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A note on a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation ☆

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ABSTRACT

In this paper, we show there is a stationary distribution of a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation and it has ergodic property.

1. Introduction

In the paper [5], we investigate a predator–prey model with modified Leslie–Gower and Holling-type II schemes with stochastic perturbation

$$\begin{cases} dx(t) = x(t)\left(a - bx(t) - \dfrac{cy(t)}{m + x(t)}\right)dt + \alpha x(t)\,dB_1(t), \\ dy(t) = y(t)\left(r - \dfrac{fy(t)}{m + x(t)}\right)dt + \beta y(t)\,dB_2(t). \end{cases} \tag{1.1}$$

It is well known for the corresponding deterministic system [2], there is an interior equilibrium

$$x^* = \frac{a}{b} - \frac{cr}{bf}, \qquad y^* = \frac{r(x^* + m)}{f}, \tag{1.2}$$

when $r/f < a/c$. While for the stochastic system (1.1), there isn't a positive time independent equilibrium point. Hence, in [5], we do not explore the stability of system (1.1), but only give the long time behavior of system (1.1), which reflects stability in some extent when the intensity of white noise is small. In this paper, we further investigate some stability properties. By constructing suitable Lyapunov function [7], we show there is a stationary distribution of system (1.1) and it has ergodic property under some conditions. Ergodic property is one of important properties of Markov processes, and it has been applied in many areas, such as probability theory, statistics, harmonic analysis, Lie theory. There are lots of studies on this topic, such as [1,4,6]. In this paper, we obtain ergodic property by the theory in [4]. Then together with the result in [5], we show

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$$P\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds=\int_{R_+^2} z_1\mu(dz_1,dz_2)=\frac{a-\alpha^2/2}{b}-\frac{c(r-\beta^2/2)}{bf}\right\}=1,$$

see Section 2.

Throughout this paper, unless otherwise specified, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B_1(t)$ and $B_2(t)$ denote the independent standard Brownian motions defined on this probability space, and $R_+^2=\{(x,y)\in R^2,\ x>0,\ y>0\}$.

2. Stationary distribution and ergodicity

Before giving the main theorems, we first give a lemma (see [4]).

Let $X(t)$ be a homogeneous Markov process in E_l (E_l denotes euclidean l-space) described by the stochastic equation

$$dX(t)=b(X)\,dt+\sum_{r=1}^{k}\sigma_r(X)\,dB_r(t). \tag{2.1}$$

The diffusion matrix is

$$A(x)=\big(a_{ij}(x)\big),\quad a_{ij}(x)=\sum_{r=1}^{k}\sigma_r^i(x)\sigma_r^j(x).$$

Assumption B. There exists a bounded domain $U\subset E_l$ with regular boundary Γ, having the following properties:

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x\in E_l\setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K}E_x\tau<\infty$ for every compact subset $K\subset E_l$.

Lemma 2.1. *(See [4].) If* (B) *holds, then the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f\big(X(t)\big)\,dt=\int_{E_l} f(x)\mu(dx)\right\}=1$$

for all $x\in E_l$.

Remark 2.1. The proof can be found in [4]. Exactly, the existence of stationary distribution with density is referred to Theorem 4.1, p. 119 and Lemma 9.4, p. 138. The weak convergence and the ergodicity is obtained in Theorem 5.1, p. 121 and Theorem 7.1, p. 130.

To validate (B.1), it suffices to prove F is uniformly elliptical in U, where $Fu=b(x)\cdot u_x+[tr(A(x)u_{xx})]/2$, that is, there is a positive number M such that

$$\sum_{i,j=1}^{k}a_{ij}(x)\xi_i\xi_j\geqslant M|\xi|^2,\quad x\in U,\ \xi\in R^k$$

(see Chapter 3, p. 103 of [3] and Rayleigh's principle in [8, Chapter 6, p. 349]). To verify (B.2), it is sufficient to show that there exists some neighborhood U and a non-negative C^2-function such that and for any $E_l\setminus U$, LV is negative (details refer to [9, p. 1163]).

Remark 2.2. System (1.1) can be written as the form of system (2.1):

$$d\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}x(t)(a-bx(t)-\frac{cy(t)}{m+x(t)})\\ y(t)(r-\frac{fy(t)}{m+x(t)})\end{pmatrix}dt+\begin{pmatrix}\alpha x(t)\\ 0\end{pmatrix}dB_1(t)+\begin{pmatrix}0\\ \beta y(t)\end{pmatrix}dB_2(t).$$

Here the diffusion matrix is

$$A(x,y)=\begin{pmatrix}\alpha^2x^2 & 0\\ 0 & \beta^2y^2\end{pmatrix}.$$

Based on this lemma, we give out the main theorem in this section as follows.

Theorem 2.1. *Assume $r/f < \min\{a/c, bm/c\}$ and $\alpha > 0$, $\beta > 0$ such that*

$$\delta < \min\left\{\frac{bfm - cr}{f}\left[x^* + \frac{f}{4(bfm - cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)\right]^2, \frac{cf}{r}\left(y^*\right)^2\right\}.$$

Then there is a stationary distribution $\mu(\cdot)$ for system (1.1) with initial value $(x_0, y_0) \in R^2_+$ and it has ergodic property. Here (x^*, y^*) is defined as in (1.2) and

$$\delta = \frac{f}{16(bfm - cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)^2 + \frac{1}{2}\left(x^* + m\right)\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right).$$

Proof. We know there is a solution $(x(t), y(t)) \in R^2_+$ of (1.1) for any initial value $(x_0, y_0) \in R^2_+$, and for simplicity, we write $x(t)$ and $y(t)$ as x and y respectively. Define $V : E_l = R^2_+ \to R_+$,

$$V(x, y) = \left[x - x^* - x^* \ln\frac{x}{x^*}\right] + k\left[y - y^* - y^* \ln\frac{y}{y^*}\right] := V_1 + kV_2,$$

where k is a positive constant to be determined later. By the Itô formula, we compute

$$\begin{aligned}
dV_1 &= \left(1 - \frac{x^*}{x}\right)dx + \frac{1}{2}\frac{x^*}{x^2}(dx)^2 \\
&= \left(x - x^*\right)\left[\left(a - bx - \frac{cy}{m + x}\right)dt + \alpha\, dB_1(t)\right] + \frac{1}{2}x^*\alpha^2\, dt \\
&= \left(x - x^*\right)\left[\left(bx^* + \frac{cy^*}{m + x^*} - bx - \frac{cy}{m + x}\right)dt + \alpha\, dB_1(t)\right] + \frac{1}{2}x^*\alpha^2\, dt \\
&= \left[-b\left(x - x^*\right)^2 + \frac{cy^*(x - x^*)^2}{(m + x^*)(m + x)} - \frac{c(x - x^*)(y - y^*)}{m + x} + \frac{1}{2}x^*\alpha^2\right]dt + \alpha\left(x - x^*\right)dB_1(t) \\
&= \left[-b\left(x - x^*\right)^2 + \frac{cr(x - x^*)^2}{f(m + x)} - \frac{c(x - x^*)(y - y^*)}{m + x} + \frac{1}{2}x^*\alpha^2\right]dt + \alpha\left(x - x^*\right)dB_1(t),
\end{aligned}$$

and

$$\begin{aligned}
dV_2 &= \left(1 - \frac{y^*}{y}\right)dy + \frac{1}{2}\frac{y^*}{y^2}(dy)^2 \\
&= \left(y - y^*\right)\left[\left(r - \frac{fy}{m + x}\right)dt + \beta\, dB_2(t)\right] + \frac{1}{2}y^*\beta^2\, dt \\
&= \left(y - y^*\right)\left[\left(\frac{fy^*}{m + x^*} - \frac{fy}{m + x}\right)dt + \beta\, dB_2(t)\right] + \frac{1}{2}y^*\beta^2\, dt \\
&= \left[-\frac{f(y - y^*)^2}{m + x} + \frac{fy^*(x - x^*)(y - y^*)}{(m + x^*)(m + x)} + \frac{1}{2}y^*\beta^2\right]dt + \beta\left(y - y^*\right)dB_2(t) \\
&= \left[-\frac{f(y - y^*)^2}{m + x} + \frac{r(x - x^*)(y - y^*)}{m + x} + \frac{1}{2}y^*\beta^2\right]dt + \beta\left(y - y^*\right)dB_2(t)
\end{aligned}$$

according to (1.2). Then

$$\begin{aligned}
dV &= dV_1 + k\, dV_2 \\
&:= LV\, dt + \alpha\left(x - x^*\right)dB_1(t) + k\beta\left(y - y^*\right)dB_2(t),
\end{aligned}$$

where

$$LV = -b\left(x - x^*\right)^2 + \frac{cr(x - x^*)^2}{f(m + x)} - k\frac{f(y - y^*)^2}{m + x} - (c - kr)\frac{(x - x^*)(y - y^*)}{m + x} + \frac{x^*}{2}\alpha^2 + k\frac{y^*}{2}\beta^2.$$

Choose $k = c/r$, then

$$LV = -b(x-x^*)^2 + \frac{cr(x-x^*)^2}{f(m+x)} - \frac{cf(y-y^*)^2}{r(m+x)} + \frac{x^*}{2}\alpha^2 + \frac{cy^*}{2r}\beta^2$$
$$= -\frac{(bf(m+x)-cr)(x-x^*)^2}{f(m+x)} - \frac{cf(y-y^*)^2}{r(m+x)} + \frac{x^*}{2}\alpha^2 + \frac{cy^*}{2r}\beta^2$$
$$\leqslant -\frac{bfm-cr}{f(m+x)}(x-x^*)^2 - \frac{cf(y-y^*)^2}{r(m+x)} + \frac{x^*}{2}\alpha^2 + \frac{cy^*}{2r}\beta^2.$$

Note that

$$(m+x)LV \leqslant -\frac{bfm-cr}{f}(x-x^*)^2 - \frac{cf}{r}(y-y^*)^2 + \left(\frac{x^*}{2}\alpha^2 + \frac{cy^*}{2r}\beta^2\right)(m+x)$$
$$= -\frac{bfm-cr}{f}\left[x - x^* - \frac{f}{4(bfm-cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)\right]^2 - \frac{cf}{r}(y-y^*)^2$$
$$+ \frac{f}{16(bfm-cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)^2 + \frac{1}{2}(x^*+m)\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)$$
$$:= -\frac{bfm-cr}{f}\left[x - x^* - \frac{f}{4(bfm-cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)\right]^2 - \frac{cf}{r}(y-y^*)^2 + \delta,$$

then when $\delta < \min\{\frac{bfm-cr}{f}[x^* + \frac{f}{4(bfm-cr)}(x^*\alpha^2 + \frac{cy^*\beta^2}{r})]^2, \frac{cf}{r}(y^*)^2\}$, the ellipsoid

$$-\frac{bfm-cr}{f}\left[x - x^* - \frac{f}{4(bfm-cr)}\left(x^*\alpha^2 + \frac{cy^*\beta^2}{r}\right)\right]^2 - \frac{cf}{r}(y-y^*)^2 + \delta = 0$$

lies entirely in R^2_+. We can take U to be a neighborhood of the ellipsoid with $\bar{U} \subseteq E_l = R^2_+$, so for $x \in U \setminus E_l$, $LV < -C$ (C is a positive constant), which implies condition (B.2) in Lemma 2.1 is satisfied. Besides, there is $M > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j = \alpha^2 x^2 \xi_1^2 + \beta^2 y^2 \xi_2^2 \geqslant M|\xi^2| \quad \text{all } (x,y) \in \bar{U},\ \xi \in R^2,$$

which implies condition (B.1) is also satisfied. Therefore, the stochastic system (1.1) has a stable stationary distribution $\mu(\cdot)$ and it is ergodic. □

Lemma 2.2. *Let $(x(t), y(t))$ be a solution of system (1.1) for any initial value $(x_0, y_0) \in R^2_+$. Then we have*

$$\limsup_{t\to\infty} E[x^p(t)] \leqslant L(p) \quad \text{for all } p > 1,$$

where

$$L(p) = \left[\frac{2a + \alpha^2(p-1)}{2b}\right]^p.$$

Proof. By the Itô formula, we have

$$d(x^p) = px^{p-1}\,dx + \frac{1}{2}p(p-1)x^{p-2}(dx)^2$$
$$= px^p\left[\left(a - bx - \frac{cy}{m+x}\right)dt + \alpha\,dB_1(t)\right] + \frac{1}{2}p(p-1)x^p\alpha^2\,dt$$
$$\leqslant px^p\left(a + \frac{\alpha^2}{2}(p-1) - bx\right)dt + \alpha px^p\,dB_1(t),$$

and

$$x^p(t) \leqslant x_0^p + \int_0^t px^p(s)\left(a + \frac{\alpha^2}{2}(p-1) - bx(s)\right)ds + \int_0^t \alpha px^p(s)\,dB_1(s). \tag{2.2}$$

Taking the expectation of both sides of (2.2), we have

$$E\big[x^p(t)\big] \leqslant x_0^p + \int_0^t \bigg[p\bigg(a + \frac{\alpha^2}{2}(p-1)\bigg)E\big[x^p(s)\big] - bpE\big[x^{p+1}(s)\big]\bigg]ds,$$

and

$$\frac{dE[x^p]}{dt} \leqslant p\bigg(a + \frac{\alpha^2}{2}(p-1)\bigg)E\big[x^p\big] - bpE\big[x^{p+1}\big]$$
$$\leqslant p\bigg(a + \frac{\alpha^2}{2}(p-1)\bigg)E\big[x^p\big] - bp\big\{E\big[x^p\big]\big\}^{p+1/p}.$$

Let $u(t) = E[x^p(t)]$, and then we have

$$\frac{du(t)}{dt} \leqslant p\bigg(a + \frac{\alpha^2}{2}(p-1)\bigg)u(t) - bpu^{p+1/p}(t).$$

Note that the solution of equation

$$\begin{cases} \dfrac{dz(t)}{dt} = pz(t)\bigg[\bigg(a + \dfrac{\alpha^2}{2}(p-1)\bigg) - bz^{1/p}(t)\bigg], \\ z(0) = x_0 \end{cases}$$

is

$$z(t) = \bigg\{x_0^{-1}e^{-[a+\alpha^2(p-1)/2]t} + \frac{2b}{2a+\alpha^2(p-1)}\big[1 - e^{-[a+\alpha^2(p-1)/2]t}\big]\bigg\}^{-p}.$$

Letting $t \to \infty$, yields

$$z(t) \to \bigg[\frac{2a+\alpha^2(p-1)}{2b}\bigg]^p.$$

Thus by the comparison argument we get

$$\limsup_{t\to\infty} u(t) \leqslant \bigg[\frac{2a+\alpha^2(p-1)}{2b}\bigg]^p := L(p).$$

Therefore, we obtain

$$\limsup_{t\to\infty} E\big[x^p(t)\big] \leqslant L(p). \qquad \square$$

Remark 2.3. From Lemma 2.2, there is a $T > 0$ such that

$$E\big[x^p(t)\big] \leqslant 2L(p) \quad \text{for all } t \geqslant T.$$

Besides, note that $E[x^p(t)]$ is continuous, then there is a $\tilde{L}(p,T) > 0$ such that

$$E\big[x^p(t)\big] \leqslant \tilde{L}(p,T) \quad \text{for } t \in [0,T].$$

Let

$$K(p) = \max\big\{2L(p), \tilde{L}(p,T)\big\},$$

and then we have

$$E\big[x^p(t)\big] \leqslant K(p) \quad \text{for all } t \in [0,\infty).$$

In other words, the pth moment of $x(t)$ is bounded.

By the ergodic property, for $m > 0$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \big[x^p(s) \wedge m\big]ds = \int_{R_+^2} \big(z_1^p \wedge m\big)\mu(dz_1, dz_2) \quad \text{a.s.} \tag{2.3}$$

On the other hand, by dominated convergence theorem, we can get

$$E\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t \left[x^p(s)\wedge m\right]ds\right]=\lim_{t\to\infty}\frac{1}{t}\int_0^t E\left[x^p(s)\wedge m\right]ds\leqslant K(p),$$

which together with (2.3) implies

$$\int_{R^2_+}\left(z_1^p\wedge m\right)\mu(dz_1,dz_2)\leqslant K(p).$$

Letting $m\to\infty$, we get

$$\int_{R^2_+} z_1^p\mu(dz_1,dz_2)\leqslant K(p),$$

which implies the function $f(x)=x^p$ is integrable with respect to the measure μ. Therefore, these arguments together with Theorem 2.1 and Theorem 3.2 in [5] imply:

Theorem 2.2. *Assume $r/f<\min\{a/c, bm/c\}$ and $a>\alpha^2/2>0$, $r>\beta^2/2>0$, $(a-\alpha^2/2)/c>(r-\beta^2/2)/f$ such that*

$$\delta<\min\left\{\frac{bfm-cr}{f}\left[x^*+\frac{f}{4(bfm-cr)}\left(x^*\alpha^2+\frac{cy^*\beta^2}{r}\right)\right]^2,\ \frac{cf}{r}\left(y^*\right)^2\right\},$$

where (x^*, y^*) is defined as in (1.2),

$$\delta=\frac{f}{16(bfm-cr)}\left(x^*\alpha^2+\frac{cy^*\beta^2}{r}\right)^2+\frac{1}{2}\left(x^*+m\right)\left(x^*\alpha^2+\frac{cy^*\beta^2}{r}\right).$$

Then for any initial value $(x_0, y_0)\in R^2_+$, $x(t)$ has the property

$$P\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds=\int_{R^2_+} z_1\mu(dz_1,dz_2)=\frac{a-\alpha^2/2}{b}-\frac{c(r-\beta^2/2)}{bf}\right\}=1.$$

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