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Paraholomorphic B-manifold and its properties ☆

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Abstract

This paper is concerned with problem of the geometry of B-manifolds. We give some properties of Riemannian curvature tensors of paraholomorphic B-manifolds. Finally, we consider some examples of paraholomorphic B-manifolds.

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1. Introduction

Let M_n be a Riemannian manifold with metric g, which is not necessarily positive definite. We denote by $\Im_q^p(M_n)$ the set of all tensor fields of type (p,q) on M_n. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

An almost paracomplex manifold is an almost product manifold (M_n,φ), $\varphi^2=\mathrm{id}$, such that the two eigenbundles T^+M_n and T^-M_n associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the following set of affinors on $M_n : \{\mathrm{id},\varphi\}$, $\varphi^2=\mathrm{id}$, which form a bases of a representation of the algebra of order 2 over the field of real numbers R, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j)=\{a_0+a_1 j \mid j^2=1;\ a_0,a_1\in R\}$. Obviously, it is associative, commutative and it admits principal unit 1. The canonical bases of this algebra has the form $\{1, j\}$. Structural constants of an algebra are defined by the multiplication law of the base units of this algebra: $e_i e_j = C_{ij}^k e_k$. The components of C_{ij}^k are given by $C_{11}^1=C_{12}^2=C_{21}^2=C_{22}^1=1$, all the others being zero, with respect to the canonical bases of $R(j)$.

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Consider $R(j)$ endowed with the usual topology of R^2 and a domain U of $R(j)$. Let

$$X = x^1 + jx^2$$

be a variable in $R(j)$, where x^i are real coordinates of a point of a certain domain U for $i = 1, 2$. Using two real-valued functions $f^i(x^1, x^2)$, $i = 1, 2$, we introduce a paracomplex function

$$F = f^1 + jf^2$$

of variable X. It is said to be paraholomorphic if we have

$$dF = F'(X)\,dX$$

for the differentials $dX = dx^1 + j\,dx^2$, $dF = df^1 + j\,df^2$ and the derivative $F'(X)$. The paraholomorphy of the function $F = f^1 + jf^2$ in the variable $X = x^1 + jx^2$ is equivalent to the fact that the Jacobian matrix $D = (\partial_k f^i)$ commutes with the matrix

$$C_2 = \left(C^k_{2j}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [10, p. 87]). It follows that F is paraholomorphic if and only if f^1 and f^2 satisfy the para-Cauchy–Riemann equations:

$$\frac{\partial f^1}{\partial x^1} = \frac{\partial f^2}{\partial x^2}, \qquad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1}.$$

For almost paracomplex structure the integrability is equivalent to the vanishing of the Nijenhuis tensor

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X, Y].$$

On the other hand, in order that an almost paracomplex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection such that $\nabla\varphi = 0$. A paracomplex manifold is an almost paracomplex manifold (M_{2k}, φ) such that the G-structure defined by the affinor field φ is integrable. We can give another-equivalent-definition of paracomplex manifold in terms of local homeomorphisms in the space $R^k(j) = \{(X^1, \ldots, X^k) \mid X^i \in R(j),\ i = 1, \ldots, k\}$ and paraholomorphic changes of charts in a way similar to [3] (for more details see [10] or [4]), i.e. a manifold M_{2k} with an integrable paracomplex structure φ is a real realization of the paraholomorphic manifold $M_k(R(j))$ over the algebra $R(j)$. Let $\overset{*}{t}$ be a paracomplex tensor field on $M_k(R(j))$. The real model of such a tensor field is a tensor field on M_{2k} of the same order that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [5,8–10,12]). In particular, being applied to a $(0, q)$-tensor field ω, the purity means that for any $X_1, \ldots, X_q \in \Im^1_0(M_{2k})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \cdots = \omega(X_1, X_2, \ldots, \varphi X_q).$$

We define an operator

$$\phi_\varphi : \Im^0_q(M_{2k}) \to \Im^0_{q+1}(M_{2k})$$

applied to the pure tensor field ω by [12]

$$\begin{aligned}(\phi_\varphi\omega)(X, Y_1, Y_2, \ldots, Y_q) &= (\varphi X)\big(\omega(Y_1, Y_2, \ldots, Y_q)\big) - X\big(\omega(\varphi Y_1, Y_2, \ldots, Y_q)\big) \\ &\quad + \omega\big((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q\big) + \cdots + \omega\big(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X\big),\end{aligned} \tag{1.1}$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a paracomplex structure on M_{2k} and the tensor field $\phi_\varphi\omega$ vanishes, the paracomplex tensor field $\overset{*}{\omega}$ on $M_k(R(j))$ is said to be paraholomorphic [5]. Thus a paraholomorphic tensor field $\overset{*}{\omega}$ on $M_k(R(j))$ is realized on M_{2k} in the form of a pure tensor field ω, such that

$$(\phi_\varphi\omega)(X, Y_1, Y_2, \ldots, Y_q) = 0 \tag{1.2}$$

for any $X, Y_1, \ldots, Y_q \in \Im^1_0(M_n)$. Therefore such a tensor field ω on M_{2k} is also called paraholomorphic tensor field.

The main results of the present paper are the following ones: Almost paracomplex manifolds endowed with a paraholomorphic Riemannian metric g are the manifolds where the Levi-Civita connection of the metric parallelizes the almost paracomplex structure (Theorem 2). In that case, the Levi-Civita connection is also the Levi-Civita connection of the metric G given by $G(X,Y) = g(\varphi X, Y)$ (Theorem 5). Moreover, in such a manifold, the Riemannian curvature tensor is pure (Theorem 6).

2. Paraholomorphic B-manifold

A *pure metric* with respect to the almost paracomplex structure is a Riemannian metric g such that

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{2.1}$$

for any $X, Y \in \Im_0^1(M_n)$. Such Riemannian metrics were studied in [11], where they were said to be *B-metrics*, since the metric tensor g with respect to the structure φ is B-tensor according to the terminology accepted in [10]. If (M_{2k}, φ) is an almost paracomplex manifold with B-metric, we say that (M_{2k}, φ, g) is an *almost B-manifold*. If φ is integrable, we say that (M_{2k}, φ, g) is an *B-manifold*.

In a B-manifold a B-metric is called a *paraholomorphic* if

$$(\phi_\varphi g)(X, Y, Z) = 0.$$

If (M_{2k}, φ, g) is a B-manifold with paraholomorphic B-metric, we say that (M_{2k}, φ, g) is a *paraholomorphic B-manifold* (in some papers, (almost) B-manifolds are simply (almost) paracomplex manifolds, as in [4]). Now we establish a formula for the B-metric for an almost B-manifold.

Theorem 1. *Let g be a B-metric of almost B-manifold. Then*

$$g\big(Z, (\nabla_Y \varphi)(X)\big) = g\big((\nabla_Y \varphi)(Z), X\big),$$

where ∇ denotes the operator of the Riemannian covariant derivative with respect to g.

Proof. By virtue of (2.1) and

$$Yg(Z, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X),$$

from $Yg(\varphi Z, X) = Yg(Z, \varphi X)$, we have

$$g(\nabla_Y \varphi Z, X) + g(\varphi Z, \nabla_Y X) = g(\nabla_Y Z, \varphi X) + g(Z, \nabla_Y \varphi X)$$

or

$$g(\varphi Z, \nabla_Y X) - g(Z, \nabla_Y \varphi X) = g(\nabla_Y Z, \varphi X) - g(\nabla_Y \varphi Z, X)$$

and consequently

$$g\big(Z, \varphi(\nabla_Y X) - \nabla_Y \varphi X\big) = g\big(\varphi(\nabla_Y Z) - \nabla_Y \varphi Z, X\big)$$

from which by virtue of formula

$$(\nabla_Y \varphi)X = (\nabla \varphi)(X, Y) = \nabla_Y \varphi X - \varphi(\nabla_Y X) \tag{2.2}$$

we see that the proof is completed. □

In some aspects, paraholomorphic B-manifolds are similar to Kahler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kahler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 2. *An almost B-manifold is paraholomorphic B-manifold if and only if the almost paracomplex structure is parallel with respect to the Levi-Civita connection ∇.*

Proof. Putting $(g\circ\varphi)(X,Y)=g(\varphi X,Y)$, we get from (1.1)

$$\begin{aligned}(\phi_\varphi g)(X,Z_1,Z_2) &= \big(L_{\varphi X}g - L_X(g\circ\varphi)\big)(Z_1,Z_2)+g(Z_1,\varphi L_X Z_2)-g(\varphi Z_1,L_X Z_2)\\
&= (\varphi X)g(Z_1,Z_2)-g(\nabla_{\varphi X}Z_1-\nabla_{Z_1}\varphi X,Z_2)-g(Z_1,\nabla_{\varphi X}Z_2-\nabla_{Z_2}\varphi X)\\
&\quad - Xg(\varphi Z_1,Z_2)+(g\circ\varphi)(\nabla_X Z_1-\nabla_{Z_1}X,Z_2)-(g\circ\varphi)(Z_1,\nabla_X Z_2-\nabla_{Z_2}X)\\
&\quad + g\big(Z_1,\varphi(\nabla_X Z_2-\nabla_{Z_2}X)\big)-g(\varphi Z_1,\nabla_X Z_2-\nabla_{Z_2}X)\\
&= (\varphi X)g(Z_1,Z_2)-Xg(\varphi Z_1,Z_2)-g(\nabla_{\varphi X}Z_1,Z_2)+g(\nabla_{Z_1}\varphi X,Z_2)\\
&\quad - g(Z_1,\nabla_{\varphi X}Z_2)+g(Z_1,\nabla_{Z_2}\varphi X)+g\big(\varphi(\nabla_X Z_1),Z_2\big)-g\big(\varphi(\nabla_{Z_1}X),Z_2\big)\\
&\quad + g(\varphi Z_1,\nabla_X Z_2)-g(\varphi Z_1,\nabla_{Z_2}X)+g\big(Z_1,\varphi(\nabla_X Z_2)\big)-g\big(Z_1,\varphi(\nabla_{Z_2}X)\big)\\
&\quad - g(\varphi Z_1,\nabla_X Z_2)+g(\varphi Z_1,\nabla_{Z_2}X)\\
&= (\varphi X)g(Z_1,Z_2)-Xg(\varphi Z_1,Z_2)-g(\nabla_{\varphi X}Z_1,Z_2)+g(\nabla_{Z_1}\varphi X,Z_2)\\
&\quad - g(Z_1,\nabla_{\varphi X}Z_2)+g(Z_1,\nabla_{Z_2}\varphi X)+g\big(\varphi(\nabla_X Z_1),Z_2\big)-g\big(\varphi(\nabla_{Z_1}X),Z_2\big)\\
&\quad + g(\varphi Z_1,\nabla_X Z_2)-g\big(Z_1,\varphi(\nabla_{Z_2}X)\big).\end{aligned}\tag{2.3}$$

Taking account of (2.2), we find

$$\begin{aligned}&g(\nabla_{Z_1}\varphi X,Z_2)-g\big(\varphi(\nabla_{Z_1}X),Z_2\big)+g(Z_1,\nabla_{Z_2}\varphi X)-g\big(Z_1,\varphi(\nabla_{Z_2}X)\big)\\
&\quad = g\big((\nabla\varphi)(X,Z_1),Z_2\big)+g\big(Z_1,(\nabla\varphi)(X,Z_2)\big).\end{aligned}\tag{2.4}$$

Substitution (2.4) into (2.3), (2.3) may be written as

$$\begin{aligned}(\phi_\varphi g)(X,Z_1,Z_2) &= (\varphi X)g(Z_1,Z_2)-Xg(\varphi Z_1,Z_2)+g\big((\nabla\varphi)(X,Z_1),Z_2\big)\\
&\quad + g\big(Z_1,(\nabla\varphi)(X,Z_2)\big)-g(\nabla_{\varphi X}Z_1,Z_2)-g(Z_1,\nabla_{\varphi X}Z_2)\\
&\quad + g\big(\varphi(\nabla_X Z_1),Z_2\big)+g(\varphi Z_1,\nabla_X Z_2).\end{aligned}\tag{2.5}$$

On the other hand, with respect to the Levi-Civita connection ∇, we have

$$(\varphi X)g(Z_1,Z_2)-g(\nabla_{\varphi X}Z_1,Z_2)-g(Z_1,\nabla_{\varphi X}Z_2)=(\nabla_{\varphi X}g)(Z_1,Z_2)=0\tag{2.6}$$

and

$$\begin{aligned}&-Xg(\varphi Z_1,Z_2)+g\big(\varphi(\nabla_X Z_1),Z_2\big)+g(\varphi Z_1,\nabla_X Z_2)\\
&\quad = -Xg(\varphi Z_1,Z_2)+g\big((\nabla_X\varphi Z_1),Z_2\big)+g(\varphi Z_1,\nabla_X Z_2)-g\big((\nabla_X\varphi)Z_1,Z_2\big)\\
&\quad = -(\nabla_X g)(\varphi Z_1,Z_2)-g\big((\nabla_X\varphi)Z_1,Z_2\big)=-g\big((\nabla_X\varphi)Z_1,Z_2\big).\end{aligned}\tag{2.7}$$

By virtue of (2.6) and (2.7), (2.5) reduces to

$$(\phi_\varphi g)(X,Z_1,Z_2)=-g\big((\nabla_X\varphi)Z_1,Z_2\big)+g\big((\nabla_{Z_1}\varphi)X,Z_2\big)+g\big(Z_1,(\nabla_{Z_2}\varphi)X\big).\tag{2.8}$$

Similarly, we have

$$(\phi_\varphi g)(Z_2,Z_1,X)=-g\big((\nabla_{Z_2}\varphi)Z_1,X\big)+g\big((\nabla_{Z_1}\varphi)Z_2,X\big)+g\big(Z_1,(\nabla_X\varphi)Z_2\big).\tag{2.9}$$

The sufficiency follows easily from (2.8) (or (2.9)).

By virtue of Theorem 1 we find

$$(\phi_\varphi g)(X,Z_1,Z_2)+(\phi_\varphi g)(Z_2,Z_1,X)=2g\big(X,(\nabla_{Z_2}\varphi)Z_2\big).\tag{2.10}$$

Now, putting $\phi_\varphi g=0$ in (2.10), we find $\nabla\varphi=0$ from which the necessity follows. Thus Theorem 2 is proved. □

Corollary. *The almost paracomplex structure φ on almost B-manifold is integrable if $\phi_\varphi g=0$.*

Remark. The leaves of the foliations defined by the paracomplex structure of a paraholomorphic B-manifold are totally geodesic submanifolds (see [6] or the book K. Yano and M. Kon [14, p. 420]). In the paper of A.M. Naveira [6] Riemannian almost product manifolds were classified.

Let (M_{2k}, φ, g) be an almost B-manifold. The associated B-metric of almost B-manifold is defined by

$$G(X, Y) = (g \circ \varphi)(X, Y) \tag{2.11}$$

for all vector fields X and Y on M_{2k}. One can easily prove that G is a metric, which is also called the twin metric of g (see [2]) and it plays a role similar to the Kahler form in Hermitian Geometry. We shall now apply the Tachibana operator to the pure Riemannian metric G

$$\begin{aligned}(\phi_\varphi G)(X, Y, Z) &= \big(L_{\varphi X} G - L_X(G \circ \varphi)\big)(Y, Z) + G(Y, \varphi L_X Z) - G(\varphi Y, L_X Z) \\ &= \big(L_{\varphi X}(g \circ \varphi) - L_X\big((g \circ \varphi) \circ \varphi\big)\big)(Y, Z) + (g \circ \varphi)(Y, \varphi L_X Z) - (g \circ \varphi)(\varphi Y, L_X Z) \\ &= \big((L_{\varphi X} g) \circ \varphi + g \circ L_{\varphi X}\varphi - L_X(g \circ \varphi) \circ \varphi - (g \circ \varphi) L_X \varphi\big)(Y, Z) \\ &\quad + (g \circ \varphi)(Y, \varphi L_X Z) - (g \circ \varphi)(\varphi Y, L_X Z) \\ &= \big(L_{\varphi X} g - L_X(g \circ \varphi)\big)(\varphi Y, Z) + g(\varphi Y, \varphi L_X Z) \\ &\quad - g\big(\varphi(\varphi Y), L_X Z\big) + \big(g \circ L_{\varphi X}\varphi - (g \circ \varphi) L_X \varphi\big)(Y, Z) \\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g\big((L_{\varphi X}\varphi)Y, Z\big) - g\big(\varphi\big((L_X\varphi)Y\big), Z\big) \\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g\big([\varphi X, \varphi Y] - \varphi[\varphi X, Y], Z\big) - g\big(\varphi[X, \varphi Y] - \varphi^2[X, Y], Z\big) \\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g\big([\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y], Z\big) \\ &= (\phi_\varphi g)(X, \varphi Y, Z) + g\big(N_\varphi(X, Y), Z\big).\end{aligned} \tag{2.12}$$

Thus (2.12) implies the following:

Theorem 3. *In an almost B-manifold, we have*

$$\phi_\varphi G = (\phi_\varphi g) \circ \varphi + g \circ (N_\varphi).$$

From Theorems 2 and 3 we have

Theorem 4. *Almost B-manifold with condition $\phi_\varphi G = 0$, $N_\varphi \neq 0$ i.e. analogues of the almost Kahler manifolds, does not exist.*

Corollary. *The following conditions are equivalent*:

(a) $\phi_\varphi g = 0$;
(b) $\phi_\varphi G = 0$.

We denote by ∇_g the covariant differentiation of Levi-Civita connection of B-metric g. Then, we have

$$\nabla_g G = (\nabla_g g) \circ \varphi + g \circ (\nabla_g \varphi) = g \circ (\nabla_g \varphi)$$

which implies $\nabla_g G = 0$ by virtue of Theorem 2. Therefore we have

Theorem 5. *Let (M_{2k}, φ, g) be a paraholomorphic B-manifold. Then the Levi-Civita connection of B-metric g coincides with the Levi-Civita connection of associated B-metric G.*

3. Curvature tensors in a paraholomorphic B-manifold

Let R and S be the curvature tensors formed by g and G respectively, then for the paraholomorphic B-manifold we have $R = S$ by means of the Theorem 5.

Applying the Ricci's identity to φ, we get

$$\varphi\big(R(X, Y)Z\big) = R(X, Y)\varphi Z \tag{3.1}$$

by virtue of $\nabla\varphi = 0$. Hence $R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4)$ is pure with respect to X_3 and X_4 and also pure with respect to X_1 and X_2:

$$\begin{aligned} R(X_1, X_2, \varphi X_3, X_4) &= g\big(R(X_1, X_2)\varphi X_3, X_4\big) \\ &= g\big(\varphi\big(R(X_1, X_2)X_3\big), X_4\big) \\ &= g\big(R(X_1, X_2)X_3, \varphi X_4\big) \\ &= R(X_1, X_2, X_3, \varphi X_4). \end{aligned}$$

On the other hand, S being the curvature tensor formed by associated B-metric G, if we put $S(X_1, X_2, X_3, X_4) = G(S(X_1, X_2)X_3, X_4)$, then we have

$$S(X_1, X_2, X_3, X_4) = S(X_3, X_4, X_1, X_2). \tag{3.2}$$

Taking account of (1.1), (2.11), (3.1) and $R = S$, we find that

$$\begin{aligned} S(X_1, X_2, X_3, X_4) &= G\big(S(X_1, X_2)X_3, X_4\big) \\ &= g\big(\varphi\big(S(X_1, X_2)X_3\big), X_4\big) \\ &= g\big(S(X_1, X_2)X_3, \varphi X_4\big) \\ &= g\big(R(X_1, X_2)X_3, \varphi X_4\big) \\ &= R(X_1, X_2, X_3, \varphi X_4) \end{aligned}$$

and

$$\begin{aligned} S(X_3, X_4, X_1, X_2) &= G\big(S(X_3, X_4)X_1, X_2\big) \\ &= g\big(\varphi\big(S(X_3, X_4)X_1\big), X_2\big) \\ &= g\big(S(X_3, X_4)X_1, \varphi X_2\big) \\ &= g\big(R(X_3, X_4)X_1, \varphi X_2\big) \\ &= R(X_3, X_4, X_1, \varphi X_2) \\ &= R(X_1, \varphi X_2, X_3, X_4). \end{aligned}$$

Thus Eq. (3.2) becomes

$$R(X_1, X_2, X_3, \varphi X_4) = R(X_1, \varphi X_2, X_3, X_4),$$

which shows that $R(X_1, X_2, X_3, X_4)$ is pure with respect to X_2 and X_4. Therefore $R(X_1, X_2, X_3, X_4)$ is pure.

Thus we get

Theorem 6. *In a paraholomorphic B-manifold, the Riemannian curvature tensor of B-metric is pure.*

Since the Riemannian curvature tensor R is pure, we can apply the ϕ-operator to R. By similar devices (see proof of Theorem 2), we can prove that

$$(\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) = (\nabla_{\varphi X} R)(Y_1, Y_2, Y_3, Y_4) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3, Y_4). \tag{3.3}$$

Using (3.1) and applying the Bianchi's 2nd identity to (3.3), we get

$$\begin{aligned} (\phi_\varphi R)(X, Y_1, Y_2, Y_3, Y_4) &= g\big((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - (\nabla_X R)(\varphi Y_1, Y_2, Y_3), Y_4\big) \\ &= g\big((\nabla_{\varphi X} R)(Y_1, Y_2, Y_3) - \varphi\big((\nabla_X R)(Y_1, Y_2, Y_3)\big), Y_4\big) \\ &= g\big(-(\nabla_{Y_1} R)(Y_2, \varphi X, Y_3) - (\nabla_{Y_2} R)(\varphi X, Y_1, Y_3) \\ &\quad - \varphi\big((\nabla_X R)(Y_1, Y_2, Y_3)\big), Y_4\big). \end{aligned} \tag{3.4}$$

On the other hand, using $\nabla\varphi = 0$, we find

$$\begin{aligned}(\nabla_{Y_2}R)(\varphi X,Y_1,Y_3) &= \nabla_{Y_2}\big(R(\varphi X,Y_1,Y_3)\big) - R\big(\nabla_{Y_2}(\varphi X),Y_1,Y_3\big) - R(\varphi X,\nabla_{Y_2}Y_1,Y_3) - R(\varphi X,Y_1,\nabla_{Y_2}Y_3)\\ &= (\nabla_{Y_2}\varphi)\big(R(X,Y_1,Y_3)\big) + \varphi\big(\nabla_{Y_2}R(X,Y_1,Y_3)\big)\\ &\quad - R\big((\nabla_{Y_2}\varphi)X + \varphi(\nabla_{Y_2}X),Y_1,Y_3\big) - R(\varphi X,\nabla_{Y_2}Y_1,Y_3) - R(\varphi X,Y_1,\nabla_{Y_2}Y_3)\\ &= \varphi\big(\nabla_{Y_2}R(X,Y_1,Y_3)\big) - \varphi\big(R(\nabla_{Y_2}X,Y_1,Y_3)\big)\\ &\quad - \varphi\big(R(X,\nabla_{Y_2}Y_1,Y_3)\big) - \varphi\big(R(X,Y_1,\nabla_{Y_2}Y_3)\big)\\ &= \varphi\big((\nabla_{Y_2}R)(X,Y_1,Y_3)\big).\end{aligned} \tag{3.5}$$

Similarly

$$(\nabla_{Y_1}R)(Y_2,\varphi X,Y_3) = \varphi\big((\nabla_{Y_1}R)(Y_2,X,Y_3)\big). \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4) and using again the Bianchi's 2nd identity, we obtain

$$\begin{aligned}(\phi_\varphi R)(X,Y_1,Y_2,Y_3,Y_4) &= g\big(-\varphi\big((\nabla_{Y_1}R)(Y_2,X,Y_3)\big) - \varphi\big((\nabla_{Y_2}R)(X,Y_1,Y_3)\big) - \varphi\big((\nabla_X R)(Y_1,Y_2,Y_3)\big),Y_4\big)\\ &= -g\big(\varphi\big(\sigma\big\{(\nabla_X R)(Y_1,Y_2\big\},Y_3)\big),Y_4\big)\\ &= 0,\end{aligned}$$

where σ denotes the cyclic sum with respect to X, Y_1 and Y_2. Therefore we have

Theorem 7. *In a paraholomorphic B-manifold, the Riemannian curvature tensor field is paraholomorphic tensor field.*

4. Examples

Examples 1. We suppose that the manifold M_{2n} is the tangent bundle $\pi : T(V_n) \to V_n$ of a Riemannian manifold V_n. If x^i are local coordinates on V_n, then x^i together with the fibre coordinates $x^{\bar{\imath}} = y^i$, $\bar{\imath} = n+1,\dots,2n$, form local coordinates on $T(V_n)$.

A tensor field of type $(0,q)$ on $T(V_n)$ completely determined by its action on all vector fields $\tilde{X}_i$, $i = 1,2,\dots,q$, which are of the form ${}^V X$ (vertical lift) or ${}^H X$ (horizontal lift) [13, p. 101]:

$${}^V X = X^i \frac{\partial}{\partial x^{\bar{\imath}}}, \qquad {}^H X = X^i \frac{\partial}{\partial x^i} - y^s \Gamma^i_{s\,h} X^h \frac{\partial}{\partial x^{\bar{\imath}}}.$$

Therefore, we define the Sasakian metric ${}^s g$ on $T(V_n)$ by

$$\begin{cases} {}^S g({}^H X, {}^H Y) = {}^V(g(X,Y)),\\ {}^S g({}^V X, {}^V Y) = {}^V(g(X,Y)),\\ {}^S g({}^V X, {}^H Y) = 0, \end{cases} \tag{4.1}$$

for any $X, Y \in \Im^1_0(V_n)$. ${}^S g$ has local components

$${}^S g = \begin{pmatrix} g_{ji} + g_{ts} y^k y^l \Gamma^t_{kj}\Gamma^s_{li} & y^k \Gamma^s_{kj} g_{si}\\ y^k \Gamma^s_{ki} g_{js} & g_{ji}\end{pmatrix}$$

with respect to the induced coordinates $(x^i, x^{\bar{\imath}})$ in $T(V_n)$, where Γ^k_{ij} are components of Levi-Civita connection ∇_g in V_n.

The diagonal lift ${}^D\varphi$ in $T(V_n)$ is defined by

$$\begin{cases} {}^D\varphi\,{}^H X = {}^H(\varphi X),\\ {}^D\varphi\,{}^V X = -{}^V(\varphi X), \end{cases} \tag{4.2}$$

for any $X \in \Im^1_0(V_n)$ and $\varphi \in \Im^1_1(M_n)$. The diagonal lift ${}^D I$ of the identity tensor field $I \in \Im^1_1(M_n)$ has the components

$${}^D I = \begin{pmatrix} \delta^j_i & 0\\ -2y^t \Gamma^j_{ti} & -\delta^j_i \end{pmatrix}$$

with respect to the induced coordinates and satisfies $({}^D I)^2 = I_{T(V_n)}$. Thus ${}^D I$ is an almost paracomplex structure determining the horizontal distribution and the distribution consisting of the tangent planes to fibres.

We put

$$A(\tilde{X},\tilde{Y}) = {}^S g({}^D I\tilde{X},\tilde{Y}) - {}^S g(\tilde{X},{}^D I\tilde{Y}).$$

If $A(\tilde{X},\tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ${}^V X$, ${}^V Y$ or ${}^H X$, ${}^H Y$, then $A = 0$. We have by virtue of ${}^D I\,{}^V X = -{}^V X$, ${}^D I\,{}^H X = {}^H X$ (4.1) and (4.2)

$$\begin{aligned}
&A({}^V X,{}^V Y) = {}^S g(-{}^V X,{}^V Y) - {}^S g({}^V X,-{}^V Y) = 0,\\
&A({}^V X,{}^H Y) = {}^S g(-{}^V X,{}^H Y) - {}^S g({}^V X,{}^H Y) = 0,\\
&A({}^H X,{}^V Y) = {}^S g({}^H X,{}^V Y) - {}^S g({}^H X,-{}^V Y) = 0,\\
&A({}^H X,{}^H Y) = {}^S g({}^H X,{}^H Y) - {}^S g({}^H X,{}^H Y) = 0,
\end{aligned}$$

i.e. ${}^S g$ is B-metric with respect to ${}^D I$.

We hence have:

Theorem 8. *$(T(V_n), {}^D I, {}^S g)$ is an almost B-manifold.*

Using the properties of ${}^V X$, ${}^H X$ and $\gamma R(X,Y) = y^s R^k_{ijs} X^i Y^j \frac{\partial}{\partial x^{\bar{k}}}$, we have

$$\begin{aligned}
&(\phi_{{}^D I}{}^S g)({}^V X,{}^H Y,{}^H Z) = -2({}^S g\,{}^V(\nabla_Y X),{}^H Z) + {}^S g({}^H Y,{}^V(\nabla_Z X)) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^V X,{}^H Y,{}^V Z) = -2\,{}^S g({}^H Y,[{}^V Z,{}^V X]) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^V X,{}^V Y,{}^H Z) = -2\,{}^S g([{}^V Y,{}^V X],{}^H Z) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^V X,{}^V Y,{}^V Z) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^H X,{}^H Y,{}^H Z) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^H X,{}^V Y,{}^V Z) = 2\,{}^V((\nabla_X g)(Y,Z)) = 0,\\
&(\phi_{{}^D I}{}^S g)({}^H X,{}^H Y,{}^V Z) = -2\,{}^S g(\gamma R(Y,X),{}^V Z),\\
&(\phi_{{}^D I}{}^S g)({}^H X,{}^V Y,{}^H Z) = -2\,{}^S g({}^V Y,\gamma R(Z,X)).
\end{aligned}$$

Therefore we have

Theorem 9. *The almost B-manifold $(T(V_n), {}^D I, {}^S g)$ is paraholomorphic if and only if V_n is locally Euclidean.*

Examples 2. Let now M_n be the locally product Riemannian manifold with integrable almost product structure

$$\varphi = \begin{pmatrix} \delta^i_j & 0 \\ 0 & -\delta^{\bar{i}}_{\bar{j}} \end{pmatrix}, \quad i,j = 1,\dots,k,\ \bar{i},\bar{j} = k+1,\dots,n,$$

and let $n = 2k$. Then the paracomplex manifold M_{2k}, admit a structure of B-manifold:

$$g = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\bar{i}\bar{j}} \end{pmatrix}, \quad g_{ij} = g_{ij}(x^t, x^{\bar{t}}),\ g_{\bar{i}\bar{j}} = g_{\bar{i}\bar{j}}(x^t, x^{\bar{t}}).$$

Suppose that the metric of the locally product Riemannian manifold M_{2k} has the form

$$ds^2 = g_{ij}(x^t)\,dx^i\,dx^j + g_{\bar{i}\bar{j}}(x^{\bar{t}})\,dx^{\bar{i}}\,dx^{\bar{j}}, \quad i,j,t = 1,\dots,k,\ \bar{i},\bar{j},\bar{t} = k+1,\dots,2k,$$

that is $g_{ij}(x)$ are functions of x^t only, $g_{i\bar{j}} = 0$, and $g_{\bar{i}\bar{j}}(x)$ are functions of $x^{\bar{t}}$ only, then we call the manifold a locally *decomposable* Riemannian manifold. A necessary and sufficient condition for a locally product Riemannian manifold to be a locally decomposable Riemannian manifold is that $\nabla_g \varphi = 0$ [14, p. 420]. Then from Theorem 2 we have

Theorem 10. *A locally decomposable Riemannian manifold M_{2k} is a paraholomorphic B-manifold.*

Examples 3. Let (M_{2k}, ω) be a symplectic manifold and let D be a Lagrangian distribution, which is a k-dimensional distribution having $\omega/D = 0$. Then, M may be endowed with an almost B-structure.

First of all, we shall prove that there exist a transversal Lagrangian distribution. Taking into account that (M, ω) is an almost symplectic manifold one can find (see [1] or [7]) an almost Hermitian structure (J, G) on M such that $\omega(X, Y) = G(JX, Y)$. Let $D^\perp$ the G-orthogonal distribution to D. Then one has:

(1) If $X, Y \in D$, then $G(JX, Y) = \omega(X, Y) = 0$, thus proving that $J(D) = D^\perp$.
(2) $D^\perp$ is a Lagrangian distribution, because $\omega(JX, JY) = \omega(X, Y)$, for all $X, Y \in \Im^1_0(M_n)$.

Let F be the almost product structure defined by D and $D^\perp$, i.e., $F^+ = D$ and $F^- = D^\perp$. Then, one easily check that $J \circ F = -F \circ J$. Moreover, one can prove that (M, F, G) is a Riemannian almost product manifold:

If $X \in \Im^1_0(M_n)$, then $X = X_1 + X_2$, where $X_1 \in F^+ = D$ and $X_2 \in F^- = D^\perp = J(D)$, and one can write $X_2 = J(X_3)$, with $X_3 \in F^+$. Using this notation we obtain:

$$G(X, Y) = G(X_1 + JX_3, Y_1 + JY_3) = G(X_1, Y_1) + G(JX_3, JY_3) = G(X_1, Y_1) + G(X_3, Y_3)$$

and

$$G(FX, FY) = G(X_1 - JX_3, Y_1 - JY_3) = G(X_1, Y_1) + G(JX_3, JY_3) = G(X_1, Y_1) + G(X_3, Y_3)$$

thus proving $G(X, Y) = G(FX, FY)$.

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