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Vague Groups

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1. INTRODUCTION

Denoting the group operation by the multiplication symbol "·" on a nonvoid set G, Rosenfeld introduced the fuzzy subgroup of G as a fuzzy subset of G satisfying two additional conditions [6]. Many authors have worked to present the fuzzy setting of various algebraic concepts based on his approach [1, 2, 5]. In Rosenfeld's work, only the subsets are fuzzy, but the group operation is crisp. A natural question, how the group structure on G can be constructed whenever the group operation "·" on G is fuzzy, arises from the essence of fuzzy logic. The concept of fuzzy equality and fuzzy function given in [3] provides a good tool for fuzzifying the group operation on a crisp set.

In this paper, taking the group operation on a crisp set as a fuzzy function in the sense of [3], we establish the group structure on a crisp set and investigate the validity of the classical results in this setting.

2. PRELIMINARIES

The notation $\wedge$ always stands for the minimum operation between two real numbers, and X represents a crisp (usual) and nonempty set in this paper. For a given fuzzy subset A of X and for a crisp subset H of X, the fuzzy subset B of H, defined by $\mu_B(x) = \mu_A(x)$ for each $x \in H$, is called the restriction of A on H, and it will be denoted by $A|_H$.

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A mapping $E_X\colon X \times X \to [0,1]$ is called a *fuzzy equality* on X [3] iff the following conditions are satisfied:

(E.1) $E_X(x, y) = 1 \Leftrightarrow x = y$, $\forall x, \forall y \in X$.

(E.2) $E_X(x, y) = E_X(y, x)$, $\forall x, \forall y \in X$.

(E.3) $E_X(x, y) \wedge E_X(y, z) \leq E_X(x, z)$, $\forall x, \forall y, \forall z \in X$.

For $x, y \in X$, the real number $E_X(x, y)$ shows the degree of the equality of x and y. One can always define a fuzzy equality on X w.r.t. classical equality of the elements of X. The mapping $E_X^*\colon X \times X \to [0,1]$, defined by

$$E_X^*(x, y) = \begin{Bmatrix} 1: & x = y \\ 0: & x \neq y \end{Bmatrix} \qquad \text{for all } x, y \in X,$$

is obviously a fuzzy equality on X.

For two nonempty crisp sets X and Y, let E_X and E_Y be two fuzzy equalities on X and Y, respectively. Then a fuzzy relation f on $X \times Y$ (a fuzzy subset f of $X \times Y$) is called a fuzzy function from X to Y w.r.t. fuzzy equalities E_X and E_Y [3], denoted by the usual notation $f\colon X \to Y$, iff the characteristic function $\mu_F\colon X \times Y \to [0,1]$ of f holds the following two conditions:

(F.1) $\forall x \in X$, $\exists y \in Y$ such that $\mu_f(x, y) > 0$.

(F.2) $\forall x, \forall y \in X, \forall z, \forall w \in Y$, $\mu_f(x, z) \wedge \mu_f(y, w) \wedge E_X(x, y) \leq E_Y(z, w)$.

A fuzzy function f is called a strong fuzzy function iff it additionally satisfies

(F.3) $\forall x \in X$, $\exists y \in Y$, such that $\mu_f(x, y) = 1$.

It is noticed that if E_X, E_Y, and μ_f are chosen such that $E = E_X^*$, $E_Y = E_Y^*$, and $\mu_f(X \times Y) \subseteq \{0,1\}$, then the fuzzy function f one-to-one way corresponds to a classical function. In this case, a fuzzy function is called a *crisp function*.

DEFINITION 2.1. (i) A strong fuzzy function $f\colon X \times X \to X$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X is said to be a vague binary operation on X w.r.t. $E_{X\times X}$ and E_X.

(ii) A vague binary operation f on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of first order iff

(T.1) $(\forall a, \forall b, \forall c, \forall d \in X)(\mu_f(a, b, c) \wedge E_X(c, d) \leq \mu_f(a, b, d))$.

(iii) A vague binary operation f on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of second order iff

(T.2) $(\forall a, \forall b, \forall c, \forall d \in X)(\mu_f(a, b, c) \wedge E_X(b, d) \leq \mu_f(a, d, c))$.

It can easily be seen that every crisp function $f\colon X \times X \to X$ is a vague binary operation on X w.r.t. $E^*_{X\times X}$ and E^*_X, and it is transitive of both first order and second order.

For a given fuzzy equality E_X on X and for a crisp subset H of X, the restriction of mapping E_X on $H \times H$, denoted by E^H_X, is obviously a fuzzy equality on H. For a given vague binary operation f on X, we say that a crisp subset B of X is vague closed under f iff

(VGC) $\mu_f(a, b, c) = 1 \Rightarrow c \in B, \forall a, \forall b \in B, \forall c \in X.$

For a given vague binary operation f on X w.r.t. $E_{X\times X}$ and E_X, if a crisp subset H of X is vague closed under f, then it is not difficult to observe that $f|_{H\times H\times H}$ is a vague binary operation on H and $f|_{H\times H\times H}$ preserves the transitivity properties of f.

3. DEFINITION OF VAGUE GROUPS AND THE PROPERTIES OF VAGUE GROUPS

For the construction of vague groups, we first need to introduce the following well-known definition in classical group theory [4].

DEFINITION 3.1. X together with a binary operation $\circ$, denoted by $(X, \circ)$, is a semigroup iff the following associative property is satisfied:

(G.1) $a \circ (b \circ c) = (a \circ b) \circ c, \forall a, \forall b, \forall c \in X.$

A semigroup $(X, \circ)$ is a monoid iff

(G.2) There exists an element $e \in X$, called the (two-sided) identity element of $(X, \circ)$, such that $e \circ a = a$ and $a \circ e = a$ for each $a \in X$.

A monoid $(X, \circ)$ is a group iff

(G.3) For each $a \in X$, there exists an element of X, denoted by a^{-1} and called the (two-sided) inverse element of a, such that $a^{-1} \circ a = e$ and $a \circ a^{-1} = e$.

A semigroup $(X, \circ)$ is said to be abelian (commutative) iff the binary operation $\circ$ has the following property:

(G.4) $a \circ b = b \circ a, \forall a, \forall b \in X.$

The conditions (G.1) and (G.4) can be written in the following equivalent statements, respectively:

(G.1′) $(\forall a, \forall b, \forall c, \forall d, \forall m, \forall q, \forall w \in X)(((b \circ c = d)$ and $(a \circ d = m)$ and $(a \circ b = q)$ and $(q \circ c = w)) \Rightarrow (m = w))$.

(G.4′) $(\forall a, \forall b, \forall m, \forall w \in X)(((a \circ b = m)$ and $(b \circ a = w)) \Rightarrow (m = w))$.

The binary operation $\circ$ can be conceivable as a special vague binary operation $\circ$ on X w.r.t. $E^*_{X\times X}$ and E^*_X satisfying the condition $\mu_\circ(X \times X \times X) \subseteq \{0, 1\}$. Then, for $a, b, m \in X$, the classical notation $a \circ b = m$ means that $\mu_\circ(a, b, m) = 1$, or equivalently, $\mu_\circ(a, b, m) > 0$. Therefore, regarding (G.1′) and (G.4′) instead of (G.1) and (G.4), respectively, we observe that (G.1−G.4) can be respectively represented in the following statements:

(VG.1) $(\forall a, \forall b, \forall c, \forall d, \forall m, \forall q, \forall w)((\mu_\circ(b, c, d) \wedge \mu_\circ(a, d, m) \wedge \mu_\circ(a, b, q) \wedge \mu_\circ(q, c, w)) \le E_X(m, w))$.

(VG.2) There exists an (two sided) identity element $e \in X$ such that $\mu_\circ(e, a, a) \wedge \mu_\circ(a, e, a) = 1$ for each $a \in X$.

(VG.3) For each $a \in X$, there exists an (two-sided) inverse element $a^{-1} \in X$ such that $\mu_\circ(a^{-1}, a, e) \wedge \mu_\circ(a, a^{-1}, e) = 1$.

(VG.4) $(\forall a, \ \forall b, \ \forall m, \ \forall w \in X)((\mu_\circ(a, b, m) \wedge \mu_\circ(b, a, w)) \le E_X(m, w))$.

This motivation leads us to the following definition.

DEFINITION 3.2. Let $\circ$ be a vague binary operation on X w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X. Then

(i) X together with $\circ$, denoted by $(X, \circ)$, is called a vague semigroup iff the characteristic function $\mu_\circ \colon X \times X \times X \to [0, 1]$ of $\circ$ fulfills the condition (VG.1).

(ii) A vague semigroup $(X, \circ)$ is a vague monoid iff the condition (VG.2) is satisfied by $\circ$.

(iii) A vague monoid $(X, \circ)$ is a vague group iff $\circ$ holds the condition (VG.3).

(iv) A vague semigroup $(X, \circ)$ is said to be abelian (commutative) iff the condition (VG.4) is satisfied by $\circ$.

In particular, if $\circ$ is a vague binary operation on X w.r.t. $E^*_{X\times X}$ on $X \times X$ and E^*_X on X such that $\mu_\circ(X \times X \times X) \subseteq \{0, 1\}$, then a vague group $(X, \circ)$ one-to-one way corresponds to a group in the classical sense. In this case, a vague group is simply called a *crisp group*. In the following example, for a given classical group $(X, \cdot)$, it is shown that an infinite number of nontrivial vague groups can be defined on X.

EXAMPLE 3.3. For a classical group $(X, \cdot)$, for fixed real numbers α, β, θ satisfying $0 < \theta \le \alpha \le \beta < 1$ and for $x, y, z, w \in X$, defining the fuzzy equalities on X and $X \times X$ such that

$$E_X(x, y) = \begin{cases} 1: & x = y \\ \beta: & x \neq y \end{cases},$$

$$E_{X\times X}((x, y), (z, w)) = \begin{cases} 1: & (x, y) = (z, w) \\ \alpha: & (x, y) \neq (z, w) \end{cases},$$

and considering the fuzzy relation $*$ on $X \times X \times X$ given by

$$\mu_*(x, y, z) = \begin{cases} 1: & z = x \cdot y \\ \theta: & z \neq x \cdot y \end{cases},$$

it is easily checked that $(X, *)$ is a vague semigroup. Furthermore, the identity element of $(X, *)$ and the inverse of an element a of $(X, *)$ are, respectively, the identity element of $(X, \cdot)$ and the inverse of that element a of $(X, \cdot)$. Thus, $(X, *)$ is a vague group. If $(X, \cdot)$ is commutative, so is $(X, *)$. It should also be noticed that $*$ is neither transitive of first order nor transitive of second order for the case of $\theta < \beta$, and that when $\theta = \beta = \alpha$, $*$ is both transitive of first and second order w.r.t. $E_{X\times X}$ and E_X.

For a vague semigroup $(X, \circ)$, if there exists $e_L \in X$ ($e_R \in X$) such that

$$\mu_\circ(e_L, a, a) = 1 \quad (\mu_\circ(a, e_R, a) = 1) \qquad \text{for all } a \in X,$$

then we say that $e_L(e_R)$ is a left (right) identity element of $(X, \circ)$. Furthermore, for a vague semigroup $(X, \circ)$ with the left (right) identity element e_L (e_R) and for each $a \in X$, if there exists an element $a_L^{-1} \in X$ $(a_R^{-1} \in X)$ such that

$$\mu_\circ(a_L^{-1}, a, e_L) = 1 \quad (\mu_\circ(a, a_R^{-1}, e_R) = 1),$$

then it is said that a_L^{-1} (a_R^{-1}) is a left (right) inverse of a.

For a given vague group $(X, \circ)$, the uniqueness of the identity and the inverse of an element of $(X, \circ)$ can be easily seen, and it can be also easily verified that, for each $a \in X$, $(a^{-1})^{-1} = a$.

PROPOSITION 3.4. *For a given vague group $(X, \circ)$, there exists a binary operation in the classical sense, denoted by $\sigma_\circ$, on X such that $(X, \sigma_\circ)$ is a group in the classical sense.*

Proof. For a given vague group $(X, \circ)$, denoting the crisp subset $\{(a, b, c) \in X \times X \times X: \mu_\circ(a, b, c) = 1\}$ of $X \times X \times X$ by the notation $\sigma_\circ$, one can easily check that $\sigma_\circ$ is a binary operation on X in the classical sense. Considering the definition of $\sigma_\circ$ and using the condition

(VG.1−VG.3), we easily see that $(X, \sigma_\circ)$ holds (G.1), and the identity of $(X, \sigma_\circ)$ and the inverse of $a \in X$ w.r.t. $\sigma_\circ$ are, respectively, the identity of $(X, \circ)$ and the inverse of $a \in X$ w.r.t. $\circ$.

For a given vague group $(X, \circ)$, we call the group $(X, \sigma_\circ)$ occurring in Proposition 3.4 an *induced classical group* of $(X, \circ)$.

PROPOSITION 3.5. *Let $(X, \circ)$ w.r.t. fuzzy equalities $E_{X\times X}$ on $X \times X$ and E_X on X be a vague semigroup. If $(X, \circ)$ has a left (right) identity element e_L (e_R) and, for each $a \in X$, there exists a left (right) inverse a_L^{-1} (a_R^{-1}) of a, then, for each $c \in X$, $\mu_\circ(c, c, c) \le E_X(c, e_L(e_R))$.*

Proof. Let us suppose that $(X, \circ)$ has a left identity element e_L, and, for each $a \in X$, there exists a left inverse a_L^{-1} of a. For $c \in X$, since $\mu_\circ(c_L^{-1}, c, e_L) = \mu_\circ(e_L, c, c) = 1$, and using (VG.1), we observe that

$$\mu_\circ(c, c, c) = \mu_\circ(c, c, c) \wedge \mu_\circ\left(c_L^{-1}, c, e_L\right)$$
$$\wedge\ \mu_\circ\left(c_L^{-1}, c, e_L\right) \wedge \mu_\circ(e_L, c, c) \le E_X(c, e_L).$$

For the case of e_R and the right inverse, the required inequality is similarly obtained.

THEOREM 3.6 (Vague Cancellation Law). *Let $(X, \circ)$ w.r.t. fuzzy equalities $E_{X\times X}$ on$X \times X$ and E_X on X be a vague group. Then*

(i) $\mu_\circ(a, b, u) \wedge \mu_\circ(a, c, u) \le E_X(b, c)$, $\forall a, \forall b, \forall c, \forall u \in X$.

(ii) $\mu_\circ(b, a, u) \wedge \mu_\circ(c, a, u) \le E_X(b, c)$, $\forall a, \forall b, \forall c, \forall u \in X$.

Proof. (i) Let $a, b, c, u \in X$. Since $\circ: X \times X \to X$ is a strong fuzzy function, $\exists v \in X$ such that $\mu_\circ(a^{-1}, u, v) = 1$. From (VG.1−VG.3) we have

$$\mu_\circ(a, b, u) = \mu_\circ(a, b, u) \wedge \mu_\circ(a^{-1}, u, v) \wedge \mu_\circ(a^{-1}, a, e)$$
$$\wedge\ \mu_\circ(e, b, b) \le E_X(v, b)$$

and

$$\mu_\circ(a, c, u) = \mu_\circ(a, c, u) \wedge \mu_\circ(a^{-1}, u, v) \wedge \mu_\circ(a^{-1}, a, e)$$
$$\wedge\ \mu_\circ(e, c, c) \le E_X(v, c).$$

Thus

$$\mu_\circ(a, b, u) \wedge \mu_\circ(a, c, u) \le E_X(b, v) \wedge E_X(v, c) \le E_X(b, c).$$

(ii) is similarly proved.

THEOREM 3.7. *Let* $(X, \circ)$ *w.r.t. fuzzy equalities* $E_{X\times X}$ *on* $X \times X$ *and* E_X *on* X *be a vague group. Then*

(i) *If the vague binary operation* $\circ$ *is transitive of first order, then*

$$\mu_\circ(b^{-1}, a^{-1}, u) \wedge \mu_\circ(a, b, v) \leq E_X(u, v^{-1}) \wedge E_X(v, u^{-1}), \quad \forall a, \forall b, \forall u, \forall v \in X.$$

(ii) *If the vague binary operation* $\circ$ *is transitive of second order, then*

$$E_X(a, b) = E_X(a^{-1}, b^{-1}), \qquad \forall a, \forall b \in X.$$

Proof. (i) Let us assume that $\circ$ is transitive of first order and $a, b, u, v \in X$. Since $\circ$ is a fuzzy function, $\exists k$, $\exists w$, $\exists r$, $\exists t \in X$ such that $\mu_\circ(u, a, k) = \mu_\circ(v, b^{-1}, w) = \mu_\circ(v, u, r) = \mu_\circ(u, v, t) = 1$. Then considering the conditions (VG.1–VG.3), we may write

$$\begin{aligned}\mu_\circ(b^{-1}, a^{-1}, u) &= \mu_\circ(a^{-1}, a, e) \wedge \mu_\circ(b^{-1}, e, b^{-1}) \\ &\quad \wedge \mu_\circ(b^{-1}, a^{-1}, u) \wedge \mu_\circ(u, a, k) \\ &\leq E_X(k, b^{-1})\end{aligned}$$

and

$$\begin{aligned}\mu_\circ(a, b, v) &= \mu_\circ(b, b^{-1}, e) \wedge \mu_\circ(a, e, a) \\ &\quad \wedge \mu_\circ(a, b, v) \wedge \mu_\circ(v, b^{-1}, w) \leq E_X(a, w).\end{aligned}$$

By the first-order transitivity of $\circ$ we also have

$$E_X(k, b^{-1}) = \mu_\circ(u, a, k) \wedge E_X(k, b^{-1}) \leq \mu_\circ(u, a, b^{-1})$$

and

$$E_X(a, w) = \mu_\circ(v, b^{-1}, w) \wedge E_X(a, w) \leq \mu_\circ(v, b^{-1}, a)$$

Therefore we find that

$$\mu_\circ(b^{-1}, a^{-1}, u) \leq \mu_\circ(u, a, b^{-1}) \quad \text{and} \quad \mu_\circ(a, b, v) \leq \mu_\circ(v, b^{-1}, a). \tag{1}$$

On the other hand,

$$\begin{aligned}&\mu_\circ(b^{-1}, a^{-1}, u) \wedge \mu_\circ(v, b^{-1}, a) \\ &\quad = \mu_\circ(b^{-1}, a^{-1}, u) \wedge \mu_\circ(v, u, r) \wedge \mu_\circ(v, b^{-1}, a) \wedge \mu_\circ(a, a^{-1}, e) \\ &\quad \leq E_X(r, e)\end{aligned}$$

and

$$\mu_\circ(a,b,v) \wedge \mu_\circ(u,a,b^{-1}) = \mu_\circ(a,b,v) \wedge \mu_\circ(u,v,t) \wedge \mu_\circ(u,a,b^{-1}) \wedge \mu_\circ(b^{-1},b,e) \leq E_X(t,e).$$

Furthermore, the first-order transitivity of $\circ$ yields

$$E_X(r,e) = \mu_\circ(v,u,r) \wedge E_X(r,e) \leq \mu_\circ(v,u,e)$$

and

$$E_X(t,e) = \mu_\circ(u,v,t) \wedge E_X(t,e) \leq \mu_\circ(u,v,e).$$

Thus we get

$$\mu_\circ(b^{-1},a^{-1},u) \wedge \mu_\circ(v,b^{-1},a) \leq \mu_\circ(v,u,e) \quad \text{and}$$
$$\mu_\circ(a,b,v) \wedge \mu_\circ(u,a,b^{-1}) \leq \mu_\circ(u,v,e). \tag{2}$$

Combining (1) and (2) we see that

$$\mu_\circ(b^{-1},a^{-1},u) \wedge \mu_\circ(a,b,v) = \left(\mu_\circ(b^{-1},a^{-1},u) \wedge \mu_\circ(u,a,b^{-1})\right) \wedge \left(\mu_\circ(a,b,v) \wedge \mu_\circ(v,b^{-1},a)\right) \leq \mu_\circ(u,v,e) \wedge \mu_\circ(v,u,e). \tag{3}$$

By Theorem 3.6(i), we possess the inequalities

$$\mu_\circ(u,v,e) = \mu_\circ(u,v,e) \wedge \mu_\circ(u,u^{-1},e) \leq E_X(u^{-1},v)$$

and

$$\mu_\circ(v,u,,e) = \mu_\circ(v,u,e) \wedge \mu_\circ(v,v^{-1},e) \leq E_X(u,v^{-1}), \quad \text{i.e.,}$$
$$\mu_\circ(u,v,e) \wedge \mu_\circ(v,u,e) \leq E_X(u^{-1},v) \wedge E_X(u,v^{-1}). \tag{4}$$

Hence the required inequality immediately is deduced form (3) and (4).

(ii) Let $\circ$ be transitive of second order, and $a,b \in X$. By the assumption on $\circ$ and Theorem 3.6(ii), it can easily be written that

$$E_X(a,b) = \mu_\circ(a^{-1},a,e) \wedge E_X(a,b) \leq \mu_\circ(a^{-1},b,e) = \mu_\circ(a^{-1},b,e) \wedge \mu_\circ(b^{-1},b,e) \leq E_X(a^{-1},b^{-1})$$

and

$$E_X(a^{-1},b^{-1}) = \mu_\circ(a,a^{-1},e) \wedge E_X(a^{-1},b^{-1})$$
$$\leq \mu_\circ(a,b^{-1},e) = \mu_\circ(a,b^{-1},e) \wedge \mu_\circ(b,b^{-1},e)$$
$$\leq E_X(a,b), \quad \text{i.e.,}$$
$$E_X(a,b) = E_X(a^{-1},b^{-1}).$$

THEOREM 3.8. *Let $(X,\circ)$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X\times X$ and a fuzzy equality E_X on X be a vague semigroup. Then $(X,\circ)$ is a vague group $\Leftrightarrow$ $(X,\circ)$ has a left (right) identity element e_L (e_R) and, for each $a\in X$, there exists a left (right) inverse a_L^{-1} (a_R^{-1}) of a.*

Proof. We shall give the proof of the required equivalency for only the case where $(X,\circ)$ has a left identity e_L and, for each $a\in X$, there exists a left inverse a_L^{-1} of a. In a fashion similar to that of this proof, the proof of the required equivalency for other case can easily be constructed.

The implication $(\Rightarrow)$ is obvious. To confirm the converse implication $(\Leftarrow)$, let us suppose that the vague semigroup $(X,\circ)$ w.r.t. fuzzy equalities $E_{X\times X}$ on $X\times X$ and E_X on X possesses a left identity element e_L, and, for each $a\in X$, there exists a left inverse a_L^{-1} of a. Since $\circ$ is a fuzzy function, for $a\in X$, $\exists u$, $\exists v$, $\exists w$, $\exists r$, $\exists t\in X$, such that

$$\mu_\circ\big(a,a_L^{-1},u\big) = \mu_\circ(u,u,v) = \mu_\circ(a,e_L,w) = \mu_\circ(u,a,r)$$
$$= \mu_\circ\big(w,a_L^{-1},t\big) = 1.$$

Then using the hypothesis and (VG.1) we observe that

$$\mu_\circ\big(a_L^{-1},a,e_L\big) \wedge \mu_\circ(a,e_L,w) \wedge \mu_\circ\big(a,a_L^{-1},u\big) \wedge \mu_\circ(u,a,r)$$
$$= 1 \leq E_X(r,w), \quad \text{i.e.,}$$
$$r = w, \quad \text{i.e.,} \quad \mu_\circ(u,a,w) = 1$$

and

$$\mu_\circ\big(e_L,a_L^{-1},a_L^{-1}\big) \wedge \mu_\circ\big(a,a_L^{-1},u\big) \wedge \mu_\circ(a,e_L,w) \wedge \mu_\circ\big(w,a_L^{-1},t\big)$$
$$= 1 \leq E_X(t,u), \quad \text{i.e.,}$$
$$t = u, \quad \text{i.e.} \quad \mu_\circ\big(w,a_L^{-1},u\big) = 1.$$

Therefore,

$$\mu_\circ\big(a,a_L^{-1},u\big) \wedge \mu_\circ(u,u,v) \wedge \mu_\circ(u,a,w) \wedge \mu_\circ\big(w,a_L^{-1},u\big)$$
$$= 1 \leq E_X(v,u), \quad \text{i.e.,}$$
$$\mu_\circ(u,u,u) = 1.$$

From Proposition 3.5 it follows that $u = e_L$. Thus

$$\mu_\circ\left(a, a_L^{-1}, e_L\right) = 1. \tag{5}$$

Now let us confirm that $\mu_\circ(a, e_L, a) = 1$ for each $a \in X$, i.e., e_L is also a right identity element of the vague semigroup $(X, \circ)$, i.e., e_L is a two-sided identity element of $(X, \circ)$, i.e., (VG.2) is satisfied by $(X, \circ)$. For $a \in X$, it is obvious that $\exists u \in X$ such that $\mu_\circ(a, e_L, u) = 1$. Then considering the hypothesis and (5) we see that

$$\begin{aligned} &\mu_\circ\left(a_L^{-1}, a, e_L\right) \wedge \mu_\circ(a, e_L, u) \wedge \mu_\circ\left(a, a_L^{-1}, e_L\right) \wedge \mu_\circ(e_L, a, a) \\ &\quad = 1 \le E_X(u, a), \quad \text{i.e.,} \\ &\qquad u = a, \quad \text{i.e.} \quad \mu_\circ(a, e_L, a) = 1. \end{aligned}$$

Since e_L is a two-sided identity element of $(X, \circ)$, considering the hypothesis and (5), it is straightforward that (VG.3) is satisfied. Hence $(X, \circ)$ is a vague group.

THEOREM 3.9. *Let $(X, \circ)$ w.r.t a fuzzy equality $E_{X \times X}$ on $X \times X$ and a fuzzy equality E_X on X be a vague semigroup. Then*

$$\begin{aligned} &(X, \circ) \textit{ is a vague group} \\ &\quad \Leftrightarrow \Big(\big[(\forall a, \forall b \in X)(\exists x \in X)(\mu_\circ(a, x, b) = 1)\big] \quad \textit{and} \\ &\qquad \big[(\forall a, \forall b \in X)(\exists y \in X)(\mu_\circ(y, a, b) = 1)\big]\Big). \end{aligned}$$

Proof. $(\Rightarrow)$ Let $(X, \circ)$ w.r.t. $E_{X \times X}$ and E_X be a vague group. Then, for $a, b \in X$, $\exists x$, $\exists u \in X$ such that $\mu_\circ(a^{-1}, b, x) = \mu_\circ(a, x, u) = 1$. Now we may write

$$\begin{aligned} &\mu_\circ(a^{-1}, b, x) \wedge \mu_\circ(a, x, u) \wedge \mu_\circ(a, a^{-1}, e) \wedge \mu_\circ(e, b, b) \\ &\quad = 1 \le E_X(u, b), \quad \text{i.e.,} \quad u = b, \\ &\qquad \text{i.e.,} \quad \mu_\circ(a, x, b) = 1. \end{aligned}$$

On the other hand, for $a, b \in X$, $\exists y$, $\exists v \in X$ such that $\mu_\circ(b, a^{-1}, y) = \mu_\circ(y, a, v) = 1$. Thus

$$\begin{aligned} &\mu_\circ(a^{-1}, a, e) \wedge \mu_\circ(b, e, b) \wedge \mu_\circ(b, a^{-1}, y) \wedge \mu_\circ(y, a, v) \\ &\quad = 1 \le E_X(v, b), \quad \text{i.e.,} \quad \mathrm{v} = \mathrm{b}, \quad \textit{i.e.,} \\ &\qquad \mu_\circ(y, a, b) = 1. \end{aligned}$$

($\Leftarrow$) For a given vague semigroup $(X, \circ)$ w.r.t. $E_{X\times X}$ and E_X, suppose that $\forall a$, $\forall b \in X$, $\exists x \in X$, such that $\mu_\circ(a, x, b) = 1$ and $\forall a$, $\forall b \in X$, $\exists y \in X$ such that $\mu_\circ(y, a, b) = 1$. Then, by this assumption, for a fixed $m \in X$ and for an arbitrary $a \in X$, there exist $e^*, x \in X$ such that

$$\mu_\circ(e^*, m, m) = \mu_\circ(m, x, a) = 1.$$

Furthermore, since $\circ$ is a fuzzy function, it is obvious that, for $a \in X$, there exists $u \in X$ such that $\mu_\circ(e^*, a, u) = 1$. Therefore we obtain the following inequality:

$$\begin{aligned} &\mu_\circ(m, x, a) \wedge \mu_\circ(e^*, a, u) \wedge \mu_\circ(e^*, m, m) \wedge \mu_\circ(m, x, a) \\ &\quad = 1 \le E_X(u, a), \quad \text{i.e.,} \quad u = a, \\ &\qquad\qquad \text{i.e.,} \quad \mu_\circ(e^*, a, a) = 1, \end{aligned}$$

i.e., e^* is a left identity element of $(X, \circ)$.

On the other hand, by the hypothesis, for each $a \in X$, there exists $w \in X$ such that $\mu_\circ(w, a, e^*) = 1$, i.e., w is a left inverse of a. Hence the required result immediately follows from Theorem 3.8.

4. VAGUE SUBGROUPS AND VAGUE HOMOMORPHISMS

DEFINITION 4.1. Let $(X, \circ)$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X be a vague group, and let H be a nonempty and crisp subset of X that is vague closed under $\circ$. Then H is said to be a vague subgroup of X iff $(H, \circ|_{H\times H\times H})$ is itself a vague group.

THEOREM 4.2. *Let $(X, \circ)$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X be a vague group. Then, a nonempty and crisp subset H of X is a vague subgroup of $X \Leftrightarrow (\forall a, \forall b \in H)(\forall c \in X)(\mu_\circ(a, b^{-1}, c) = 1 \Rightarrow c \in H)$.*

Proof. ($\Rightarrow$) Let H be a vague subgroup of X. Denoting an identity of the vague groups $(X, \circ)$ and $(H, \circ|_{H\times H\times H})$ by e_X and e_H, respectively, it is obvious that, for $a \in H$,

$$\mu_{\circ|H\times H\times H}(a, e_H, a) = \mu_\circ(a, e_H, a) = \mu_\circ(a, e_X, a) = 1.$$

From Theorem 3.6 we may write

$$\mu_\circ(a, e_H, a) \wedge \mu_\circ(a, e_X, a) = 1 \le E_X(e_H, e_X), \quad \text{i.e.,} \quad e_H = e_X.$$

Similarly, for $b \in H$, denoting the inverse of b in H w.r.t. $\circ|_{H\times H\times H}$ by b_H^{-1}, it is obvious that

$$\mu_{\circ|H\times H\times H}\big(b, b_H^{-1}, e_H\big) = \mu_\circ\big(b, b_H^{-1}, e_H\big) = \mu_0\big(b, b^{-1}, e_H\big) = 1$$

and

$$\mu_\circ\big(b, b_H^{-1}, e_H\big) \wedge \mu_\circ\big(b, b^{-1}, e_H\big) = 1 \leq E_X\big(b_H^{-1}, b^{-1}\big), \text{ i.e., } b_H^{-1} = b^{-1}.$$

Now, for $a, b \in H$ and $c \in X$, if $\mu_\circ(a, b^{-1}, c) = 1$, then since $a, b^{-1} \in H$ and H is vague closed under $\circ$, we directly get $c \in H$.

($\Leftarrow$) Suppose that $(\forall a,\ \forall b \in H)(\forall c \in X)(\mu_\circ(a, b^{-1}, c) = 1 \Rightarrow c \in H)$. Since $H \neq \emptyset$, there exists $u \in H$. Since $\mu_\circ(u, u^{-1}, e_X) = 1$, and by the hypothesis, we have $e_X \in H$. Then, for $a, e_X \in H$, since $\mu_\circ(e_X, a^{-1}, a^{-1}) = 1$, and by the hypothesis, we directly get $a^{-1} \in H$. Now we prove that the nonempty subset H of X is vague closed under $\circ$.

For $a, b \in H$, $c \in X$, let $\mu_\circ(a, b, c) = 1$. Considering the equality $\mu_\circ(a, b, c) = \mu_\circ(a, (b^{-1})^{-1}, c)$ and since $b^{-1} \in H$, the hypothesis directly implies $c \in H$. Thus H is vague closed under $\circ$.

Since $(X, \circ)$ is a vague group, it can easily be seen that (VG.1−VG.3) are satisfied by $(H, \circ|_{H\times H\times H})$ w.r.t. the fuzzy equalities $E_{X\times X}^{H\times H}$ on $H \times H$ and E_X^H on H. Hence H is a vague subgroup of X.

THEOREM 4.3. *Let $(X, \circ)$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X be a vague group. Then, a nonempty and crisp subset H of X is a vague subgroup of X iff*

(i) *H is vague closed under $\circ$.*

(ii) *For each $a \in H$, $a^{-1} \in H$.*

Proof. The proof of this result can be obtained in a manner similar to that of the classical case [4]. For this reason, it is omitted here.

COROLLARY 4.4. *Let $(X, \circ)$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X be a vague group. If $\{H_i: i \in I\}$ is a nonempty family of vague subgroups of X such that $\bigcap_{i\in I} H_i \neq \emptyset$, then $\bigcap_{i\in I} H_i$ is a vague subgroup of X.*

DEFINITION 4.5. Let $(X, \circ)$ be a vague group and let H be a crisp and nonempty subset of X. Let $\{H_i: i \in I\}$ be the family of all vague subgroups of X containing H. Then $\bigcap_{i\in I} H_i$ is called the vague subgroup of X generated by the set H, and it is denoted by $\langle H \rangle$.

DEFINITION 4.6. Let $(X, \circ)$ and $(Y, \odot)$ be two vague semigroups. A function (in the classical sense) $\Phi: X \to Y$ is called a *vague homomorphism* iff

$$\mu_\circ(a, b, c) \leq \mu_\odot(\Phi(a), \Phi(b), \Phi(c)), \qquad \forall a, \forall b, \forall c \in X.$$

PROPOSITION 4.7. *Let* $(X, \circ)$ *w.r.t. fuzzy equalities* $E_{X \times X}$ *on* $X \times X$, E_X *on* X *and* $(Y, \odot)$ *w.r.t. fuzzy equalities* $E_{Y \times Y}$ *on* $Y \times Y$, E_Y *on* Y, *be two vague groups, and* $\Phi\colon X \to Y$ *a vague homomorphism. Then*

(i) *If* e_X *and* e_Y *are identities of* $(X, \circ)$ *and* $(Y, \odot)$, *respectively, then* $\Phi(e_X) = e_Y$.

(ii) *For each* $a \in X$, $\Phi(a)^{-1} = \Phi(a^{-1})$.

Proof. (i) Let e_X and e_Y be identities of $(X, \circ)$ and $(Y, \odot)$, respectively. For $a \in X$, since $\mu_0(a, e_X, a) = 1$ and $\Phi\colon X \to Y$ is a vague homomorphism, we have $\mu_{\odot}(\Phi(a), \Phi(e_X), \Phi(a)) = 1$. Furthermore, $\mu_{\odot}(\Phi(a), e_Y, \Phi(a)) = 1$ and, by Theorem 3.6,

$$\mu_{\odot}\big(\Phi(a), \Phi(e_X), \Phi(a)\big) \wedge \mu_{\odot}\big(\Phi(a), e_Y, \Phi(a)\big) = 1 \le E_Y\big(\Phi(e_X), e_Y\big),$$

$$\text{i.e.,} \qquad \Phi(e_X) = e_Y.$$

(ii) For each $a \in X$, since $\mu_{\circ}(a, a^{-1}, e_X) = 1$ and $\Phi\colon X \to Y$ is a vague homomorphism, we have $\mu_{\odot}(\Phi(a), \Phi(a^{-1}), \Phi(e_X)) = 1$. From (i) we may write $\mu_{\odot}(\Phi(a), \Phi(a^{-1}), e_Y) = 1$. Applying Theorem 3.6 we find

$$\mu_{\odot}\big(\Phi(a), \Phi(a^{-1}), e_Y\big) \wedge \mu_{\odot}\big(\Phi(a), \Phi(a)^{-1}, e_Y\big)$$

$$= 1 \le E_Y\big(\Phi(a)^{-1}, \Phi(a^{-1})\big), \quad \text{i.e.,}$$

$$\Phi(a)^{-1} = \Phi(a^{-1}).$$

DEFINITION 4.8. Let $(X, \circ)$ w.r.t. fuzzy equalities $E_{X \times X}$ on $X \times X$, E_X on X and $(Y, \odot)$ w.r.t. fuzzy equalities $E_{Y \times Y}$ on $Y \times Y$, E_Y on Y be two vague groups, and let $\Phi\colon X \to Y$ be a vague homomorphism. The crisp set $\{a \in X\colon \Phi(a) = e_Y\}$ is called a *vague kernel* of Φ, and it is denoted by $V \ker \Phi$.

DEFINITION 4.9. Let E_X and E_Y be, respectively, fuzzy equalities on X and Y. A function $g\colon X \to Y$ is said to be vague injective w.r.t. E_X and E_Y if

$$E_Y(g(a), g(b)) \le E_X(a, b), \qquad \forall a, \forall b \in X.$$

It can be noted that a vague injective function is obviously injective in the classical sense.

PROPOSITION 4.10. *Let* $(X, \circ)$ *w.r.t. fuzzy equalities* $E_{X \times X}$ *on* $X \times X$, E_X *on* X *and* $(Y, \odot)$ *w.r.t. fuzzy equalities* $E_{Y \times Y}$ *on* $Y \times Y$, E_Y *on* Y *be two vague groups, and let* $\Phi\colon X \to Y$ *be a vague homomorphism. Let* e_X *be the* *identity of the vague group* $(X, \circ)$. *Then*

(i) Φ *is injective* $\Leftrightarrow V \ker \Phi = \{e_X\}$.

(ii) *If* $\circ$ *is transitive of first order and* Φ *is vague injective and surjective, then the function* $\Phi^{-1}: Y \to X$ *is a vague homomorphism.*

Proof. The proof of (i) is the analogue of classical case [4].

(ii) Let Φ be a vague injective and surjective function, i.e., Φ is bijective and vague injective. Furthermore, let us suppose that $\circ$ is transitive of first order. For $u, v, w \in Y$, our aim is to show that $\mu_{\odot}(u, v, w) \le \mu_{\circ}(\Phi^{-1}(u), \Phi^{-1}(v), \Phi^{-1}(w))$.

For $u, v, w \in Y$, $\exists a$, $\exists b$, $\exists c \in X$ such that $a = \Phi^{-1}(u)$, $b = \Phi^{-1}(v)$, and $\mu_{\circ}(a, b, c) = 1$. Since Φ is a vague homomorphism we may write

$$\mu_{\odot}(\Phi(a), \Phi(b), \Phi(c)) = \mu_{\odot}(u, v, \Phi(c)) = 1.$$

Then since $\odot$ is a fuzzy function and applying the condition (F.2) we may write

$$\mu_{\odot}(u, v, w) = \mu_{\odot}(u, v, w) \wedge \mu_{\odot}(u, v, \Phi(c)) \le E_Y(w, \Phi(c)). \quad (6)$$

Using the bijectivity and vague injectivity of Φ and considering the first-order transitivity of $\circ$, we observe that

$$\begin{aligned} E_Y(w, \Phi(c)) &= E_Y\big(\Phi(\Phi^{-1}(w)), \Phi(c)\big) \le E_X\big(\Phi^{-1}(w), c\big) \\ &= \mu_{\circ}\big(\Phi^{-1}(u), \Phi^{-1}(v), c\big) \wedge E_X\big(\Phi^{-1}(w), c\big) \qquad (7) \\ &\le \mu_{\circ}\big(\Phi^{-1}(u), \Phi^{-1}(v), \Phi^{-1}(w)\big). \end{aligned}$$

The required inequality is acquired directly from (6) and (7).

THEOREM 4.11. *Let* $(X, \circ)$ *w.r.t. fuzzy equalities* $E_{X \times X}$ *on* $X \times X$, E_X *on* X *and* $(Y, \odot)$ *w.r.t. fuzzy equalities* $E_{Y \times Y}$ *on* $Y \times Y$, E_Y *on* Y *be two vague groups, and let* $\Phi: X \to Y$ *be a vague homomorphism. Then*

(i) $V \ker \Phi$ *is a vague subgroup of* X.

(ii) *For a vague subgroup* A *of* X, $\Phi(A)$ *is a vague subgroup of* Y.

(iii) *For a vague subgroup* B *of* Y, $\Phi^{-1}(B)$ *is a vague subgroup of* X.

Proof. (i) For $a, b \in V \ker \Phi$, $c \in X$, let $\mu_{\circ}(a, b^{-1}, c) = 1$. Then

$$\Phi(a) = \Phi(b), \quad \text{i.e.,} \quad \Phi(a)^{-1} = \Phi(b)^{-1}, \quad \text{i.e.,}$$

$$\mu_{\odot}\big(\Phi(a), \Phi(a)^{-1}, e_Y\big) = \mu_{\odot}\big(\Phi(a), \Phi(b)^{-1}, e_Y\big) = 1.$$

Since Φ is a vague homomorphism and using Proposition 4.7(ii), the assumption $\mu_{\circ}(a, b^{-1}, c) = 1$ implies

$$\mu_{\odot}\big(\Phi(a), \Phi(b^{-1}), \Phi(c)\big) = \mu_{\odot}\Phi(a), \Phi(b)^{-1}, \Phi(c) = 1.$$

Therefore,

$$\begin{aligned} &\mu_{\odot}\big(\Phi(a), \Phi(b)^{-1}, e_Y\big) \wedge \mu_{\odot}\big(\Phi(a), \Phi(b)^{-1}, \Phi(c)\big) \\ &\qquad = 1 \leq E_X\big(\Phi(c), e_Y\big), \quad \text{i.e.,} \\ &\qquad\qquad \Phi(c) = e_Y, \quad \text{i.e.,} \quad c \in V \ker \Phi. \end{aligned}$$

Hence the required result is straightforward from Theorem 4.2.

(ii) Let A be a vague subgroup of X. For $a, b \in \Phi(A)$, $c \in Y$, let $\mu_{\odot}(a, b^{-1}, c) = 1$. Then, $\exists u$, $\exists v \in A$, $\exists w \in X$, such that $\Phi(u) = a$, $\Phi(v) = b$, and $\mu_{\circ}(u, v^{-1}, w) = 1$. A is a vague subgroup of X and, by Theorem 4.2, we have $w \in A$, i.e., $\Phi(w) \in \Phi(A)$. Furthermore, since Φ is a vague homomorphism and considering Proposition 4.7(ii), we may write

$$\begin{aligned} \mu_{\odot}\big(\Phi(u), \Phi(v^{-1}), \Phi(w)\big) &= \mu_{\odot}\big(\Phi(u), \Phi(v)^{-1}, \Phi(w)\big) \\ &= \mu_{\odot}\big(a, b^{-1}, \Phi(w)\big) = 1. \end{aligned}$$

Therefore we get

$$\begin{aligned} \mu_{\odot}\big(a, b^{-1}, \Phi(w)\big) \wedge \mu_{\circ}(a, b^{-1}, c) = 1 \leq E_Y\big(\Phi(w), c\big) = 1,& \\ \text{i.e.,} \quad c = \Phi(w) \in \Phi(A).& \end{aligned}$$

Hence the required result follows from Theorem 4.2 at once.

(iii) This can be verified in a fashion similar to the proof of (ii).

REFERENCES

1. M. Akgul, Some properties of fuzzy groups, *J. Math Anal. Appl.* **133** (1988), 93–100.
2. P. S. Das, Fuzzy groups and level subgroups, *J. Math. Anal. Appl.* **84** (1981), 264–269.
3. M. Demirci, Fuzzy function and their fundamental properties, *Fuzzy Sets and Systems*, to appear.
4. T. W. Hungerford "Algebra," Holt, Rinehart and Winston, New York, 1974.
5. N. P. Mukherjee and P. Bhattacharya, Fuzzy normal subgroups and fuzzy costs, *Inform. Sci.*, **34** (1984), 225–239.
6. A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971), 512–517.