2,i,1}\|_\infty, \|q_{1,i,2}\|_\infty \right).$$

Assume here that

$$\phi^*(b) := \sum_{i=1}^{r} \left[ \frac{M_{1,i}}{2} + \frac{M_{2,i}}{\sqrt{2}} \right] \left( \frac{(b - a)^{\nu - \gamma}}{\Gamma(\nu - \gamma) \sqrt{2^{\nu - \gamma} - 2\gamma - 1}} \right) < 1.$$  

Then, if system (98) has two pairs of solutions $(f_1, f_2)$ and $(f_1^*, f_2^*)$, we prove that $f_j = f_j^*, j = 1, 2$, that is, we have uniqueness of solution.

PROOF. Assume there are two pairs of solutions $(f_1, f_2)$, $(f_1^*, f_2^*)$ to system (98). Set $g_j := f_j - f_j^*, j = 1, 2$. Then, $g_j^{(i)} = f_j^{(i)} - f_j^{*(i)}$ and $g_j^{(i)}(a) = 0, i = 0, 1, \ldots, n - 1; j = 1, 2$. It holds

$$\langle D_{a+}^\nu g_j \rangle(t) = F_j(t, \{(D_{a}^\nu f_1)(t)\}_{i=1}^r, \{(D_{a}^\nu f_2)(t)\}_{i=1}^r) - F_j(t, \{(D_{a}^\nu f_1^*)(t)\}_{i=1}^r, \{(D_{a}^\nu f_2^*)(t)\}_{i=1}^r).$$

Hence, by (99), we have

$$|\langle D_{a+}^\nu g_j \rangle(t)\| \leq \sum_{i=1}^{r} \left[ |q_{1,i,j}(t)| \langle D_{a+}^\nu g_1 \rangle(t)\| + |q_{2,i,j}(t)| \langle D_{a+}^\nu g_2 \rangle(t)\| \right].$$

Thus,

$$|\langle D_{a+}^\nu g_j \rangle(t)\| \leq \sum_{i=1}^{r} \left[ \|q_{1,i,j}\|_\infty \langle D_{a+}^\nu g_1 \rangle(t)\| + \|q_{2,i,j}\|_\infty \langle D_{a+}^\nu g_2 \rangle(t)\| \right].$$

The last implies that

$$|\langle D_{a+}^\nu g_j \rangle(t)\|^2 \leq \sum_{i=1}^{r} \left[ \|q_{1,i,j}\|_\infty |\langle D_{a+}^\nu g_1 \rangle(t)\| \langle D_{a+}^\nu g_1 \rangle(t)\| + \|q_{2,i,j}\|_\infty |\langle D_{a+}^\nu g_2 \rangle(t)\| \langle D_{a+}^\nu g_2 \rangle(t)\| \right].$$
Consequently, we observe
\[I := \int_a^b \left( (D^\nu g_1)(t))^2 + (D^\nu g_2)(t))^2 \right) \, dt\]
\[\leq \sum_{i=1}^r \left[ \|q_{1,i,1}\|_\infty \int_a^b (D^\nu g_1)(t) \, dt + \|q_{2,i,1}\|_\infty \int_a^b (D^\nu g_2)(t) \, dt \right] + \sum_{i=1}^r \left[ \|q_{1,i,2}\|_\infty \int_a^b (D^\nu g_2)(t) \, dt + \|q_{2,i,2}\|_\infty \int_a^b (D^\nu g_2)(t) \, dt \right]\]
\[\leq \sum_{i=1}^r M_{1,i} \int_a^b \left[ (D^\nu g_1)(t) \right] \left[ (D^\nu g_2)(t) \right] \, dt \]
\[+ \sum_{i=1}^r M_{2,i} \int_a^b \left[ (D^\nu g_2)(t) \right] \left[ (D^\nu g_1)(t) \right] \, dt =: (*) .\]

However, by Corollary 2, we obtain
\[\int_a^b \left[ (D^\nu g_1)(t) \right] \left[ (D^\nu g_1)(t) \right] \, dt \leq \left( \frac{(b-a)^{\nu-\gamma}}{2 \Gamma (\nu-\gamma) \sqrt{\nu-\gamma} \sqrt{2\nu-2\gamma-1}} \right) . I. \] (102)

Also, by Corollary 4, we find
\[\int_a^b \left[ (D^\nu g_2)(t) \right] \left[ (D^\nu g_2)(t) \right] \, dt \leq \left( \frac{(b-a)^{\nu-\gamma}}{\sqrt{2 \Gamma (\nu-\gamma) \sqrt{\nu-\gamma} \sqrt{2\nu-2\gamma-1}}} \right) . I. \] (103)

Therefore, by (102),(103), we get
\[(*) \leq \sum_{i=1}^r \left[ M_{1,i} \left( \frac{(b-a)^{\nu-\gamma}}{2 \Gamma (\nu-\gamma) \sqrt{\nu-\gamma} \sqrt{2\nu-2\gamma-1}} \right) . I + M_{2,i} \left( \frac{(b-a)^{\nu-\gamma}}{\sqrt{2 \Gamma (\nu-\gamma) \sqrt{\nu-\gamma} \sqrt{2\nu-2\gamma-1}}} \right) . I \right] = \phi^*(b) . I. \]

We have established that
\[I \leq \phi^*(b) . I. \] (104)

If \(I \neq 0\), then, \(\phi^*(b) \geq 1\), a contradiction by assumption (101), that \(\phi^*(b) \leq 1\). Therefore, \(I = 0\), implying that
\[((D^\nu g_1)(t))^2 + (D^\nu g_2)(t))^2 = 0, \quad \text{a.e. in } [a,b] .\]

I.e.,
\[((D^\nu g_1)(t))^2 = 0, \quad (D^\nu g_2)(t))^2 = 0, \quad \text{a.e. in } [a,b] .\]

That is,
\[(D^\nu g_1)(t) = 0, \quad (D^\nu g_2)(t) = 0, \quad \text{a.e. in } [a,b] .\]

But, for \(j = 1, 2\), we got that
\[g_j^{(i)}(a) = 0, \quad 0 \leq i \leq n-1 .\]

Hence, from fractional Taylor's Theorem 1, see (7), we find that \(g_j(t) = 0\) on \([a,b]\). That is, \(f_j \equiv f_j^*, j = 1, 2\), proving the uniqueness claim of the theorem.

Next, we give upper bounds on \(D^\nu a f_j\), solutions \(f_j\), etc., all involved in a system of fractional differential equations.
THEOREM 13. Let $\nu, \gamma_i \geq 1, \nu - \gamma_i \geq 1, i = 1, \ldots, r \in \mathbb{N}$, $n := [\nu]$, $f_j \in C_r^\nu([a,b])$, $j = 1, 2$; $f_j^{(i)}(a) = 0, i = 0, 1, \ldots, n-1$, and

$$(D^\nu f_j)(a) = A_j \in \mathbb{R}.$$ 

Furthermore, for $a \leq t \leq b$, we have, holding the system of fractional differential equations,

$$(D^\nu f_j)'(t) = F_j(t, \{(D^\nu f_j)(t)\}_{i=1}^r, (D^\nu f_j)(t), \{(D^\nu f_2)(t)\}_{i=1}^r, (D^\nu f_2)(t)), \quad (105)$$

for $j = 1, 2$. Here, $F_j$ are continuous functions on $[a, b] \times \mathbb{R}^{r+1} \times \mathbb{R}^{r+1}$, such that

$$(|F_2(t,x_1,x_2,\ldots,x_r,y_1,\ldots,y_r+1)| \leq \sum_{i=1}^r |q_{1,i,j}(t)|x_i| + |q_{2,i,j}(t)|y_i|, \quad (106)$$

where $q_{1,i,j}(t), q_{2,i,j}(t) \geq 0, 1 \leq i \leq r, j = 1, 2$, are continuous functions on $[a, b]$. Call

$$M_{1,i} := \max(\|q_{1,i,1}\|_\infty, \|q_{2,i,2}\|_\infty) \quad \text{and} \quad M_{2,i} := \max(\|q_{2,i,1}\|_\infty, \|q_{1,i,2}\|_\infty). \quad (107)$$

Also, we set ($a \leq x \leq b$)

$$\theta(x) := (\|D^\nu f_1\|(x))^2 + (\|D^\nu f_2\|(x))^2, \quad (108)$$

$$\rho := A_1^2 + A_2^2, \quad (109)$$

$$Q(x) := \sum_{i=1}^r \left[ M_{1,i} + \sqrt{2} M_{2,i} \right] \cdot \left( \frac{(x-a)^{\nu-\gamma_i}}{\Gamma(\nu-\gamma_i)\sqrt{\nu-\gamma_i}2^\nu-2\gamma_i-1} \right), \quad (110)$$

and

$$\chi(x) := \sqrt{\rho} \cdot \left\{ 1 + Q(x) \cdot e^{\int_a^x Q(s)ds} \int_a^x (e^{-\int_a^t Q(s)ds})^t dt \right\}^{1/2}. \quad (111)$$

Then,

$$\sqrt{\theta(x)} \leq \chi(x), \quad a \leq x \leq b. \quad (112)$$

Consequently, it holds

$$|D^\nu f_j(x)| \leq \chi(x), \quad (113)$$

$$|f_j(x)| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \chi(t) \, dt, \quad (114)$$

for $j = 1, 2, a \leq x \leq b$. Also, it holds that

$$|D^\nu f_j(x)| \leq \frac{1}{\Gamma(\nu-\gamma_i)} \int_a^x (x-t)^{\nu-\gamma_i-1} \chi(t) \, dt, \quad (115)$$

for $j = 1, 2; i = 1, \ldots, r, a \leq x \leq b$.

PROOF. We observe that

$$(D^\nu f_j(t) \cdot (D^\nu f_j)'(t) = (D^\nu f_j)(t) \cdot F_j(t, \{(D^\nu f_j)(t)\}_{i=1}^r, (D^\nu f_2)(t), \{(D^\nu f_2)(t)\}_{i=1}^r, (D^\nu f_2)(t)), \quad \text{all } a \leq x \leq b.$$
Then, we integrate the last equality,
\[
\int_a^x (D^\nu f_j) (t) \cdot ((D^\nu f_j)' (t)) \ dt = \int_a^x (D^\nu f_j) (t) \cdot F_j (t, \{(D^\nu f_i) (t)\}_{i=1}^r, \{(D^\nu f_i) (t)\}_{i=1}^r, \{(D^\nu f_i) (t)\}_{i=1}^r, \{(D^\nu f_i) (t)\}_{i=1}^r) \ dt.
\]
Hence, we get
\[
\left( \frac{(D^\nu f_j) (x)}{2} \right)^2 \leq \int_a^x |(D^\nu f_j) (t)| |F_j| \ dt
\]
\[
\leq \int_a^x |(D^\nu f_j) (t)| \left[ \sum_{i=1}^r \{ q_{1,j} (t) |(D^\nu f_i) (t)| + q_{2,j} (t) |(D^\nu f_i) (t)| \} \right] \ dt
\]
\[
\leq \sum_{i=1}^r \{ \| q_{1,j} \|_\infty \int_a^x |(D^\nu f_j) (t)| |(D^\nu f_i) (t)| \ dt
\]
\[
+ \| q_{2,j} \|_\infty \int_a^x |(D^\nu f_j) (t)| |(D^\nu f_i) (t)| \ dt \}.
\]
Thus, we obtain
\[
((D^\nu f_j) (x))^2 \leq A_j^2 + 2 \sum_{i=1}^r \left\{ \| q_{1,i,j} \|_\infty \cdot \int_a^x |(D^\nu f_j) (t)| |(D^\nu f_i) (t)| \ dt
\]
\[
+ \| q_{2,i,j} \|_\infty \cdot \int_a^x |(D^\nu f_j) (t)| |(D^\nu f_i) (t)| \ dt \}.
\]
Consequently, we write
\[
\theta (x) \leq \rho + 2 \sum_{i=1}^r \left\{ M_{1,i} \int_a^x \| (D^\nu f_i) (t) \| \|(D^\nu f_i) (t)\| \ dt
\]
\[
+ M_{2,i} \int_a^x \left\{ (D^\nu f_i) (t) + (D^\nu f_i) (t) \right\} \ dt \left\{ \frac{1}{2} \left( \frac{a - t}{(x - a)} \right)^{\nu - \gamma_i} \right\} \ dt \}
\]
\[
\leq \rho + 2 \sum_{i=1}^r \left\{ M_{1,i} \left( \frac{a - t}{(x - a)} \right)^{\nu - \gamma_i} \right\} \left( \int_a^x \theta (t) \ dt \right)
\]
\[
+ M_{2,i} \left( \frac{a - t}{(x - a)} \right)^{\nu - \gamma_i} \left( \int_a^x \theta (t) \ dt \right)
\]
\[
= \rho + Q (x) \int_a^x \theta (t) \ dt.
\]
That is, we get
\[
\theta (x) \leq \rho + Q (x) \int_a^x \theta (t) \ dt,
\]
(116)
all \( a \leq x \leq b \). Here, \( \rho \geq 0, Q (x) \geq 0, Q (a) = 0, \theta (x) \geq 0, \) all \( a \leq x \leq b \). Solving the integral inequality (116), exactly as in [5, pp. 224,225, Application 3.2 and inequalities (44),(47)], we find that
\[
\sqrt{\theta (x)} \leq \chi (x), \quad a \leq x \leq b.
\]
(112)
Then, (113) is obvious.

Next, from (9), we have
\[
|f_j (x)| \leq \frac{1}{\Gamma (\nu)} \int_a^x (x - t)^{\nu - 1} |D^\nu f_j (t)| \ dt \leq \frac{1}{\Gamma (\nu)} \int_a^x (x - t)^{\nu - 1} \chi (t) \ dt,
\]
all \( a \leq x \leq b, j = 1, 2 \), proving (114).
Finally, from (13), we find
\[
\frac{1}{\Gamma(\nu - \gamma_i)} \int_a^x (x - t)^{\nu - \gamma_i - 1} |(D_\alpha^\nu f_j) (t)| \, dt
\]
\[
\leq \frac{1}{\Gamma(\nu - \gamma_i)} \int_a^x (x - t)^{\nu - \gamma_i - 1} \chi(t) \, dt,
\]
all \(a \leq x \leq b, j = 1, 2, i = 1, \ldots, r\), proving (115). \(\square\)

In the final application, we give similar upper bounds to Theorem 13, but under different conditions.

**THEOREM 14.** Let \(a \neq b, \nu \geq 3, \gamma_i \geq 1, \nu - \gamma_i \geq 1, i = 1, \ldots, r \in \mathbb{N}, n := [\nu], f_j \in C_\alpha^\nu([a, b]), j = 1, 2; f_j^{(1)}(a) = 0, i = 0, 1, \ldots, n - 1, and \)
\[
(D_\alpha^\nu f_j)(a) = A_j \in \mathbb{R}.
\]
Furthermore, for \(a \leq t \leq b\), we have, holding the system of fractional differential equations,
\[
(D_\alpha^\nu f_j)'(t) = F_j(t, \{(D_\alpha^\nu f_1)(t)\}_{i=1}^r, \{(D_\alpha^\nu f_2)(t)\}_{i=1}^r), \quad \text{for } j = 1, 2.
\]
For fixed \(i \in \{1, \ldots, r\}\), we assume that \(\gamma_{i+1} = \gamma_i + 1\), and \(\nu - \gamma_i \geq 2\), where \(\gamma_i, \gamma_{i+1} \in \{\gamma_i, \ldots, \gamma_r\}\). Call \(k := \gamma_i, \gamma := \gamma_i + 1\), i.e., \(\gamma = k + 1\). Here, \(F_j\) are continuous functions on \([a, b] \times \mathbb{R}^{r+1} \times \mathbb{R}^{r+1}\), such that
\[
|F_j(t, x_1, \ldots, x_r, y_1, \ldots, y_r)| \leq q_{1,j}(t) |x_i| + q_{2,j}(t) |y_i|, \quad i = 1, \ldots, r+1,
\]
where
\[
q_{1,j}, q_{2,j} \neq 0, \quad q_{1,j}(t), q_{2,j}(t) \geq 0
\]
continuous functions over \([a, b]\). Put
\[
M := \max \left(\|q_{1,1}\|_\infty, \|q_{2,1}\|_\infty, \|q_{1,2}\|_\infty, \|q_{2,2}\|_\infty\), \quad \theta(x) := |(D_\alpha^\nu f_1)(x)| + |(D_\alpha^\nu f_2)(x)|, \quad a \leq x \leq b,
\]
\[
\rho := |A_1| + |A_2|.
\]
Also, we define
\[
Q(x) := 2M \cdot \left(\sqrt{\frac{2}{3\nu - 3k - 2}}\right)^{1/2} \cdot \frac{(x - a)^{(3\nu - 3k - 1)/2}}{\Gamma(\nu - k)^{3/2}},
\]
\[
\sigma := \|Q(x)\|_\infty, \quad a \leq x \leq b.
\]
We assume that
\[
(b - a) \sigma \sqrt{\rho} < 2.
\]
Call
\[
\tilde{\varphi}(x) := \frac{(x - a) \cdot (\sigma - Q(x)) \cdot \left[\sigma \rho^2 (x - a) - 4 \rho^{3/2}\right] + 4 \rho}{2 - \sigma \sqrt{\rho} (x - a)^{3/2}},
\]
all \(a \leq x \leq b\). Then,
\[
\theta(x) \leq \tilde{\varphi}(x),
\]
in particular, it holds

\[ |(D_a^\nu f_j)(x)| \leq \varphi(x), \quad j = 1, 2, \]  

all \( a \leq x \leq b \). Furthermore, we find

\[ |f_j(x)| \leq \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \varphi(t) \, dt, \]

\[ |(D_a^\nu f_j)(x)| \leq \frac{1}{\Gamma(\nu-\gamma_i)} \int_a^x (x-t)^{\nu-\gamma_i-1} \varphi(t) \, dt, \]

\( j = 1, 2, i = 1, \ldots, r \), all \( a \leq x \leq b \).

**Proof.** For \( a \leq x \leq b \), we get

\[ \int_a^x (D_a^\nu f_j)'(t) \, dt = \int_a^x f_j(t, (D_a^\nu f_1)(t))_{i=1}^r, (D_a^\nu f_1)(t), ((D_a^\nu f_2)(t))_{i=1}^r, (D_a^\nu f_2)(t)) \, dt. \]

That is,

\[ (D_a^\nu f_j)(x) = A_j + \int_a^x F_j(t, \ldots) \, dt. \]

Then, we observe

\[ |(D_a^\nu f_j)(x)| \leq |A_j| + \int_a^x |F_j(t, \ldots)| \, dt \]

\[ \leq |A_j| + \int_a^x q_{1,j}(t) |(D_a^k f_1)(t)| \sqrt{|(D_a^{k+1} f_2)(t)|} \]

\[ + q_{2,j}(t) |(D_a^k f_2)(t)| \sqrt{|(D_a^{k+1} f_1)(t)|} \, dt \]

\[ \leq |A_j| + M \left( \int_a^x |(D_a^k f_1)(t)| \sqrt{|(D_a^{k+1} f_2)(t)|} \right. \]

\[ \left. + |(D_a^k f_2)(t)| \sqrt{|(D_a^{k+1} f_1)(t)|} \right) \, dt. \]

Hence,

\[ \theta(x) \leq \rho + 2M \left( \int_a^x \left[ |(D_a^k f_1)(t)| \sqrt{|(D_a^{k+1} f_2)(t)|} + |(D_a^k f_2)(t)| \sqrt{|(D_a^{k+1} f_1)(t)|} \right] \, dt \right) \]

\[ \leq \rho + 2M \cdot \sqrt{\frac{2}{3\nu - 3k - 2}} \cdot \frac{(x-a)^{(3\nu-3k-1)/2}}{(\Gamma(\nu-k))^{3/2}} \]

\[ \cdot \left( \int_a^x |(D_a^k f_1)(t)|^{3/2} + |(D_a^k f_2)(t)|^{3/2} \, dt \right) \]

\[ \leq \rho + 2M \cdot \sqrt{\frac{2}{3\nu - 3k - 2}} \cdot \frac{(x-a)^{(3\nu-3k-1)/2}}{(\Gamma(\nu-k))^{3/2}} \]

\[ \cdot \left( \int_a^x |(D_a^k f_1)(t)| + |(D_a^k f_2)(t)| \right)^{3/2} \, dt \].

We have proved that

\[ \theta(x) \leq \rho + Q(x) \cdot \int_a^x (\theta(t))^{3/2} \, dt, \]

all \( a \leq x \leq b \). Notice that \( \theta(x) \geq 0, \rho \geq 0, Q(x) \geq 0, \) and \( Q(a) = 0 \), also, it is \( \sigma > 0 \).

Call

\[ 0 \leq w(x) := \int_a^x (\theta(t))^{3/2} \, dt, \quad w(a) = 0, \quad \text{all} \ a \leq x \leq b. \]
That is,
\[ w'(x) = (\theta(x))^{3/2} \geq 0 \]
and
\[ \theta(x) = (w'(x))^{2/3}, \quad \text{all } a \leq x \leq b. \]  
(132)
Hence, we rewrite (130) as
\[ (w'(x))^{2/3} \leq \rho + Q(x)w(x), \quad a \leq x \leq b, \]  
(133)
so that
\[ (w'(x))^{2/3} \leq \rho + \sigma w(x) < \rho + \varepsilon + \sigma w(x), \]
for \( \varepsilon > 0 \), arbitrarily small. Hence,
\[ w'(x) < (\rho + \varepsilon + \sigma w(x))^{3/2}, \quad a \leq x \leq b. \]
Here \( (\rho + \varepsilon + \sigma w(x))^{3/2} > 0 \). In particular, it holds
\[ w'(t) < (\rho + \varepsilon + \sigma w(t))^{3/2}, \quad a \leq t \leq x, \]
and
\[ \frac{w'(t)}{(\rho + \varepsilon + \sigma w(t))^{3/2}} < 1. \]  
(134)
The last is same as
\[ \left( \frac{2}{\sigma} (\rho + \varepsilon + \sigma w(t))^{-1/2} \right)' < 1. \]
Therefore, after integration, we have
\[ -\int_a^x d \left( (\rho + \varepsilon + \sigma w(t))^{-1/2} \right) \leq \frac{\sigma}{2} (x - a) \]
and
\[ (\rho + \varepsilon + \sigma w(t))^{-1/2} \bigg|_a^x \leq \frac{\sigma}{2} (x - a). \]
That is,
\[ (\rho + \varepsilon)^{-1/2} - (\rho + \varepsilon + \sigma w(x))^{-1/2} \leq \frac{\sigma}{2} (x - a) \]
and
\[ (\rho + \varepsilon)^{-1/2} - \frac{\sigma}{2} (x - a) \leq (\rho + \varepsilon + \sigma w(x))^{-1/2}, \]
i.e.,
\[ \frac{2 - \sigma (\rho + \varepsilon)^{1/2} (x - a)}{2 (\rho + \varepsilon)^{1/2}} \leq \frac{1}{(\rho + \varepsilon + \sigma w(x))^{1/2}}. \]  
(135)
By assumption (124), we get
\[ (x - a) \sigma \sqrt{\rho} < 2, \quad \text{all } a \leq x \leq b. \]
Then, for sufficiently small \( \varepsilon > 0 \), we still have
\[ (x - a) \sigma (\rho + \varepsilon)^{1/2} < 2. \]
That is,
\[ 2 - (x - a) \sigma (\rho + \varepsilon)^{1/2} > 0, \quad \text{all } a \leq x \leq b. \]  
(136)
From (135) and (136), we get

\[(\rho + \varepsilon + \sigma w(x))^{1/2} \leq \frac{2(\rho + \varepsilon)^{1/2}}{2 - \sigma(\rho + \varepsilon)^{1/2}(x - a)}, \quad \text{all } a < x < b. \tag{137}\]

Solving (137) for \(w(x)\), we find

\[w(x) \leq \left(\frac{\rho + \varepsilon}{\sigma}\right) \left[\frac{4}{(2 - \sigma(\rho + \varepsilon)(x - a))^2} - 1\right], \tag{138}\]

for \(\varepsilon > 0\), sufficiently small and, for all \(a < x < b\).

Letting \(\varepsilon \to 0\), we obtain

\[w(x) \leq \left(\frac{\rho}{\sigma}\right) \left[\frac{4}{(2 - \sigma\rho(x - a))^2} - 1\right], \tag{139}\]

for all \(a < x < b\).

That is,

\[w(x) \leq \frac{4\rho^{3/2}(x - a) - \sigma\rho^2(x - a)^2}{(2 - \sigma\rho(x - a))^2}, \tag{140}\]

for all \(a < x < b\). Then, by (130) and (140), we find

\[\theta(x) \leq \rho + Q(x) \left[\frac{4\rho^{3/2}(x - a) - \sigma\rho^2(x - a)^2}{(2 - \sigma\rho(x - a))^2}\right] = \phi(x), \tag{141}\]

for all \(a < x < b\). That is, (141) implies (126). Then, by (120),(126), inequality (127) is obvious.

Finally, by (9),(13) and (127), the inequalities (128),(129) are clear, respectively.

REFERENCES