Theta Series of Ternary Quadratic Forms

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We establish the linear independence of the theta series arising from the classes of even positive definite ternary forms of discriminant $2p^2$ which have a nontrivial automorphism group. $p$ is assumed to be an odd prime number. We also consider even positive ternary forms over $\mathbb{Z}[\frac{1 + \sqrt{-1}}{2}]$ of discriminant 2. In the case $p \equiv 5 \pmod{8}$, we show that those forms that represent 2 have independent generalized theta series.

Unless otherwise indicated, we shall adopt the terminology and notations of [7]. All quadratic forms will be even positive definite and all genera will be uniquely determined by their discriminants. Let $G(n, d)$ denote the genus of $n$-ary quadratic forms over $\mathbb{Z}$ of discriminant $d$, and $G(n, d)$ the corresponding genus over the ring of integers of $\mathbb{Q}(\sqrt{p})$, where $p$ throughout will be an arbitrarily fixed prime $\equiv 1 \pmod{4}$. In an earlier paper [1], Hsia and the author established a few results on the linear independence of theta series associated with certain classes of forms in $G(3, 2p)$ and $G(4, p)$. These genera, as it turns out, are not the only instances to which our arithmetic method applies, and similar results can be obtained (see [2]) for other genera as well as for theta series over number fields. The objective here is to investigate the theta series arising from the forms in the ternary genera $G(3, 2p^2)$ and $G(3, 2)$.

Our approach is similar to that of [1, 2]. Some technical features must be overcome by analyzing the arithmetic structures of the forms in each genus. In the case of $G(3, 2p^2)$, this is accomplished by establishing a one-to-one correspondence between its classes and those in $G(3, 2p)$. Such a correspondence has been proved in [8] via the arithmetic of quaternion algebra. We present here a lattice-theoretic proof which will be useful to our study of the theta series. The second half of this paper is devoted to the study of the forms in $G(3, 2)$ and their associated theta series. For this we make use of the known structures of the forms in $G(4, 1)$, which was treated in [2]. We shall adopt the geometric language of quadratic spaces and lattices in the following discussion.
1. Even Positive Definite Ternary Lattices of Discriminant $2p^2$

We fix a prime $p \equiv 1 \pmod{4}$. Let $K$ be an even positive definite ternary lattice of discriminant $2p^2$. Then $K$ is $2\mathbb{Z}$-maximal and the Hasse symbol of $\mathbb{Q}K$ at 2 is $-1$. Since $S_r(\mathbb{Q}K)=1$ for all primes $r \neq 2$, $p$, we have $S_p(\mathbb{Q}K)=-1$ also by the Hilbert reciprocity law. It follows that there is just one genus $G(3, 2p^2)$ of such ternary lattices. Locally at the prime $p$, we have

$$K_p \cong \langle -2\Delta \rangle \perp \langle p \rangle \perp \langle -\Delta p \rangle,$$

where $\Delta$ is a nonsquare $p$-adic unit.

**Proposition 1.1.** Let $K \in G(3, 2p^2)$. Then $K$ contains a unique sublattice $\bar{K}$ such that $[K: \bar{K}]=p$ and $n(\bar{K}_p)=p\mathbb{Z}_p$.

**Proof.** Locally at $p$, $K_p=\mathbb{Z}_p x_1 \perp \mathbb{Z}_p x_2 \perp \mathbb{Z}_p x_3$, where $Q(x_1)=-2\Delta$, $Q(x_2)=p$, and $Q(x_3)=-\Delta p$. By lattice theory, there is a lattice $\bar{K}$ such that

$$\bar{K}_r = \begin{cases} K_r, & r \neq p, \\ (\mathbb{Z}_p(p x_1) \perp \mathbb{Z}_p x_2 \perp \mathbb{Z}_p x_3), & r = p. \end{cases}$$

Clearly, $K$ contains $\bar{K}$ with $[K: \bar{K}]=p$ and $n(\bar{K}_p)=p\mathbb{Z}_p$. To show the uniqueness, let $J$ be a sublattice of $K$ with the stated properties. Then $J_r=K_r=\bar{K}_r$ for all $r \neq p$. It suffices to show that $J_p=\bar{K}_p$. If $u$ is any vector in $J_p$, then $u=\alpha x_1 + \beta x_2 + \gamma x_3$ for some $\alpha, \beta, \gamma \in \mathbb{Z}_2$. Since $Q(u)=-2\Delta x^2 + \beta^2 p - \Delta \gamma^2 p \in p\mathbb{Z}_p$, we have $\alpha \in p\mathbb{Z}_p$, hence $u \in K_p$. It follows at once that $J=K$.

**Remark 1.2.** If the lattice $\bar{K}$ is scaled by a factor $p^{-1}$ then one obtains a lattice $K'$ in $G(3, 2p)$. Conversely, given $K' \in G(3, 2p)$, we let $\bar{K}=(K')^p$. Then $\bar{K}$ has discriminant $2p^4$ and locally at $p$, $\bar{K}_p \cong \langle p \rangle \perp \langle -\Delta p \rangle \perp \langle -2\Delta p^2 \rangle$. By a construction which is the inverse process of that in Proposition 1.1, one obtains a unique lattice $K$ in $G(3, 2p^2)$ containing $\bar{K}$ such that $[K: \bar{K}]=p$. We shall reserve the symbols $K$, $\bar{K}$ and $K'$ for the lattices just mentioned.

**Corollary 1.3.** The mapping $K \mapsto K'$ induces a one-to-one correspondence between the classes in $G(3, 2p^2)$ and those in $G(3, 2p)$.

**Remark 1.4.** It is easy to see from the proof of Proposition 1.1 that any vector $u$ in $K$ of length $Q(u)$ which is divisible by $p$ actually lies in $\bar{K}$. In particular, if $u \in K$ has length $2p$, then $u \in \bar{K}$, hence the number of representations of $2p$ by $K$ (denoted by $a_k(2p)$) is the same as those by $\bar{K}$. On the
other hand, it is clear that \( a_K(2p) = a_K(2) \), since \( K' \) is just the lattice \( \bar{K} \) scaled by \( p^{-1} \). It follows therefore that \( a_K(2p) = a_K(2) \).

**Proposition 1.5.** Let \( K, \bar{K}, K' \) be as in Remark 1.2. Then \( O(K) = O(\bar{K}) = O(K') \).

**Proof.** Since \( \bar{K} \) and \( K' \) have the same underlying lattice structure, it is clear that \( O(\bar{K}) = O(K') \). To see that \( O(K) = O(\bar{K}) \), let \( \sigma \in O(K) \). Since both \( \sigma \bar{K} \) and \( \bar{K} \) are sublattices of index \( p \) in \( K \) with their norms contained in \( p\mathbb{Z} \), it follows by Proposition 1.1 that \( \sigma \bar{K} = \bar{K} \). Conversely, if \( \sigma \in O(\bar{K}) \), then \( \sigma K \) contains \( \bar{K} \) as a sublattice of index \( p \). Therefore \( \sigma K = K \) by the uniqueness of such lattice (see Corollary 1.3).

**Remark 1.6.** It is now clear from the structure of \( O(K') \) as given in [4] that \( O(K) \) is generated by symmetries and \( \pm 1 \). A typical symmetry of \( O(K) \) is of the form \( S_u \), where \( u \) is a vector in \( K \) of length \( 2p \). The order of \( O(K) \) is determined by the number of such vectors of length \( 2p \). In particular, \( |O(K)| = 2, 4, 8, 12 \) accordingly as \( a_K(2p) = 0, 2, 4, 6 \), respectively.

In the following, \( G'(n, d) \) will denote those lattices in \( G(n, d) \) which have an improper automorphism. The same notation will also apply to the genus \( \mathcal{G}(n, d) \). By Remark 1.6, \( G'(3, 2p^2) \) are exactly those lattices in \( G(3, 2p^2) \) which represent \( 2p \). In view of the close relationship between these lattices and their counterparts in \( G'(3, 2p) \), we shall now examine their theta series. Specifically, we shall prove their linear independence. Recall in the proof of the independence for \( G'(3, 2p) \) that a key ingredient is the existence of a prime \( q_i \) for each class \( K_i \) such that \( a_{\bar{K}_i}(q_i) \equiv 2 \pmod{4} \) but \( a_{K_j}(q_i) \equiv 0 \pmod{4} \) for \( j \neq i \) (see [1, p. 155]), where \( \bar{K}_i \) is the reciprocal of \( K_i \) (i.e., the dual \( (K')^\# \) scaled by the factor \( dK'/\Omega \), where \( \Omega \) is the g.c.d. of the entries in the adjoint matrix of \( K' \)). It would now be sufficient, in order to prove the independence for \( G'(3, 2p^2) \), to show that \( a_{\bar{K}_i}(q_i) = a_{K_j}(p^2q_i) \) for all \( i, j \), where \( K_i \) and \( K_j \) are corresponding classes in \( G'(3, 2p) \) and \( G'(3, 2p^2) \). This can be accomplished by linking the representations of an integer by a form to the representations of codimension 1 forms by its reciprocal. For a detailed statement of this classical duality, see Proposition 1.2.3, [2]. In particular, if \( K', K \) are corresponding lattices in \( G'(3, 2p) \) and \( G'(3, 2p^2) \), and \( q \) a prime \( \neq p \), then

\[
\frac{1}{4}a_{K'}(q) = \text{the number of (primitive) binary sublattices of } K'
\]

of discriminant \( q \)

and

\[
\frac{1}{4}a_K(p^2q) = \text{the number of primitive binary sublattices of } K
\]

of discriminant \( p^2q \).
Our proof will be completed if we show that these two numbers coincide. It is clear that every binary sublattice of discriminant $q$ in $K'$ gives rise to a sublattice of discriminant $p^2q$ in $K$ by scaling. Conversely, every binary sublattice of discriminant $p^2q$ in $K$ comes from one of discriminant $q$ in $K'$:

**Proposition 1.7.** Let $K$, $\overline{K}$, $K'$ be as in Remark 1.2. Suppose $K$ contains a binary sublattice $M$ of discriminant $p^2q$, where $q$ is an odd prime $\not\equiv p$. Then $M \subset \overline{K}$, hence $K'$ contains the lattice $N = M_{p^{-1}}$.

**Proof.** We first show that $M_p$ is $p$-modular. Suppose this is not the case; then $M_p = \mathbb{Z}_p \perp \mathbb{Z}_p \mathbf{v}$, where $Q(u)$ is a $p$-adic unit and $Q(v) \in p^2 \mathbb{Z}_p$. It follows that $u$ splits $K_p$; hence $K_p = \mathbb{Z}_p \perp B$ for some binary $B$. It is easy to see that $B$ is $p$-modular and $Q_B$ is anisotropic. By [7, 63:15], $B$ represents only elements of odd order. This is absurd, since $v \in B$. Now we have $nM \subseteq 2p\mathbb{Z}$, hence by Remark 1.4, $M \subset \overline{K}$.

Summarizing the above, we obtain:

**Theorem 1.8.** The theta series $\Theta_K(z)$ for lattices $K$ coming from the classes in $G'(3, 2p^2)$ are linearly independent over $\mathbb{C}$.

**Remark 1.9.** The results in this section remain valid for the case $p \equiv 3 \pmod{4}$. The principal difference here is the existence of decomposable forms. Some modifications to the proof of Theorem 1.8 are therefore needed to deal simultaneously with such forms and the indecomposable ones.

2. **Even Positive Definite Ternary Lattices of Discriminant 2 over $\mathbb{Q}(\sqrt{p})$**

Fix a prime $p \equiv 1 \pmod{4}$. Let $F = \mathbb{Q}(\sqrt{p})$, $\mathcal{O}$ be the ring of integers of $F$ and $\mathcal{O}^\times$ be the group of units of $\mathcal{O}$. Let $V$ be a positive definite ternary space of discriminant 2 over $F$. We assume that $V$ admits an even lattice $K$ over $\mathcal{O}$ such that $K_p$ is unimodular at all nondyadic primes and $dK_p = 2$ for dyadic primes. Let $\mathfrak{A}$ be the quaternion algebra of discriminant $p^2$ over $\mathbb{Q}$ and $\mathfrak{A}_F = \mathfrak{A} \otimes F$; then $V$ is isometric to the pure part of $\mathfrak{A}_F$, where the quadratic map on $\mathfrak{A}_F$ is $2N(-)$, $N$ the reduced norm. The $\mathfrak{O}$-maximal lattices $K$ on $V$ are free (see [3, Appendix]) and constitute a single genus $G(3, 2)$. They are closely related to the unimodular lattices on $\mathfrak{A}_F$, specifically, to those lattices $L \in G'(4, 1)$ which contain a nontrivial root system. Some basic facts about these lattices may be found in [2]. A vector $e$ will be called a minimal vector if it has quadratic length 2. By a standard construction of even quaternary lattices from even ternary ones (see Proposition I.1.1, [2]), there is associated to every $K \in G(3, 2)$ a unique lattice $L$ in $G'(4, 1)$ containing $\mathfrak{O}e \perp K$. Conversely, if $L \in G'(4, 1)$ and $e$ is a
minimal vector in \( L \), then \( K = (\mathcal{O}e)^\perp \in \mathcal{B}(3, 2) \) (here the orthogonal complement is taken in \( L \)). We shall assume in the following that \( p > 5 \).

**Proposition 2.1.** Let \( L \) be any lattice in \( \mathcal{B}'(4, 1) \); then the minimal vectors in \( L \) are transitively permuted by its automorphism group \( O(L) \).

**Proof.** By the structures of 2-lattices over real quadratic integers as given in [6], any two minimal vectors \( e_1, e_2 \) in \( L \) are either orthogonal or they satisfy \( B(e_1, e_2) = \pm 1 \). If \( e_1 \) and \( e_2 \) are orthogonal, then there exist a vector \( u \) in \( (\mathcal{O}e_1 \perp \mathcal{O}e_2)^\perp \) and a principal prime ideal \( \mathfrak{q} = \mathcal{O}\pi \) such that \( Q(u) = 2\pi \). It follows that \( L \supset \mathcal{O}e_1 \perp \mathcal{O}e_2 \perp \mathcal{O}u \perp \mathcal{O}v \) for some vector \( v \) of length \( Q(v) = 2\pi \). The mapping \( \sigma \) defined by \( \sigma(e_1) = e_2, \sigma(e_2) = e_1, \sigma(u) = v \) and \( \sigma(v) = -u \) is an automorphism of \( L \). If \( e_1 \) and \( e_2 \) are not orthogonal, then we may assume that \( B(e_1, e_2) = 1 \). Clearly, the automorphism \( S_{e_1} \cdot -e_2 \) permutes \( e_1 \) and \( e_2 \).

It follows from Proposition 2.1 that the mapping \( K \mapsto L \supset \mathcal{O}e \perp K \) induces a one-to-one correspondence between the classes in \( \mathcal{B}(3, 2) \) and those in \( \mathcal{B}'(4, 1) \). The type of roots system of \( K \) can be easily determined by the type of the corresponding lattice \( L \). In fact, \( K \) is of type \( \phi, A_1, A_2 \) or \( A_1 \oplus A_1 \oplus A_1 \) as the type of \( L \) is \( A_1 \) or \( A_2, A_1 \oplus A_1, A_2 \oplus A_2, \) or \( D_4 \), respectively.

**Proposition 2.2.** Let \( K \in \mathcal{B}(3, 2) \) and \( L \supset \mathcal{O}e \perp K \) be the corresponding lattice in \( \mathcal{B}'(4, 1) \); then \( |O(K)| = \sqrt{|O(L)|} \).

**Proof.** Since \( O(L) \) permutes the minimal vectors of \( L \), we have for any minimal vector \( e' \) in \( L \) an automorphism \( \sigma \in O(L) \) such that \( \sigma(e) = e' \) and \( \sigma K = K' \), where \( K' = (\mathcal{O}e')^\perp \). Conversely, if \( \eta \) is an isometry of \( K \) onto \( K' \) and \( \rho e = e' \), then \( \sigma = \rho \perp \eta \) is an automorphism of \( L \), since \( L \) is the only lattice in \( \mathcal{B}'(4, 1) \) containing \( \mathcal{O}e' \perp K' \). Now the number of isometries from \( K \) onto \( K' \) is just \(|O(K)|\), hence \(|O(L)| = a_L(2)|O(K)| \). But \( a_L(2) = \sqrt{|O(L)|} \) [2, Corollary II.2.6], hence, our results follows.

**Remark 2.3.** Let \( K \in \mathcal{B}(3, 2) \) be a lattice with an empty root system. Then \( K \) either comes from a lattice \( L \) of type \( A_1 \) or from one of type \( A_2 \). In the former case, the unit group of \( K \) is trivial by Proposition 2.2 and [2, Corollary II.2.6], whereas in the latter \(|O(K)| = 6 \). It follows that the unit group of \( K \) is not necessarily determined by its roots system. We denote by \( \mathcal{B}'(3, 2) \) those lattices which contain a nontrivial root system. By [2, Corollary II.2.6], again, \(|O(K)| = 4, 12 \) or \( 24 \) according as the type of \( K \) is \( A_1, A_2 \) or \( A_1 \oplus A_1 \oplus A_1 \).

Let \( K \) be a lattice in \( \mathcal{B}'(3, 2) \) and \( e \) a minimal vector of \( K \). Denote by \( M \) the orthogonal complement of \( e \) in \( K \). It is easy to show that \( M \) is free with discriminant \( 4 \) (see [5, Appendix]). Locally at each dyadic prime \( p \), \( M_p \) is
proper $p$-modular. By Lemma 1.6, there exist a vector $u$ in $M$ and a principal prime ideal $q = \mathcal{O}_F \pi$ of $F$ such that $Q(u) = 2\pi$. The orthogonal complement of $u$ in $M$ is again a free lattice $\mathcal{O}v$ with $Q(v) = 2\pi$. Therefore we have $K \supseteq \mathcal{O}e \perp \mathcal{O}u \perp \mathcal{O}v$. Define an isometry $\tau$ on $V$ by $\tau(e) = e$, $\tau(u) = v$ and $\tau(v) = -u$.

**Proposition 2.4.** Let $p \equiv 5 \pmod{8}$, then there are four lattices in $\mathcal{G}'(3, 2)$ which contain $\mathcal{O}e \perp \mathcal{O}u \perp \mathcal{O}v$, and they are transitively permuted by $S$, and $\tau$.

**Proof.** There is a unique dyadic prime $p = (2)$. Locally at $p$, there are two $2\mathcal{O}_{\mathcal{O}_p}$-maximal lattices which contain $\mathcal{O}_p e \perp \mathcal{O}_p u \perp \mathcal{O}_p v$, namely

$$(\mathcal{O}_p e \perp \mathcal{O}_p u \perp \mathcal{O}_p v) + \mathcal{O}_p \frac{1}{2}(e + au)$$

and

$$(\mathcal{O}_p e \perp \mathcal{O}_p u \perp \mathcal{O}_p v) + \mathcal{O}_p \frac{1}{2}(e + av),$$

where $a \in \mathcal{O}_p^\times$. At the prime $q = \mathcal{O}_a \pi$, there are also two $\mathcal{O}_a$-maximal lattices which contain $\mathcal{O}_a e \perp \mathcal{O}_a u \perp \mathcal{O}_a v$. They are

$$(\mathcal{O}_a e \perp \mathcal{O}_a u \perp \mathcal{O}_a v) + \mathcal{O}_a \frac{1}{\pi} (u + bv)$$

and

$$(\mathcal{O}_a e \perp \mathcal{O}_a u \perp \mathcal{O}_a v) + \mathcal{O}_a \frac{1}{\pi} (u - bv), \quad \text{where } b \in \mathcal{O}_a^\times.$$
nonisometric lattices \( \mathcal{G}(3,2) \). Each \( K_i \) contains a minimal vector \( e_i \). As before, we consider a unary free lattice \( J_i \) in the orthogonal complement of \( e_i \) with discriminant \( 2\pi_i \), where \( q_i = \theta \pi_i \) is a principal prime ideal of \( F \). If \( a_{ij} \) is the number of isometric embeddings of \( J_i \) into \( K_j \), then

\[
a_{ij} \equiv 0 \pmod{2^2}, \quad i \neq j
\]
\[
a_{ij} \equiv 0 \pmod{2}, \quad \text{but} \quad a_{ij} \not\equiv 0 \pmod{2^2}
\]

for the lattices \( K_j \) of type \( A_1 \) or \( A_2 \). If \( K_j \) is of the type \( A_1 \oplus A_1 \oplus A_1 \), then we have

\[
a_{ij} \equiv 0 \pmod{2^3}, \quad i \neq j
\]
\[
a_{ij} \not\equiv 0 \pmod{2^3}.
\]

Now let \( \sum c_j \Theta_{K_j}(z) = 0 \) be a nontrivial linear relation, where the coefficients \( c_j \) are relatively prime integers. By evaluating at each \( J_i \) and considering mod 2 and mod \( 2^2 \), we obtain \( c_j \equiv 0 \pmod{2} \) for all \( j \). This is a contradiction, and we have proved:

**Theorem 2.6.** Let \( p \equiv 5 \pmod{8} \). The generalized theta series \( \Theta_K(z) \) of degree one for lattices \( K \) coming from the classes in \( \mathcal{G}(3,2) \) are linearly independent.

**References**