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Linearity defects of face rings

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Abstract

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K, and $E = \bigwedge\langle y_1, \dots, y_n\rangle$ an exterior algebra. The *linearity defect* $\mathrm{ld}_E(N)$ of a finitely generated graded E-module N measures how far N departs from "componentwise linear". It is known that $\mathrm{ld}_E(N) < \infty$ for all N. But the value can be arbitrary large, while the similar invariant $\mathrm{ld}_S(M)$ for an S-module M is always at most n. We will show that if I_Δ (resp. J_Δ) is the squarefree monomial ideal of S (resp. E) corresponding to a simplicial complex $\Delta \subset 2^{\{1,\dots,n\}}$, then $\mathrm{ld}_E(E/J_\Delta) = \mathrm{ld}_S(S/I_\Delta)$. Moreover, except some extremal cases, $\mathrm{ld}_E(E/J_\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$ of Δ. We also show that, when $n \geqslant 4$, $\mathrm{ld}_E(E/J_\Delta) = n - 2$ (this is the largest possible value) if and only if Δ is an n-gon.

Keywords: Stanley–Reisner ring; Exterior face ring; Linearity defect; Weakly Koszul module; Componentwise linear; Sequentially Cohen–Macaulay; Squarefree module

1. Introduction

Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a graded (not necessarily commutative) noetherian algebra over a field $K (\cong A_0)$. Let M be a finitely generated graded left A-module, and $P_\bullet$ its minimal free resolution. Eisenbud et al. [3] defined the *linear part* $\mathrm{lin}(P_\bullet)$ of $P_\bullet$, which is the complex obtained by erasing all terms of degree $\geqslant 2$ from the matrices representing the differen-

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tial maps of $P_\bullet$ (hence $\operatorname{lin}(P_\bullet)_i = P_i$ for all i). Following Herzog and Iyengar [6], we call $\operatorname{ld}_A(M) = \sup\{i \mid H_i(\operatorname{lin}(P_\bullet)) \neq 0\}$ the *linearity defect* of M. This invariant and related concepts have been studied by several authors (e.g., [3,6,9,12,18]). Following [5], we say a finitely generated graded A-module M is *componentwise linear* (or (*weakly*) *Koszul* in some literature) if $M_{\langle i\rangle}$ has a linear free resolution for all i. Here $M_{\langle i\rangle}$ is the submodule of M generated by its degree i part M_i. Then we have

$$\operatorname{ld}_A(M) = \min\{i \mid \text{the } i\text{th syzygy of } M \text{ is componentwise linear}\}.$$

For this invariant, a remarkable result holds over an exterior algebra $E = \bigwedge\langle y_1, \dots, y_n\rangle$. In [3, Theorem 3.1], Eisenbud et al. showed that any finitely generated graded E-module N satisfies $\operatorname{ld}_E(N) < \infty$ while $\operatorname{proj.dim}_E(N) = \infty$ in most cases. (We also remark that Martinez-Villa and Zacharia [9] proved the same result for many selfinjective Koszul algebras.) If $n \geqslant 2$, then we have $\sup\{\operatorname{ld}_E(N) \mid N \text{ a finitely generated graded } E\text{-module}\} = \infty$. But Herzog and Römer proved that if $J \subset E$ is a *monomial* ideal then $\operatorname{ld}_E(E/J) \leqslant n-1$ (cf. [12]).

A monomial ideal of $E = \bigwedge\langle y_1, \dots, y_n\rangle$ is always of the form $J_\Delta := (\prod_{i\in F} y_i \mid F \notin \Delta)$ for a simplicial complex $\Delta \subset 2^{\{1,\dots,n\}}$. Similarly, we have the *Stanley–Reisner ideal*

$$I_\Delta := \left(\prod_{i\in F} x_i \mid F \notin \Delta\right)$$

of a polynomial ring $S = K[x_1, \dots, x_n]$. In this paper, we will show the following.

Theorem 1.1. *With the above notation, we have* $\operatorname{ld}_E(E/J_\Delta) = \operatorname{ld}_S(S/I_\Delta)$. *Moreover, if* $\operatorname{ld}_E(E/J_\Delta) > 0$ (*equivalently,* $\Delta \neq 2^T$ *for any* $T \subset [n]$), *then* $\operatorname{ld}_E(E/J_\Delta)$ *is a topological invariant of the geometric realization* $|\Delta^\vee|$ *of the Alexander dual* $\Delta^\vee$. (*But* $\operatorname{ld}(E/J_\Delta)$ *may depend on* $\operatorname{char}(K)$.)

By virtue of the above theorem, we can put $\operatorname{ld}(\Delta) := \operatorname{ld}_E(E/J_\Delta) = \operatorname{ld}_S(S/I_\Delta)$. If we set $d := \min\{i \mid [I_\Delta]_i \neq 0\} = \min\{i \mid [J_\Delta]_i \neq 0\}$, then $\operatorname{ld}(\Delta) \leqslant \max\{1, n-d\}$. But, if $d = 1$ (i.e., $\{i\} \notin \Delta$ for some $1 \leqslant i \leqslant n$), then $\operatorname{ld}(\Delta) \leqslant \max\{1, n-3\}$. Hence, if $n \geqslant 3$, we have $\operatorname{ld}(\Delta) \leqslant n-2$ for all Δ.

Theorem 1.2. *Assume that* $n \geqslant 4$. *Then* $\operatorname{ld}(\Delta) = n-2$ *if and only if* Δ *is an* n*-gon.*

While we treat S and E in most part of the paper, some results on S can be generalized to a normal semigroup ring, and this generalization makes the topological meaning of $\operatorname{ld}(\Delta)$ clear. So Section 2 concerns a normal semigroup ring. But, in this case, we use an irreducible resolution (something analogous to an injective resolution), not a projective resolution.

2. Linearity defects for irreducible resolutions

Let $C \subset \mathbb{Z}^n \subset \mathbb{R}^n$ be an affine semigroup (i.e., C is a finitely generated additive submonoid of $\mathbb{Z}^n$), and $R := K[\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C] \subset K[x_1^{\pm1}, \dots, x_n^{\pm1}]$ the semigroup ring of C over the field K. Here $\mathbf{x}^{\mathbf{c}}$ for $\mathbf{c} = (c_1, \dots, c_n) \in C$ denotes the monomial $\prod_{i=1}^n x_i^{c_i}$. Let $\mathbf{P} := \mathbb{R}_{\geqslant 0}C \subset \mathbb{R}^n$ be the polyhedral cone spanned by C. We always assume that $\mathbb{Z}C = \mathbb{Z}^n$, $\mathbb{Z}^n \cap \mathbf{P} = C$ and $C \cap (-C) =$

$\{0\}$. Thus R is a normal Cohen–Macaulay integral domain of dimension n with a maximal ideal $\mathfrak{m} := (\mathbf{x}^{\mathbf{c}} \mid 0 \neq \mathbf{c} \in C)$.

Clearly,

$$R = \bigoplus_{\mathbf{c} \in C} K\mathbf{x}^{\mathbf{c}}$$

is a $\mathbb{Z}^n$-graded ring. We say a $\mathbb{Z}^n$-graded ideal of R is a *monomial ideal*. Let $^*\mathrm{mod}\, R$ be the category of finitely generated $\mathbb{Z}^n$-graded R-modules and degree preserving R-homomorphisms. As usual, for $M \in {}^*\mathrm{mod}\, R$ and $\mathbf{a} \in \mathbb{Z}^n$, $M_{\mathbf{a}}$ denotes the degree $\mathbf{a}$ component of M, and $M(\mathbf{a})$ denotes the shifted module of M with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$.

Let $\mathbf{L}$ be the set of non-empty faces of the polyhedral cone $\mathbf{P}$. Note that $\{0\}$ and $\mathbf{P}$ itself belong to $\mathbf{L}$. For $F \in \mathbf{L}$, $P_F := (\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C \setminus F)$ is a prime ideal of R. Conversely, any monomial prime ideal is of the form P_F for some $F \in \mathbf{L}$. Note that $P_{\{0\}} = \mathfrak{m}$ and $P_{\mathbf{P}} = (0)$. Set $K[F] := R/P_F \cong K[\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C \cap F]$ for $F \in \mathbf{L}$. The Krull dimension of $K[F]$ equals the dimension $\dim F$ of the polyhedral cone F.

For a point $u \in \mathbf{P}$, we always have a unique face $F \in \mathbf{L}$ whose relative interior contains u. Here we denote $s(u) = F$.

Definition 2.1. *(See [16].)* We say a module $M \in {}^*\mathrm{mod}\, R$ is *squarefree*, if it is C-graded (i.e., $M_{\mathbf{a}} = 0$ for all $\mathbf{a} \notin C$), and the multiplication map $M_{\mathbf{a}} \ni y \mapsto \mathbf{x}^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in C$ with $s(\mathbf{a}+\mathbf{b}) = s(\mathbf{a})$.

For a monomial ideal I, R/I is a squarefree R-module if and only if I is a radical ideal (i.e., $\sqrt{I} = I$). Regarding $\mathbf{L}$ as a partially ordered set by inclusion, we say $\Delta \subset \mathbf{L}$ is an *order ideal*, if $\Delta \ni F \supset F' \in \mathbf{L}$ implies $F' \in \Delta$. If Δ is an order ideal, then $I_\Delta := (\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C, s(\mathbf{c}) \notin \Delta) \subset R$ is a radical ideal. Conversely, any radical monomial ideal is of the form I_Δ for some Δ. Set $K[\Delta] := R/I_\Delta$. Clearly,

$$K[\Delta]_{\mathbf{a}} \cong \begin{cases} K & \text{if } \mathbf{a} \in C \text{ and } s(\mathbf{a}) \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $\Delta = \mathbf{L}$ (resp. $\Delta = \{\{0\}\}$), then $I_\Delta = 0$ (resp. $I_\Delta = \mathfrak{m}$) and $K[\Delta] = R$ (resp. $K[\Delta] = K$). When R is a polynomial ring, $K[\Delta]$ is nothing else than the Stanley–Reisner ring of a simplicial complex Δ. (If R is a polynomial ring, then the partially ordered set $\mathbf{L}$ is isomorphic to the power set $2^{\{1,\dots,n\}}$, and Δ can be seen as a simplicial complex.)

For each $F \in \mathbf{L}$, take some $\mathbf{c}(F) \in C \cap \mathrm{rel} - \mathrm{int}(F)$ (i.e., $s(\mathbf{c}(F)) = F$). For a squarefree R-module M and $F, G \in \mathbf{L}$ with $G \supset F$, [16, Theorem 3.3] gives a K-linear map $\varphi^M_{G,F} : M_{\mathbf{c}(F)} \to M_{\mathbf{c}(G)}$. They satisfy $\varphi^M_{F,F} = \mathrm{Id}$ and $\varphi^M_{H,G} \circ \varphi^M_{G,F} = \varphi^M_{H,F}$ for all $H \supset G \supset F$. We have $M_{\mathbf{c}} \cong M_{\mathbf{c}'}$ for $\mathbf{c}, \mathbf{c}' \in C$ with $s(\mathbf{c}) = s(\mathbf{c}')$. Under these isomorphisms, the maps $\varphi^M_{G,F}$ do not depend on the particular choice of $\mathbf{c}(F)$'s.

Let $\mathrm{Sq}(R)$ be the full subcategory of $^*\mathrm{mod}\, R$ consisting of squarefree modules. As shown in [16], $\mathrm{Sq}(R)$ is an abelian category with enough injectives. For an indecomposable squarefree module M, it is injective in $\mathrm{Sq}(R)$ if and only if $M \cong K[F]$ for some $F \in \mathbf{L}$. Each $M \in \mathrm{Sq}(R)$ has a minimal injective resolution in $\mathrm{Sq}(R)$, and we call it a *minimal irreducible resolution* (see [10,19] for further information). A minimal irreducible resolution is unique up to isomorphism, and its length is at most n.

Let ω_R be the $\mathbb{Z}^n$-graded canonical module of R. It is well known that ω_R is isomorphic to the radical monomial ideal $(\mathbf{x}^{\mathbf{c}} \mid \mathbf{c} \in C, s(c) = \mathbf{P})$. Since we have $\operatorname{Ext}^i_R(M^\bullet, \omega_R) \in \operatorname{Sq}(R)$ for all $M^\bullet \in \operatorname{Sq}(R)$, $\mathbf{D}(-) := \operatorname{RHom}_R(-, \omega_R)$ gives a duality functor from the derived category $D^b(\operatorname{Sq}(R))(\cong D^b_{\operatorname{Sq}(R)}({}^*\mathrm{mod}\, R))$ to itself.

In the sequel, for a K-vector space V, V^* denotes its dual space. But, even if $V = M_{\mathbf{a}}$ for some $M \in {}^*\mathrm{mod}\, R$ and $\mathbf{a} \in \mathbb{Z}^n$, we set the degree of V^* to be 0.

Lemma 2.2. *(See [19, Lemma 3.8].) If $M \in \operatorname{Sq}(R)$, then $\mathbf{D}(M)$ is quasi-isomorphic to the complex $D^\bullet : 0 \to D^0 \to D^1 \to \cdots \to D^n \to 0$ with*

$$D^i = \bigoplus_{\substack{F \in \mathbf{L} \\ \dim F = n-i}} (M_{\mathbf{c}(F)})^* \otimes_K K[F].$$

Here the differential is the sum of the maps

$$\left(\pm\varphi^M_{F,F'}\right)^* \otimes \mathrm{nat} : (M_{\mathbf{c}(F)})^* \otimes_K K[F] \to (M_{\mathbf{c}(F')})^* \otimes_K K\left[F'\right]$$

for $F, F' \in \mathbf{L}$ with $F \supset F'$ and $\dim F = \dim F' + 1$, and nat *denotes the natural surjection $K[F] \to K[F']$. We can also describe $\mathbf{D}(M^\bullet)$ for a complex $M^\bullet \in D^b(\operatorname{Sq}(R))$ in a similar way.*

Convention. In the sequel, as an explicit complex, $\mathbf{D}(M^\bullet)$ for $M^\bullet \in D^b(\operatorname{Sq}(R))$ means the complex described in Lemma 2.2.

Since $\mathbf{D} \circ \mathbf{D} \cong \mathrm{Id}_{D^b(\operatorname{Sq}(R))}$, $\mathbf{D} \circ \mathbf{D}(M)$ is an irreducible resolution of M, but it is far from being minimal. Let $(I^\bullet, \partial^\bullet)$ be a minimal irreducible resolution of M. For each $i \in \mathbb{N}$ and $F \in \mathbf{L}$, we have a natural number $\nu_i(F, M)$ such that

$$I^i \cong \bigoplus_{F \in \mathbf{L}} K[F]^{\nu_i(F,M)}.$$

Since $I^\bullet$ is minimal, $z \in K[F] \subset I^i$ with $\dim F = d$ is sent to

$$\partial^i(z) \in \bigoplus_{\substack{G \in \mathbf{L} \\ \dim G < d}} K[G]^{\nu_{i+1}(G,M)} \subset I^{i+1}.$$

The above observation on $\mathbf{D} \circ \mathbf{D}(M)$ gives the formula [16, Theorem 4.15]

$$\nu_i(F, M) = \dim_K \left[\operatorname{Ext}_R^{n-i-\dim F}(M, \omega_R)\right]_{\mathbf{c}(F)}.$$

For each $l \in \mathbb{N}$ with $0 \leqslant l \leqslant n$, we define the *$l$-linear strand* $\mathrm{lin}_l(I^\bullet)$ of $I^\bullet$ as follows: The term $\mathrm{lin}_l(I^\bullet)^i$ of cohomological degree i is

$$\bigoplus_{\dim F = l-i} K[F]^{\nu_i(F,M)},$$

which is a direct summand of I^i, and the differential $\mathrm{lin}_l(I^\bullet)^i \to \mathrm{lin}_l(I^\bullet)^{i+1}$ is the corresponding component of the differential $\partial^i : I^i \to I^{i+1}$ of $I^\bullet$. By the minimality of $I^\bullet$, we can see that $\mathrm{lin}_l(I^\bullet)$ are cochain complexes. Set $\mathrm{lin}(I^\bullet) := \bigoplus_{0 \leqslant l \leqslant n} \mathrm{lin}_l(I^\bullet)$. Then we have the following.

For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the pth translation of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$.

Theorem 2.3. *(See [19, Theorem 3.9].) With the above notation, we have*

$$\operatorname{lin}_l\bigl(I^\bullet\bigr) \cong \mathbf{D}\bigl(\operatorname{Ext}_R^{n-l}(M, \omega_R)\bigr)[n-l].$$

Hence

$$\operatorname{lin}\bigl(I^\bullet\bigr) \cong \bigoplus_{i\in\mathbb{Z}} \mathbf{D}\bigl(\operatorname{Ext}_R^{i}(M, \omega_R)\bigr)[i].$$

Definition 2.4. Let $I^\bullet$ be a minimal irreducible resolution of $M \in \operatorname{Sq}(R)$. We call $\max\{i \mid H^i(\operatorname{lin}(I^\bullet)) \neq 0\}$ the *linearity defect of the minimal irreducible resolution* of M, and denote it by $\operatorname{ld.irr}_R(M)$.

Corollary 2.5. *With the above notation, we have*

$$\max\bigl\{i \mid H^i\bigl(\operatorname{lin}_l\bigl(I^\bullet\bigr)\bigr) \neq 0\bigr\} = l - \operatorname{depth}_R\bigl(\operatorname{Ext}_R^{n-l}(M, \omega_R)\bigr),$$

and hence

$$\operatorname{ld.irr}_R(M) = \max\bigl\{i - \operatorname{depth}_R\bigl(\operatorname{Ext}_R^{n-i}(M, \omega_R)\bigr) \mid 0 \leqslant i \leqslant n\bigr\}.$$

Here we set the depth of the 0 *module to be* $+\infty$.

Proof. By Theorem 2.3, we have $H^i(\operatorname{lin}_l(I^\bullet)) = \operatorname{Ext}_R^{i+n-l}(\operatorname{Ext}_R^{n-l}(M, \omega_R), \omega_R)$. Since $\operatorname{depth}_R N = \min\{i \mid \operatorname{Ext}_R^{n-i}(N, \omega_R) \neq 0\}$ for a finitely generated graded R-module N, the assertion follows. $\square$

Definition 2.6. *(See Stanley [14].)* Let $M \in {}^*\mathrm{mod}\, R$. We say M is *sequentially Cohen–Macaulay* if there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

of M by graded submodules M_i satisfying the following conditions.

(a) Each quotient M_i/M_{i-1} is Cohen–Macaulay.
(b) $\dim(M_i/M_{i-1}) < \dim(M_{i+1}/M_i)$ for all i.

Remark that the notion of sequentially Cohen–Macaulay module is also studied under the name of a "Cohen–Macaulay filtered module" [13].

Sequentially Cohen–Macaulay property is getting important in the theory of Stanley–Reisner rings. It is known that $M \in {}^*\mathrm{mod}\, R$ is sequentially Cohen–Macaulay if and only if $\operatorname{Ext}_R^{n-i}(M, \omega_R)$ is a zero module or a Cohen–Macaulay module of dimension i for all i (cf. [14, III. Theorem 2.11]). Let us go back to Corollary 2.5. If $N := \operatorname{Ext}_R^{n-i}(M, \omega_R) \neq 0$, then $\operatorname{depth}_R N \leqslant \dim_R N \leqslant i$. Hence $\operatorname{depth}_R N = i$ if and only if N is a Cohen–Macaulay module of dimension i. Thus, as stated in [19, Corollary 3.11], $\operatorname{ld.irr}_R(M) = 0$ if and only if M is sequentially Cohen–Macaulay.

Let $I^\bullet : 0 \to I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \to \cdots$ be an irreducible resolution of $M \in \operatorname{Sq}(R)$. Then it is easy to see that $\ker(\partial^i)$ is sequentially Cohen–Macaulay if and only if $i \geqslant \operatorname{ld.irr}_R(M)$. In particular,

$$\operatorname{ld.irr}_R(M) = \min\bigl\{i \mid \ker\bigl(\partial^i\bigr) \text{ is sequentially Cohen–Macaulay}\bigr\}.$$

We have a hyperplane $H \subset \mathbb{R}^n$ such that $B := H \cap \mathbf{P}$ is an $(n-1)$-dimensional polytope. Clearly, B is homeomorphic to a closed ball of dimension $n-1$. For a face $F \in \mathbf{L}$, set $|F|$ to be the relative interior of $F \cap H$. If $\Delta \subset \mathbf{L}$ is an order ideal, then $|\Delta| := \bigcup_{F \in \Delta} |F|$ is a closed subset of B, and $\bigcup_{F \in \Delta} |F|$ is a *regular cell decomposition* (cf. [1, §6.2]) of $|\Delta|$. Up to homeomorphism, (the regular cell decomposition of) $|\Delta|$ does not depend on the particular choice of the hyperplane H. The dimension $\dim |\Delta|$ of $|\Delta|$ is given by $\max\{\dim |F| \mid F \in \Delta\}$. Here $\dim |F|$ denotes the dimension of $|F|$ as a cell (we set $\dim \emptyset = -1$), that is, $\dim |F| = \dim F - 1 = \dim K[F] - 1$. Hence we have $\dim K[\Delta] = \dim |\Delta| + 1$.

If $F \in \Delta$, then $U_F := \bigcup_{F' \supset F} |F'|$ is an open set of B. Note that $\{U_F \mid \{0\} \neq F \in \mathbf{L}\}$ is an open covering of B. In [17], from $M \in \operatorname{Sq}(R)$, the second author constructed a sheaf M^+ on B. (For the sheaf theory used below, consult [7].) More precisely, the assignment

$$\Gamma\bigl(U_F, M^+\bigr) = M_{\mathbf{c}(F)}$$

for each $F \neq \{0\}$ and the map

$$\varphi^M_{F,F'} \colon \Gamma\bigl(U_{F'}, M^+\bigr) = M_{\mathbf{c}(F')} \to M_{\mathbf{c}(F)} = \Gamma\bigl(U_F, M^+\bigr)$$

for $F, F' \neq \{0\}$ with $F \supset F'$ (equivalently, $U_{F'} \supset U_F$) defines a sheaf. Note that M^+ is a *constructible sheaf* with respect to the cell decomposition $B = \bigcup_{F \in \mathbf{L}} |F|$. In fact, for all $\{0\} \neq F \in \mathbf{L}$, the restriction $M^+|_{|F|}$ of M^+ to $|F| \subset B$ is a constant sheaf with coefficients in $M_{\mathbf{c}(F)}$. Note that $M_{\mathbf{0}}$ is "irrelevant" to M^+, where $\mathbf{0}$ denotes $(0, 0, \ldots, 0) \in \mathbb{Z}^n$.

It is easy to see that $K[\Delta]^+ \cong j_* \underline{K}_{|\Delta|}$, where $\underline{K}_{|\Delta|}$ is the constant sheaf on $|\Delta|$ with coefficients in K, and j denotes the embedding map $|\Delta| \hookrightarrow B$. Similarly, we have that $(\omega_R)^+ \cong h_! \underline{K}_{B^\circ}$, where $\underline{K}_{B^\circ}$ is the constant sheaf on the relative interior B° of B, and h denotes the embedding map $B^\circ \hookrightarrow B$. Note that $(\omega_R)^+$ is the orientation sheaf of B over K.

Theorem 2.7. *(See [17, Theorem 3.3].) For $M \in \operatorname{Sq}(R)$, we have an isomorphism*

$$H^i\bigl(B; M^+\bigr) \cong \bigl[H^{i+1}_{\mathfrak{m}}(M)\bigr]_{\mathbf{0}} \quad \text{for all } i \geqslant 1,$$

and an exact sequence

$$0 \to \bigl[H^0_{\mathfrak{m}}(M)\bigr]_{\mathbf{0}} \to M_{\mathbf{0}} \to H^0\bigl(B; M^+\bigr) \to \bigl[H^1_{\mathfrak{m}}(M)\bigr]_{\mathbf{0}} \to 0.$$

In particular, we have $[H^{i+1}_{\mathfrak{m}}(K[\Delta])]_{\mathbf{0}} \cong \tilde{H}^i(|\Delta|; K)$ for all $i \geqslant 0$, where $\tilde{H}^i(|\Delta|; K)$ denotes the ith reduced cohomology of $|\Delta|$ with coefficients in K.

Let $\Delta \subset \mathbf{L}$ be an order ideal and $X := |\Delta|$. Then X admits Verdier's dualizing complex $\mathcal{D}^\bullet_X$, which is a complex of sheaves of K-vector spaces. For example, $\mathcal{D}^\bullet_B$ is quasi-isomorphic to $(\omega_R)^+[n-1]$.

Theorem 2.8. *(See [17, Theorem 4.2].) With the above notation, if* $\mathrm{ann}(M) \supset I_\Delta$ *(equivalently,* $\mathrm{supp}(M^+) := \{x \in B \mid (M^+)_x \neq 0\} \subset X$*), then we have*

$$\mathrm{supp}\big(\mathrm{Ext}^i_R(M, \omega_R)^+\big) \subset X \quad \textit{and} \quad \mathrm{Ext}^i_R(M, \omega_R)^+|_X \cong \mathcal{E}xt^{i-n+1}\big(M^+|_X, \mathcal{D}^\bullet_X\big).$$

Theorem 2.9. *Let M be a squarefree R-module with $M \neq 0$ and $[H^1_\mathfrak{m}(M)]_\mathbf{0} = 0$, and X the closure of* $\mathrm{supp}(M^+)$*. Then* $\mathrm{ld.irr}_R(M)$ *only depends on the sheaf $M^+|_X$ (also independent from R).*

Proof. We use Corollary 2.5. In the notation there, the case when $i = 0$ is always unnecessary to check. Moreover, by the present assumption, we have $\mathrm{depth}_R(\mathrm{Ext}^{n-1}_R(M, \omega_R)) \geqslant 1$ (in fact, $\mathrm{Ext}^{n-1}_R(M, \omega_R)$ is either the 0 module, or a 1-dimensional Cohen–Macaulay module). So we may assume that $i > 1$.

Recall that

$$\mathrm{depth}_R\big(\mathrm{Ext}^{n-i}_R(M, \omega_R)\big) = \min\big\{j \mid \mathrm{Ext}^{n-j}_R\big(\mathrm{Ext}^{n-i}_R(M, \omega_R), \omega_R\big) \neq 0\big\}.$$

By Theorem 2.8, $[\mathrm{Ext}^{n-j}_R(\mathrm{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_\mathbf{a}$ can be determined by $M^+|_X$ for all i, j and all $\mathbf{a} \neq 0$. If $j > 1$, then $[\mathrm{Ext}^{n-j}_R(\mathrm{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_\mathbf{0}$ is isomorphic to

$$\begin{aligned}\big[H^j_\mathfrak{m}\big(\mathrm{Ext}^{n-i}_R(M, \omega_R)\big)\big]^*_\mathbf{0} &\cong H^{j-1}\big(B; \mathrm{Ext}^{n-i}_R(M, \omega_R)^+\big)^* \\ &\cong H^{j-1}\big(X; \mathcal{E}xt^{-i-1}\big(M^+|_X; \mathcal{D}^\bullet_X\big)\big)^*\end{aligned}$$

(the first and the second isomorphisms follow from Theorems 2.7 and 2.8, respectively), and determined by $M^+|_X$. So only $[\mathrm{Ext}^{n-j}_R(\mathrm{Ext}^{n-i}_R(M, \omega_R), \omega_R)]_\mathbf{0}$ for $j = 0, 1$ remain. As above, they are isomorphic to $[H^j_\mathfrak{m}(\mathrm{Ext}^{n-i}_R(M, \omega_R))]^*_\mathbf{0}$. But, by [19, Lemma 5.11], we can compute $[H^j_\mathfrak{m}(\mathrm{Ext}^{n-i}_R(M, \omega_R))]_\mathbf{0}$ for $i > 1$ and $j = 0, 1$ from the sheaf $M^+|_X$. So we are done. □

Theorem 2.10. *For an order ideal $\Delta \subset \mathbf{L}$ with $\Delta \neq \emptyset$,* $\mathrm{ld.irr}_R(K[\Delta])$ *depends only on the topological space $|\Delta|$.*

Note that $\mathrm{ld.irr}_R(K[\Delta])$ may depend on $\mathrm{char}(K)$. For example, if $|\Delta|$ is homeomorphic to a real projective plane, then $\mathrm{ld.irr}_R(K[\Delta]) = 0$ if $\mathrm{char}(K) \neq 2$, but $\mathrm{ld.irr}_R(K[\Delta]) = 2$ if $\mathrm{char}(K) = 2$.

Similarly, some other invariants and conditions (e.g., the Cohen–Macaulay property of $K[\Delta]$) studied in this paper depend on $\mathrm{char}(K)$. But, since we fix the base field K, we always omit the phrase “over K”.

Proof. If $|\Delta|$ is not connected, then $[H^1_\mathfrak{m}(K[\Delta])]_\mathbf{0} \neq 0$ by Theorem 2.7, and we cannot use Theorem 2.9 directly. But even in this case, $\mathrm{depth}_R(\mathrm{Ext}^{n-i}_R(K[\Delta], \omega_R))$ can be computed for all $i \neq 1$ by the same way as in Theorem 2.9. In particular, they only depend on $|\Delta|$. So the assertion follows from the next lemma. □

Lemma 2.11. *We have* $\mathrm{depth}_R(\mathrm{Ext}^{n-1}_R(K[\Delta], \omega_R)) \in \{0, 1, +\infty\}$*, and*

$$\mathrm{depth}_R\big(\mathrm{Ext}^{n-1}_R\big(K[\Delta], \omega_R\big)\big) = 0 \quad \textit{if and only if} \quad |\Delta'| \textit{ is not connected.}$$

Here $\Delta' := \Delta \setminus \{F \mid F$ *is a maximal element of* Δ *and* $\dim|F| = 0\}$.

Proof. Since $\dim_R \operatorname{Ext}_R^{n-1}(K[\Delta], \omega_R) \leqslant 1$, the first statement is clear. If $\dim|\Delta| \leqslant 0$, then $|\Delta'| = \emptyset$ and $\operatorname{depth}_R(\operatorname{Ext}_R^{n-1}(K[\Delta], \omega_R)) \geqslant 1$. So, to see the second statement, we may assume that $\dim|\Delta| \geqslant 1$. Set $J := I_{\Delta'}/I_\Delta$ to be an ideal of $K[\Delta]$. Note that either J is a 1-dimensional Cohen–Macaulay module or $J = 0$. From the short exact sequence $0 \to J \to K[\Delta] \to K[\Delta'] \to 0$, we have an exact sequence

$$0 \to \operatorname{Ext}_R^{n-1}\big(K[\Delta'], \omega_R\big) \to \operatorname{Ext}_R^{n-1}\big(K[\Delta], \omega_R\big) \to \operatorname{Ext}_R^{n-1}(J, \omega_R) \to 0.$$

Since $\operatorname{Ext}_R^{n-1}(J, \omega_R)$ has positive depth,

$$\operatorname{depth}_R\big(\operatorname{Ext}_R^{n-1}\big(K[\Delta'], \omega_R\big)\big) = 0$$

if and only if $\operatorname{depth}_R(\operatorname{Ext}_R^{n-1}(K[\Delta], \omega_R)) = 0$. But, since $K[\Delta']$ does not have 1-dimensional associated primes, $\operatorname{Ext}_R^{n-1}(K[\Delta'], \omega_R)$ is an artinian module. Hence we have the following.

$$\begin{aligned}
\operatorname{depth}_R\big(\operatorname{Ext}_R^{n-1}\big(K[\Delta'], \omega_R\big)\big) = 0 &\iff \big[\operatorname{Ext}_R^{n-1}\big(K[\Delta'], \omega_R\big)\big]_{\mathbf{0}} \neq 0 \\
&\iff \big[H_{\mathfrak{m}}^1\big(K[\Delta']\big)\big]_{\mathbf{0}} = \tilde{H}^0\big(|\Delta'|; K\big) \neq 0 \\
&\iff |\Delta'| \text{ is not connected.} \qquad \square
\end{aligned}$$

3. Linearity defects of symmetric and exterior face rings

Let $S := K[x_1, \dots, x_n]$ be a polynomial ring, and consider its natural $\mathbb{Z}^n$-grading. Since $S = K[\mathbb{N}^n]$ is a normal semigroup ring, we can use the notation and the results in the previous section.

Now we introduce some conventions which are compatible with the previous notation. Let $\mathbf{e}_i := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ be the ith unit vector, and $\mathbf{P}$ the cone spanned by $\mathbf{e}_1, \dots, \mathbf{e}_n$. We identify a face F of $\mathbf{P}$ with the subset $\{i \mid \mathbf{e}_i \in F\}$ of $[n] := \{1, 2, \dots, n\}$. Hence the set $\mathbf{L}$ of nonempty faces of $\mathbf{P}$ can be identified with the power set $2^{[n]}$ of $[n]$. We say $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is *squarefree*, if $a_i = 0, 1$ for all i. A squarefree vector $\mathbf{a} \in \mathbb{N}^n$ will be identified with the subset $\{i \mid a_i = 1\}$ of $[n]$. Recall that we took a vector $\mathbf{c}(F) \in C$ for each $F \in \mathbf{L}$ in the previous section. Here we assume that $\mathbf{c}(F)$ is the squarefree vector corresponding to $F \in \mathbf{L} \cong 2^{[n]}$. So, for a $\mathbb{Z}^n$-graded S-module M, we simply denote $M_{\mathbf{c}(F)}$ by M_F. In the first principle, we regard F as a subset of $[n]$, or a squarefree vector in $\mathbb{N}^n$, rather than the corresponding face of $\mathbf{P}$. For example, we write $P_F = (x_i \mid i \notin F)$, $K[F] \cong K[x_i \mid i \in F]$. And $S(-F)$ denotes the rank 1 free S-module $S(-\mathbf{a})$, where $\mathbf{a} \in \mathbb{N}^n$ is the squarefree vector corresponding to F.

Squarefree S-modules are defined by the same way as Definition 2.1. Note that the free module $S(-\mathbf{a})$, $\mathbf{a} \in \mathbb{Z}^n$, is squarefree if and only if $\mathbf{a}$ is squarefree. Let $^*\mathrm{mod}\, S$ (resp. $\mathrm{Sq}(S)$) be the category of finitely generated $\mathbb{Z}^n$-graded S-modules (resp. squarefree S-modules). Let $P_\bullet$ be a $\mathbb{Z}^n$-graded minimal free resolution of $M \in {}^*\mathrm{mod}\, S$. Then M is squarefree if and only if each P_i is a direct sum of copies of $S(-F)$ for various $F \subset [n]$. In the present case, an order ideal Δ of $\mathbf{L}(\cong 2^{[n]})$ is essentially a simplicial complex, and the ring $K[\Delta]$ defined in the previous section is nothing other than the *Stanley–Reisner ring* (cf. [1,14]) of Δ.

Let $E = \bigwedge\langle y_1, \dots, y_n\rangle$ be the exterior algebra over K. Under the *Bernstein–Gel'fand–Gel'fand correspondence* (cf. [3]), E is the counter part of S. We regard E as a $\mathbb{Z}^n$-graded ring by $\deg y_i = \mathbf{e}_i = \deg x_i$ for each i. Clearly, any monomial ideal of E is "squarefree", and of the form

$$J_\Delta := \left(\prod_{i \in F} y_i \mid F \subset [n], F \notin \Delta \right)$$

for a simplicial complex $\Delta \subset 2^{[n]}$. We say $K\langle \Delta \rangle := E/J_\Delta$ is the *exterior face ring* of Δ.

Let ${}^*\mathrm{mod}\, E$ be the category of finitely generated $\mathbb{Z}^n$-graded E-modules and degree preserving E-homomorphisms. Note that, for graded E-modules, we do not have to distinguish left modules from right ones. Hence

$$\mathbf{D}_E(-) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \mathrm{Hom}_{{}^*\mathrm{mod}\, E}\big(-, E(\mathbf{a})\big)$$

gives an exact contravariant functor from ${}^*\mathrm{mod}\, E$ to itself satisfying $\mathbf{D}_E \circ \mathbf{D}_E = \mathrm{Id}$.

Definition 3.1. *(See Römer [11].)* We say $N \in {}^*\mathrm{mod}\, E$ is *squarefree*, if $N = \bigoplus_{F \subset [n]} N_F$ (i.e., if $\mathbf{a} \in \mathbb{Z}^n$ is not squarefree, then $N_\mathbf{a} = 0$).

An exterior face ring $K\langle \Delta \rangle$ is a squarefree E-module. But, since a free module $E(\mathbf{a})$ is not squarefree for $\mathbf{a} \neq 0$, the syzygies of a squarefree E-module are *not* squarefree. Let $\mathrm{Sq}(E)$ be the full subcategory of ${}^*\mathrm{mod}\, E$ consisting of squarefree modules. If N is a squarefree E-module, then so is $\mathbf{D}_E(N)$. That is, $\mathbf{D}_E$ gives a contravariant functor from $\mathrm{Sq}(E)$ to itself.

We have functors $\mathcal{S} : \mathrm{Sq}(E) \to \mathrm{Sq}(S)$ and $\mathcal{E} : \mathrm{Sq}(S) \to \mathrm{Sq}(E)$ giving an equivalence $\mathrm{Sq}(S) \cong \mathrm{Sq}(E)$. Here $\mathcal{S}(N)_F = N_F$ for $N \in \mathrm{Sq}(E)$ and $F \subset [n]$, and the multiplication map $\mathcal{S}(N)_F \ni z \mapsto x_i z \in \mathcal{S}(N)_{F \cup \{i\}}$ for $i \notin F$ is given by

$$\mathcal{S}(N)_F = N_F \ni z \mapsto (-1)^{\alpha(i,F)} y_i z \in N_{F \cup \{i\}} = \mathcal{S}(N)_{F \cup \{i\}},$$

where $\alpha(i, F) = \#\{j \in F \mid j < i\}$. For example, $\mathcal{S}(K\langle \Delta \rangle) \cong K[\Delta]$. See [11] for detail.

Note that $\mathbf{A} := \mathcal{S} \circ \mathbf{D}_E \circ \mathcal{E}$ is an exact contravariant functor from $\mathrm{Sq}(S)$ to itself satisfying $\mathbf{A} \circ \mathbf{A} = \mathrm{Id}$. It is easy to see that $\mathbf{A}(K[F]) \cong S(-F^\mathsf{c})$, where $F^\mathsf{c} := [n] \setminus F$. We also have $\mathbf{A}(K[\Delta]) \cong I_{\Delta^\vee}$, where

$$\Delta^\vee := \big\{ F \subset [n] \mid F^\mathsf{c} \notin \Delta \big\}$$

is the *Alexander dual* complex of Δ. Since $\mathbf{A}$ is exact, it exchanges a (minimal) free resolution with a (minimal) irreducible resolution.

Eisenbud et al. [2,3] introduced the notion of the *linear strands* and the *linear part* of a minimal free resolution of a graded S-module. Let $P_\bullet : \cdots \to P_1 \to P_0 \to 0$ be a $\mathbb{Z}^n$-graded minimal S-free resolution of $M \in {}^*\mathrm{mod}\, S$. We have natural numbers $\beta_{i,\mathbf{a}}(M)$ for $i \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{Z}^n$ such that $P_i = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)}$. We call $\beta_{i,\mathbf{a}}(M)$ the *graded Betti numbers* of M. Set $|\mathbf{a}| = \sum_{i=1}^n a_i$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. For each $l \in \mathbb{Z}$, we define the *l-linear strand* $\mathrm{lin}_l(P_\bullet)$ of $P_\bullet$ as follows: The term $\mathrm{lin}_l(P_\bullet)_i$ of homological degree i is

$$\bigoplus_{|\mathbf{a}| = l+i} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)},$$

which is a direct summand of P_i, and the differential $\mathrm{lin}_l(P_\bullet)_i \to \mathrm{lin}_l(P_\bullet)_{i-1}$ is the corresponding component of the differential $P_i \to P_{i-1}$ of $P_\bullet$. By the minimality of $P_\bullet$, we can easily verify that $\mathrm{lin}_l(P_\bullet)$ are chain complexes (see also [2, §7A]). We call $\mathrm{lin}(P_\bullet) := \bigoplus_{l\in\mathbb{Z}} \mathrm{lin}_l(P_\bullet)$ the *linear part* of $P_\bullet$. Note that the differential maps of $\mathrm{lin}(P_\bullet)$ are represented by matrices of linear forms. We call

$$\mathrm{ld}_S(M) := \max\bigl\{i \mid H_i\bigl(\mathrm{lin}(P_\bullet)\bigr) \neq 0\bigr\}$$

the *linearity defect* of M.

Sometimes, we regard $M \in {}^*\mathrm{mod}\, S$ as a $\mathbb{Z}$-graded module by $M_j = \bigoplus_{|\mathbf{a}|=j} M_{\mathbf{a}}$. In this case, we set $\beta_{i,j}(M) := \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M)$. Then $\mathrm{lin}_l(P_\bullet)_i = S(-l-i)^{\beta_{i,l+i}(M)}$.

Remark 3.2. For $M \in {}^*\mathrm{mod}\, S$, it is clear that $\mathrm{ld}_S(M) \leqslant \mathrm{proj.dim}_S(M) \leqslant n$, and there are many examples attaining the equalities. In fact, $\mathrm{ld}_S(S/(x_1^2, \ldots, x_n^2)) = n$. But if $M \in \mathrm{Sq}(S)$, then we always have $\mathrm{ld}_S(M) \leqslant n-1$. In fact, for a squarefree module M, $\mathrm{proj.dim}_S(M) = n$, if and only if $\mathrm{depth}_S M = 0$, if and only if $M \cong K \oplus M'$ for some $M' \in \mathrm{Sq}(S)$. But $\mathrm{ld}_S(K) = 0$ and $\mathrm{ld}_S(M' \oplus K) = \mathrm{ld}_S(M')$. So we may assume that $\mathrm{proj.dim}_S M' \leqslant n-1$.

Proposition 3.3. *Let $M \in \mathrm{Sq}(S)$, and $P_\bullet$ its minimal graded free resolution. We have*

$$\max\bigl\{i \mid H_i\bigl(\mathrm{lin}_l(P_\bullet)\bigr) \neq 0\bigr\} = n - l - \mathrm{depth}_S\bigl(\mathrm{Ext}_S^l\bigl(\mathbf{A}(M), S\bigr)\bigr),$$

and hence

$$\mathrm{ld}_S(M) = \max\bigl\{i - \mathrm{depth}_S\bigl(\mathrm{Ext}_S^{n-i}\bigl(\mathbf{A}(M), S\bigr)\bigr) \mid 0 \leqslant i \leqslant n\bigr\}.$$

Proof. Note that $I^\bullet := \mathbf{A}(P_\bullet)$ is a minimal irreducible resolution of $\mathbf{A}(M)$. Moreover, we have $\mathbf{A}(\mathrm{lin}_l(P_\bullet)) \cong \mathrm{lin}_{n-l}(I^\bullet)$. Since $\mathbf{A}$ is exact,

$$\max\bigl\{i \mid H_i\bigl(\mathrm{lin}_l(P_\bullet)\bigr) \neq 0\bigr\} = \max\bigl\{i \mid H^i\bigl(\mathrm{lin}_{n-l}\bigl(I^\bullet\bigr)\bigr) \neq 0\bigr\},$$

and hence

$$\mathrm{ld}_S(M) = \mathrm{ld.irr}_S\bigl(\mathbf{A}(M)\bigr). \tag{3.1}$$

Hence the assertions follow from Corollary 2.5 (note that $S \cong \omega_S$ as underlying modules). □

For $N \in {}^*\mathrm{mod}\, E$, we have a $\mathbb{Z}^n$-graded minimal E-free resolution $P_\bullet$ of N. By the similar way to the S-module case, we can define the linear part $\mathrm{lin}(P_\bullet)$ of $P_\bullet$, and set $\mathrm{ld}_E(N) := \max\{i \mid H_i(\mathrm{lin}(P_\bullet)) \neq 0\}$. (In [12,18], $\mathrm{ld}_E(N)$ is denoted by $\mathrm{lpd}(N)$. "lpd" is an abbreviation for "linear part dominate".) In [3, Theorem 3.1], Eisenbud et al. showed that $\mathrm{ld}_E(N) < \infty$ for all $N \in {}^*\mathrm{mod}\, E$. Since $\mathrm{proj.dim}_E(N) = \infty$ in most cases, this is a strong result. If $n \geqslant 2$, then we have $\sup\{\mathrm{ld}_E(N) \mid N \in {}^*\mathrm{mod}\, E\} = \infty$. In fact, since E is selfinjective, we can take "cosyzygies". But, if $N \in \mathrm{Sq}(E)$, then $\mathrm{ld}_E(N)$ behaves quite nicely.

Theorem 3.4. *For $N \in \mathrm{Sq}(E)$, we have $\mathrm{ld}_E(N) = \mathrm{ld}_S(\mathcal{S}(N)) \leqslant n-1$. In particular, for a simplicial complex $\Delta \subset 2^{[n]}$, we have $\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta])$.*

Proof. Using the Bernstein–Gel'fand–Gel'fand correspondence, the second author described $\mathrm{ld}_E(N)$ in [18, Lemma 4.12]. This description is the first equality of the following computation, which proves the assertion.

$$\begin{aligned}\mathrm{ld}_E(N) &= \max\{i - \mathrm{depth}_S(\mathrm{Ext}_S^{n-i}(\mathcal{S}\circ \mathbf{D}_E(N), S)) \mid 0 \leqslant i \leqslant n\} \quad \text{(by [18])}\\ &= \max\{i - \mathrm{depth}_S(\mathrm{Ext}_S^{n-i}(\mathbf{A}\circ \mathcal{S}(N), S)) \mid 0 \leqslant i \leqslant n\} \quad \text{(see below)}\\ &= \mathrm{ld}_S(\mathcal{S}(N)) \quad \text{(by Proposition 3.3).}\end{aligned}$$

Here the second equality follows from the isomorphisms $\mathcal{S}\circ \mathbf{D}_E(N) \cong \mathcal{S}\circ \mathbf{D}_E \circ \mathcal{E}\circ \mathcal{S}(N) \cong \mathbf{A}\circ \mathcal{S}(N)$. □

Remark 3.5. Herzog and Römer showed that $\mathrm{ld}_E(N) \leqslant \mathrm{proj.dim}_S(\mathcal{S}(N))$ for $N \in \mathrm{Sq}(E)$ [12, Corollary 3.3.5]. Since $\mathrm{ld}_S(\mathcal{S}(N)) \leqslant \mathrm{proj.dim}_S(\mathcal{S}(N))$ (the inequality is strict quite often), Theorem 3.4 refines their result. Our equality might follow from the argument in [12], which constructs a minimal E-free resolution of N from a minimal S-free resolution of $\mathcal{S}(N)$. But it seems that certain amount of computation will be required.

Theorem 3.4 suggests that we may set

$$\mathrm{ld}(\Delta) := \mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_E(K\langle\Delta\rangle).$$

Theorem 3.6. *If $I_\Delta \neq (0)$ (equivalently, $\Delta \neq 2^{[n]}$), then $\mathrm{ld}_S(I_\Delta)$ is a topological invariant of the geometric realization $|\Delta^\vee|$ of the Alexander dual $\Delta^\vee$ of Δ. If $\Delta \neq 2^T$ for any $T \subset [n]$, then $\mathrm{ld}(\Delta)$ is also a topological invariant of $|\Delta^\vee|$ (also independent from the number $n = \dim S$).*

Proof. Since $\mathbf{A}(I_\Delta) = K[\Delta^\vee]$ and $\Delta^\vee \neq \emptyset$, the first assertion follows from Theorem 2.10 and the equality (3.1) in the proof of Proposition 3.3.

It is easy to see that $\Delta \neq 2^T$ for any T if and only if $\mathrm{ld}(\Delta) \geqslant 1$. If this is the case, $\mathrm{ld}(\Delta) = \mathrm{ld}_S(I_\Delta) + 1$, and the second assertion follows from the first. □

Remark 3.7. (1) For the first statement of Theorem 3.6, the assumption that $I_\Delta \neq (0)$ is necessary. In fact, if $I_\Delta = (0)$, then $\Delta = 2^{[n]}$ and $\Delta^\vee = \emptyset$. On the other hand, if we set $\Gamma := 2^{[n]} \setminus [n]$, then $\Gamma^\vee = \{\emptyset\}$ and $|\Gamma^\vee| = \emptyset = |\Delta^\vee|$. In view of Proposition 3.3, it might be natural to set $\mathrm{ld}_S(I_\Delta) = \mathrm{ld}_S((0)) = -\infty$. But, $I_\Gamma = \omega_S$ and hence $\mathrm{ld}_S(I_\Gamma) = 0$. One might think it is better to set $\mathrm{ld}_S((0)) = 0$ to avoid the problem. But this convention does not help so much, if we consider $K[\Delta]$ and $K[\Gamma]$. In fact, $\mathrm{ld}_S(K[\Delta]) = \mathrm{ld}_S(S) = 0$ and $\mathrm{ld}_S(K[\Gamma]) = \mathrm{ld}_S(S/\omega_S) = 1$.

(2) Let us think about the second statement of the theorem. Even if we forget the assumption that $\Delta \neq 2^T$, $\mathrm{ld}(\Delta)$ is almost a topological invariant. Under the assumption that $I_\Delta \neq 0$, we have the following.

- $\mathrm{ld}(\Delta) \leqslant 1$ if and only if $K[\Delta^\vee]$ is sequentially Cohen–Macaulay. Hence we can determine whether $\mathrm{ld}(\Delta) \leqslant 1$ from the topological space $|\Delta^\vee|$.
- $\mathrm{ld}(\Delta) = 0$, if and only if all facets of $\Delta^\vee$ have dimension $n-2$, if and only if $|\Delta^\vee|$ is Cohen–Macaulay and has dimension $n-2$.

Hence, if we forget the number "n", we cannot determine whether $\mathrm{ld}(\Delta) = 0$ from $|\Delta^\vee|$.

4. An upper bound of linearity defects

In the previous section, we have seen that $\mathrm{ld}_E(N) = \mathrm{ld}_S(\mathcal{S}(N))$ for $N \in \mathrm{Sq}(E)$, in particular $\mathrm{ld}_E(K\langle\Delta\rangle) = \mathrm{ld}_S(K[\Delta])$ for a simplicial complex Δ. In this section, we will give an upper bound of them, and see that the bound is sharp.

For $0 \neq N \in {}^*\mathrm{mod}\, E$, regarding N as a $\mathbb{Z}$-graded module, we set $\mathrm{indeg}_E(N) := \min\{i \mid N_i \neq 0\}$, which is called the *initial degree* of N, and $\mathrm{indeg}_S(M)$ is similarly defined as $\mathrm{indeg}_S(M) := \min\{i \mid M_i \neq 0\}$ for $0 \neq M \in {}^*\mathrm{mod}\, S$. If $\Delta \neq 2^{[n]}$ (equivalently $I_\Delta \neq 0$ or $J_\Delta \neq 0$), then we have $\mathrm{indeg}_S(I_\Delta) = \mathrm{indeg}_E(J_\Delta) = \min\{\sharp F \mid F \subset [n], F \notin \Delta\}$, where $\sharp F$ denotes the cardinal number of F. So we set

$$\mathrm{indeg}(\Delta) := \mathrm{indeg}_S(I_\Delta) = \mathrm{indeg}_E(J_\Delta).$$

Since $\mathrm{ld}(2^{[n]}) = \mathrm{ld}_S(S) = \mathrm{ld}_E(E) = 0$ holds, we henceforth exclude this trivial case; we assume that $\Delta \neq 2^{[n]}$.

We often make use of the following facts:

Lemma 4.1. *Let $0 \neq M \in {}^*\mathrm{mod}\, S$ and let $P_\bullet$ be a minimal graded free resolution of M. Then*

(1) $\mathrm{lin}_i(P_\bullet) = 0$ *for all $i < \mathrm{indeg}_S(M)$, i.e., there are only l-linear strands with $l \geqslant \mathrm{indeg}_S(M)$ in $P_\bullet$;*
(2) $\mathrm{lin}_{\mathrm{indeg}_S(M)}(P_\bullet)$ *is a subcomplex of $P_\bullet$;*
(3) *if $M \in \mathrm{Sq}(S)$, then $\mathrm{lin}(P_\bullet) = \bigoplus_{0 \leqslant l \leqslant n} \mathrm{lin}_l(P_\bullet)$, and $\mathrm{lin}_l(P_\bullet)_i = 0$ for all $i > n - l$ and all $0 \leqslant l \leqslant n$, where the subscript i is a homological degree.*

Proof. (1) and (2) are clear. (3) holds from the fact that $P_i \cong \bigoplus_{F \subset [n]} S(-F)^{\beta_{i,F}}$. □

Theorem 4.2. *For $0 \neq N \in \mathrm{Sq}(E)$, it follows that*

$$\mathrm{ld}_E(N) \leqslant \max\bigl\{0, n - \mathrm{indeg}_E(N) - 1\bigr\}.$$

By Theorem 3.4 this is equivalent to say that for $M \in \mathrm{Sq}(S)$,

$$\mathrm{ld}_S(M) \leqslant \max\bigl\{0, n - \mathrm{indeg}_S(M) - 1\bigr\}.$$

Proof. It suffices to show the assertion for $M \in \mathrm{Sq}(S)$. Set $\mathrm{indeg}_S(M) = d$ and let $P_\bullet$ be a minimal graded free resolution of M. The case $d = n$ is trivial by Lemma 4.1 (1), (3). Assume that $d \leqslant n - 1$. Observing that $\mathrm{lin}_l(P_\bullet)_i = S(-l-i)^{\beta_{i,i+l}}$, where $\beta_{i,i+l}$ are $\mathbb{Z}$-graded Betti numbers of M, Lemma 4.1 (1), (3) implies that the last few steps of $P_\bullet$ are of the form

$$0 \to S(-n)^{\beta_{n-d,n}} \to S(-n)^{\beta_{n-d-1,n}} \oplus S(-n+1)^{\beta_{n-d-1,n-1}} \to \cdots.$$

Hence $\mathrm{lin}_d(P_\bullet)_{n-d} = S(-n)^{\beta_{n-d,n}} = P_{n-d}$. Since $\mathrm{lin}_d(P_\bullet)$ is a subcomplex of the acyclic complex $P_\bullet$ by Lemma 4.1(2), we have $H_{n-d}(\mathrm{lin}_d(P_\bullet)) = 0$, so that $\mathrm{ld}_S(M) \leqslant n - d - 1$. □

Note that $J_\Delta \in \mathrm{Sq}(E)$ (resp. $I_\Delta \in \mathrm{Sq}(S)$). Since $\mathrm{ld}(\Delta) \leqslant \mathrm{ld}_E(J_\Delta) + 1$ (resp. $\mathrm{ld}(\Delta) \leqslant \mathrm{ld}_S(I_\Delta) + 1$) holds, we have a bound for $\mathrm{ld}(\Delta)$, applying Theorem 4.2 to J_Δ (resp. I_Δ).

Corollary 4.3. *For a simplicial complex Δ on $[n]$, we have*

$$\mathrm{ld}(\Delta) \leqslant \max\bigl\{1, n - \mathrm{indeg}(\Delta)\bigr\}.$$

Let Δ, Γ be simplicial complexes on $[n]$. We denote $\Delta * \Gamma$ for the join

$$\{F \cup G \mid F \in \Delta,\ G \in \Gamma\}$$

of Δ and Γ, and for our convenience, set

$$\mathrm{ver}(\Delta) := \bigl\{v \in [n] \mid \{v\} \in \Delta\bigr\}.$$

Lemma 4.4. *Let Δ be a simplicial complex on $[n]$. Assume that* $\mathrm{indeg}(\Delta) = 1$*, or equivalently* $\mathrm{ver}(\Delta) \neq [n]$*. Then we have*

$$\mathrm{ld}(\Delta) = \mathrm{ld}\bigl(\Delta * \{v\}\bigr)$$

for $v \in [n] \setminus \mathrm{ver}(\Delta)$.

Proof. We may assume that $v = 1$. Let $P_\bullet$ be a minimal graded free resolution of $K[\Delta * \{1\}]$ and $\mathcal{K}(x_1)$ the Koszul complex

$$0 \to S(-1) \xrightarrow{x_1} S \to 0$$

with respect to x_1. Consider the mapping cone $P_\bullet \otimes_S \mathcal{K}(x_1)$ of the map $P_\bullet(-1) \xrightarrow{x_1} P_\bullet$. There is the short exact sequence

$$0 \to P_\bullet \to P_\bullet \otimes_S \mathcal{K}(x_1) \to P_\bullet(-1)[-1] \to 0,$$

whence we have $H_i(P_\bullet \otimes_S \mathcal{K}(x_1)) = 0$ for all $i \geqslant 2$ and the exact sequence

$$0 \to H_1\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr) \to H_0\bigl(P_\bullet(-1)\bigr) \xrightarrow{x_1} H_0(P_\bullet).$$

But since $H_0(P_\bullet) = K[\Delta * \{1\}]$ and x_1 is regular on it, we have $H_1(P_\bullet \otimes_S \mathcal{K}(x_1)) = 0$. Thus $P_\bullet \otimes_S \mathcal{K}(x_1)$ is acyclic and hence a minimal graded free resolution of $K[\Delta]$. Note that $\mathrm{lin}(P_\bullet \otimes_S \mathcal{K}(x_1)) = \mathrm{lin}(P_\bullet) \otimes_S \mathcal{K}(x_1)$: in fact, we have

$$\begin{aligned}
\mathrm{lin}_l\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)_i &= \mathrm{lin}_l(P_\bullet \otimes_S S)_i \oplus \mathrm{lin}_l\bigl(P_\bullet[-1] \otimes_S S(-1)\bigr)_i \\
&= \bigl(\mathrm{lin}_l(P_\bullet)_i \otimes_S S\bigr) \oplus \bigl(\mathrm{lin}_l(P_\bullet)_{i-1} \otimes_S S(-1)\bigr) \\
&= \bigl(\mathrm{lin}_l(P_\bullet) \otimes_S \mathcal{K}(x_1)\bigr)_i,
\end{aligned}$$

where the subscripts i denote homological degrees, and the differential map

$$\mathrm{lin}_l\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)_i \to \mathrm{lin}_l\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)_{i-1}$$

is composed by $\partial_i^{\langle l\rangle}$, $-\partial_{i-1}^{\langle l\rangle}$, and the multiplication map by x_1, where $\partial_i^{\langle l\rangle}$ (resp. $\partial_{i-1}^{\langle l\rangle}$) is the ith (resp. $(i-1)$st) differential map of the l-linear strand of $P_\bullet$. Hence there is the short exact sequence

$$0 \to \operatorname{lin}(P_\bullet) \to \operatorname{lin}\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr) \to \operatorname{lin}(P_\bullet)(-1)[-1] \to 0,$$

which yields that $H_i(\operatorname{lin}(P_\bullet \otimes_S \mathcal{K}(x_1))) = 0$ for all $i \geqslant \operatorname{ld}(\Delta * \{1\}) + 2$, and the exact sequence

$$\begin{aligned} 0 \ \to \ & H_{\operatorname{ld}(\Delta*\{1\})+1}\bigl(\operatorname{lin}\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)\bigr) \to H_{\operatorname{ld}(\Delta*\{1\})}\bigl(\operatorname{lin}(P_\bullet)(-1)\bigr) \\ \xrightarrow{x_1} \ & H_{\operatorname{ld}(\Delta*\{1\})}\bigl(\operatorname{lin}(P_\bullet)\bigr) \to H_{\operatorname{ld}(\Delta*\{1\})}\bigl(\operatorname{lin}\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)\bigr). \end{aligned}$$

Since x_1 does not appear in any entry of the matrices representing the differentials of $\operatorname{lin}(P_\bullet)$, it is regular on $H_\bullet(\operatorname{lin}(P_\bullet))$, and hence we have

$$H_{\operatorname{ld}(\Delta*\{1\})+1}\bigl(\operatorname{lin}\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)\bigr) = 0$$

and

$$H_{\operatorname{ld}(\Delta*\{1\})}\bigl(\operatorname{lin}\bigl(P_\bullet \otimes_S \mathcal{K}(x_1)\bigr)\bigr) \neq 0,$$

since $H_{\operatorname{ld}(\Delta*\{1\})}(\operatorname{lin}(P_\bullet)) \neq 0$. Therefore $\operatorname{ld}(\Delta) = \operatorname{ld}(\Delta * \{1\})$. $\square$

Let Δ be a simplicial complex on $[n]$. For $F \subset [n]$, we set

$$\Delta_F := \{G \in \Delta \mid G \subset F\}.$$

The following fact, due to Hochster, is well known, but because of our frequent use, we mention it.

Proposition 4.5. *(Cf. [1,14].) For a simplicial complex Δ on $[n]$, we have*

$$\beta_{i,j}\bigl(K[\Delta]\bigr) = \sum_{F\subset[n],\ \sharp F = j} \dim_K \tilde{H}_{j-i-1}(\Delta_F; K),$$

where $\beta_{i,j}(K[\Delta])$ are the $\mathbb{Z}$-graded Betti numbers of $K[\Delta]$.

Now we can give a new proof of [18, Proposition 4.15], which is the latter part of the next result.

Proposition 4.6. *(Cf. [18, Proposition 4.15].) Let Δ be a simplicial complex on $[n]$. If* $\operatorname{indeg}\Delta = 1$*, then we have*

$$\operatorname{ld}(\Delta) \leqslant \max\{1, n-3\}.$$

Hence, for any Δ, we have

$$\operatorname{ld}(\Delta) \leqslant \max\{1, n-2\}.$$

Proof. The second inequality follows from the first one and Corollary 4.3. So it suffices to show the first. We set $\mathcal{V} := [n] \setminus \operatorname{ver}(\Delta)$. Our hypothesis $\operatorname{indeg}\Delta = 1$ implies that $\mathcal{V} \neq \varnothing$. By Lemma 4.4, the proof can be reduced to the case $\sharp\mathcal{V} = 1$. We may then assume that $\mathcal{V} = \{1\}$. Thus we have only to show that $\operatorname{ld}(\Delta * \{1\}) \leqslant \max\{1, n-3\}$. Since we have $\operatorname{indeg}(\Delta * \{1\}) \geqslant 2$, we may assume $n \geqslant 4$ by Corollary 4.3. The length of the 0-linear strand of $K[\Delta * \{1\}]$ is 0, and hence we concentrate on the l-linear strands with $l \geqslant 1$. Let $P_\bullet$ be a minimal graded free resolution of $K[\Delta * \{1\}]$. Since, as is well known, the cone of a simplicial complex, i.e. the join with a point, is acyclic, we have

$$\beta_{i,n}\big(K\big[\Delta * \{1\}\big]\big) = \dim_K \tilde{H}_{n-i-1}\big(\Delta * \{1\}; K\big) = 0$$

by Proposition 4.5. Thus $\operatorname{lin}_l(P_\bullet)_{n-l} = 0$ for all $l \geqslant 1$. Now applying the same argument as the last part of the proof of Theorem 4.2 (but we need to replace n by $n-1$), we have

$$H_{n-2}\big(\operatorname{lin}(P_\bullet)\big) = 0,$$

and so $\operatorname{ld}(\Delta * \{1\}) \leqslant n-3$. $\square$

According to [18, Proposition 4.14], we can construct a squarefree module $N \in \operatorname{Sq}(E)$ with $\operatorname{ld}_E(N) = \operatorname{proj.dim}_S(\mathcal{S}(N)) = n-1$. By Theorems 3.4 and 4.2, $M := \mathcal{S}(N)$ satisfies that $\operatorname{indeg}_S(M) = 0$ and $\operatorname{ld}_S(M) = n-1$. For $0 \leqslant i \leqslant n-1$, let $\Omega_i(M)$ be the ith syzygy of M. Then $\Omega_i(M)$ is squarefree, and we have that $\operatorname{ld}_S(\Omega_i(M)) = \operatorname{ld}_S(M) - i = n-i-1$ and $\operatorname{indeg}_S(\Omega_i(M)) \geqslant \operatorname{indeg}_S(M) + i = i$. Thus by Theorem 4.2, we know that $\operatorname{indeg}_S(\Omega_i(M)) = i$ and $\operatorname{ld}_S(\Omega_i(M)) = n - \operatorname{indeg}_S(\Omega_i(M)) - 1$. So the bound in Theorem 4.2 is optimal.

In the following, we will give an example of a simplicial complex Δ with $\operatorname{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$ for $2 \leqslant \operatorname{indeg}(\Delta) \leqslant n-2$, and so we know the bound in Proposition 4.3 is optimal if $\operatorname{indeg}(\Delta) \geqslant 2$, that is, $\operatorname{ver}(\Delta) = [n]$.

Given a simplicial complex Δ on $[n]$, we denote $\Delta^{(i)}$ for the ith skeleton of Δ, which is defined as

$$\Delta^{(i)} := \{F \in \Delta \mid \#F \leqslant i+1\}.$$

Example 4.7. Set $\Sigma := 2^{[n]}$, and let Γ be a simplicial complex on $[n]$ whose geometric realization $|\Gamma|$ is homeomorphic to the $(d-1)$-dimensional sphere with $2 \leqslant d < n-1$, which we denote by S^{d-1}. (For $m > d$ there exists a triangulation of S^{d-1} with m vertices. See, for example, [1, Proposition 5.2.10].) Consider the simplicial complex $\Delta := \Gamma \cup \Sigma^{(d-2)}$. We will verify that Δ is a desired complex, that is, $\operatorname{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$. For brief notation, we put $t := \operatorname{indeg}\Delta$ and $l := \operatorname{ld}(\Delta)$.

First, from our definition, it is clear that $t \geqslant d$. Thus it is enough to show that $n-d \leqslant l$; in fact we have that $l \leqslant n-t \leqslant n-d \leqslant l$ by Corollary 4.3, and hence that $t = d$ and $l = n-d$. Our aim is to prove that

$$\beta_{n-d,n}\big(K[\Delta]\big) \neq 0 \quad \text{and} \quad \beta_{n-d-1,n-1}\big(K[\Delta]\big) = 0,$$

since, in this case, we have $H_{n-d}(\operatorname{lin}_d(P_\bullet)) \neq 0$, and hence $n-d \leqslant l$.

Now, let $F \subset [n]$, and $\tilde{\mathcal{C}}_\bullet(\Delta_F; K)$, $\tilde{\mathcal{C}}_\bullet(\Gamma_F; K)$ be the augmented chain complexes of Δ_F and Γ_F, respectively. Since $\Sigma^{(d-2)}$ have no faces of dimension $\geqslant d-1$, we have $\tilde{\mathcal{C}}_{d-1}(\Delta_F; K) =$

$\tilde{\mathcal{C}}_{d-1}(\Gamma_F; K)$ and hence $\tilde{H}_{d-1}(\Delta_F; K) = \tilde{H}_{d-1}(\Gamma_F; K)$. On the other hand, our assumption that $|\Gamma| \approx S^{d-1}$ implies that Γ is Gorenstein, and hence that

$$\tilde{H}_{d-1}(\Gamma_F; K) = \begin{cases} K & \text{if } F = [n]; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by Proposition 4.5, we have that

$$\beta_{n-d,n}\big(K[\Delta]\big) = \dim_K \tilde{H}_{d-1}(\Gamma; K) = 1 \neq 0;$$
$$\beta_{n-d-1,n-1}\big(K[\Delta]\big) = \sum_{F \subset [n],\ \sharp F = n-1} \dim_K \tilde{H}_{d-1}(\Gamma_F; K) = 0.$$

5. A simplicial complex Δ with $\mathrm{ld}(\Delta) = n-2$ is an n-gon

Following the previous section, we assume that $\Delta \neq [n]$, throughout this section. We say a simplicial complex on $[n]$ is an *n-gon* if its facets are $\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}$, and $\{n,1\}$ after a suitable permutation of vertices. Consider the simplicial complex Δ on $[n]$ given in Example 4.7. If we set $d = 2$, then Δ is an n-gon. Thus if a simplicial complex Δ on $[n]$ is an n-gon, we have $\mathrm{ld}(\Delta) = n-2$. Actually, the inverse holds, that is, if $\mathrm{ld}(\Delta) = n-2$ with $n \geqslant 4$, Δ is nothing but an n-gon.

Theorem 5.1. *Let Δ be a simplicial complex on $[n]$ with $n \geqslant 4$. Then* $\mathrm{ld}(\Delta) = n-2$ *if and only if Δ is an n-gon.*

In the previous section, we introduced Hochster's formula (Proposition 4.5), but in this section, we need explicit correspondence between $[\mathrm{Tor}^S_\bullet(K[\Delta], K)]_F$ and reduced cohomologies of Δ_F, and so we will give it as follows.

Set $V := \langle x_1, \ldots, x_n \rangle = S_1$ and let $\mathcal{K}_\bullet := S \otimes_K \bigwedge V$ be the Koszul complex of S with respect to $x_1, \ldots, x_n$. Then we have

$$\big[\mathrm{Tor}^S_i\big(K[\Delta], K\big)\big]_F = H_i\big(\big[K[\Delta] \otimes_S \mathcal{K}_\bullet\big]_F\big) = H_i\Big(\Big[K[\Delta] \otimes_K \bigwedge V\Big]_F\Big)$$

for $F \subset [n]$. Furthermore, the basis of the K-vector space $[K[\Delta] \otimes_K \bigwedge V]_F$ is of the form $\mathbf{x}^G \otimes \wedge^{F \setminus G}\mathbf{x}$ with $G \in \Delta_F$, where $\mathbf{x}^G = \prod_{i \in G} x_i$ and $\wedge^{F\setminus G}\mathbf{x} = x_{i_1} \wedge \cdots \wedge x_{i_k}$ for $\{i_1, \ldots, i_k\} = F \setminus G$ with $i_1 < \cdots < i_k$. Thus the assignment

$$\varphi^i : \tilde{\mathcal{C}}^{i-1}(\Delta_F; K) \ni e^*_G \mapsto (-1)^{\alpha(G,F)}\mathbf{x}^G \otimes \wedge^{F\setminus G}\mathbf{x} \in \Big[K[\Delta] \otimes_K \bigwedge V\Big]_F$$

with $G \in \Delta_F$ gives the isomorphism $\varphi^\bullet : \tilde{\mathcal{C}}^\bullet(\Delta_F; K)[-1] \to [K[\Delta] \otimes_K \bigwedge V]_F$ of chain complexes, where $\tilde{\mathcal{C}}^{i-1}(\Delta_F; K)$ (resp. $\tilde{\mathcal{C}}_{i-1}(\Delta_F; K)$) is the $(i-1)$st term of the augmented cochain (resp. chain) complex of Δ_F over K, e_G is the basis element of $\tilde{\mathcal{C}}_{i-1}(\Delta_F; K)$ corresponding to G, and e^*_G is the K-dual base of e_G. Here we set

$$\alpha(A, B) := \sharp\big\{(a,b) \mid a > b,\ a \in A,\ b \in B\big\}$$

for $A, B \subset [n]$. Thus we have the isomorphism

$$\bar{\varphi}: \tilde{H}^{i-1}(\Delta_F; K) \to \left[\operatorname{Tor}^S_{\sharp F - i}\big(K[\Delta], K\big)\right]_F. \tag{5.1}$$

Lemma 5.2. *Let Δ be a simplicial complex on $[n]$ with $\operatorname{indeg}(\Delta) \geqslant 2$, and $P_\bullet$ a minimal graded free resolution of $K[\Delta]$. We denote $Q_\bullet$ for the subcomplex of $P_\bullet$ such that $Q_i := \bigoplus_{j \leqslant i+1} S(-j)^{\beta_{i,j}} \subset \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}} = P_i$. Assume $n \geqslant 4$. Then the following are equivalent.*

(1) $\operatorname{ld}(\Delta) = n-2$;
(2) $H_{n-2}(\operatorname{lin}_2(P_\bullet)) \neq 0$;
(3) $H_{n-3}(Q_\bullet) \neq 0$.

In the case $n \geqslant 5$, the condition (3) is equivalent to $H_{n-3}(\operatorname{lin}_1(P_\bullet)) \neq 0$.

Proof. Since $\operatorname{indeg}(\Delta) \geqslant 2$, $\operatorname{lin}_0(P_\bullet)_i = 0$ holds for $i \geqslant 1$. Clearly, $H_i(Q_\bullet) = H_i(\operatorname{lin}_1(P_\bullet))$ for $i \geqslant 2$. Since $\operatorname{lin}_l(P_\bullet)_i = 0$ for $i \geqslant n-2$ and $l \geqslant 3$ by Lemma 4.1 and that $\operatorname{ld}(\Delta) \leqslant n-2$ by Proposition 4.6, it suffices to show the following.

$$H_{n-2}\big(\operatorname{lin}_2(P_\bullet)\big) \cong H_{n-3}(Q_\bullet) \quad \text{and} \quad H_i(Q_\bullet) = 0 \quad \text{for } i \geqslant n-2. \tag{5.2}$$

Since $Q_\bullet$ is a subcomplex of $P_\bullet$, there exists the following short exact sequence of complexes.

$$0 \to Q_\bullet \to P_\bullet \to \tilde{P}_\bullet := P_\bullet / Q_\bullet \to 0,$$

which induces the exact sequence of homology groups

$$H_i(P_\bullet) \to H_i(\tilde{P}_\bullet) \to H_{i-1}(Q_\bullet) \to H_{i-1}(P_\bullet).$$

Hence the acyclicity of $P_\bullet$ implies that $H_i(\tilde{P}_\bullet) \cong H_{i-1}(Q_\bullet)$ for all $i \geqslant 2$. Now $H_i(\tilde{P}_\bullet) = 0$ for $i \geqslant n-1$ by Lemma 4.1 and the fact that $\tilde{P}_i = \bigoplus_{l \geqslant 2} \operatorname{lin}_l(P_\bullet)_i$. So the latter assertion of (5.2) holds, since $n-2 \geqslant 2$. The former follows from the equality $H_{n-2}(\tilde{P}_\bullet) = H_{n-2}(\operatorname{lin}_2(P_\bullet))$, which is a direct consequence of the fact that $\operatorname{lin}_2(P_\bullet)$ is a subcomplex of $\tilde{P}_\bullet$, that $\tilde{P}_{n-2} = \operatorname{lin}_2(P_\bullet)_{n-2}$, and that $\tilde{P}_{n-1} = 0$. $\square$

Let Δ be a 1-dimensional simplicial complex on $[n]$ (i.e., Δ is essentially a simple graph). A *cycle* C in Δ of length t ($\geqslant 3$) is a sequence of edges of Δ of the form $(v_1, v_2), (v_2, v_3), \ldots, (v_t, v_1)$ joining distinct vertices $v_1, \ldots, v_t$.

Now we are ready for the proof of Theorem 5.1.

Proof of Theorem 5.1. The implication "$\Leftarrow$" has been already done in the beginning of this section. So we shall show the inverse. By Proposition 4.6, we may assume that $\operatorname{indeg}(\Delta) \geqslant 2$. Let $P_\bullet$ be a minimal graded free resolution of $K[\Delta]$ and $Q_\bullet$ as in Lemma 5.2. Note that $Q_\bullet$ is determined only by $[I_\Delta]_2$ and that it follows $[I_\Delta]_2 = [I_{\Delta^{(1)}}]_2$. If the 1-skeleton $\Delta^{(1)}$ of Δ is an n-gon, then so is Δ itself. Thus by Lemma 5.2, we may assume that $\dim \Delta = 1$. Since $\operatorname{ld}(\Delta) = n-2$, by Lemma 5.2 we have

$$\tilde{H}_1(\Delta; K) \cong \tilde{H}^1(\Delta; K) \cong \left[\operatorname{Tor}^S_{n-2}\big(K[\Delta], K\big)\right]_{[n]} \neq 0,$$

and hence Δ contains at least one cycle as a subcomplex. So it suffices to show that Δ has no cycles of length $\leqslant n-1$. Suppose not, i.e., Δ has some cycles of length $\leqslant n-1$. To give a contradiction, we shall show

$$0 \to \mathrm{lin}_2(P_\bullet)_{n-2} \to \mathrm{lin}_2(P_\bullet)_{n-3} \tag{5.3}$$

is exact; in fact it follows $H_{n-2}(\mathrm{lin}_2(P_\bullet)) = 0$, which contradicts to Lemma 5.2. For that, we need some observations (this is a similar argument to that done in Theorem 4.1 of [15]). Consider the chain complex $K[\Delta] \otimes_K \bigwedge V \otimes_K S$ where V is the K-vector space with the basis $x_1, \dots, x_n$. We can define two differential map ϑ, ∂ on it as follows:

$$\vartheta\big(f \otimes \wedge^G \mathbf{x} \otimes g\big) = \sum_{i \in G} (-1)^{\alpha(i,G)} \big(x_i f \otimes \wedge^{G \setminus \{i\}} \mathbf{x} \otimes g\big);$$
$$\partial\big(f \otimes \wedge^G \mathbf{x} \otimes g\big) = \sum_{i \in G} (-1)^{\alpha(i,G)} \big(f \otimes \wedge^{G \setminus \{i\}} \mathbf{x} \otimes x_i g\big).$$

By a routine, we have that $\partial\vartheta + \vartheta\partial = 0$, and easily we can check that the ith homology group of the chain complex $(K[\Delta] \otimes_K \bigwedge V \otimes_K S, \vartheta)$ is isomorphic to the ith graded free module of a minimal free resolution $P_\bullet$ of $K[\Delta]$. Since, moreover, the differential maps of $\mathrm{lin}(P_\bullet)$ is induced by ∂ due to Eisenbud and Goto [4], Herzog, Simis and Vasconcelos [8], $\mathrm{lin}_l(P_\bullet)_i \to \mathrm{lin}_l(P_\bullet)_{i-1}$ can be identified with

$$\bigoplus_{F \subset [n],\ \sharp F = i+l} \big[\mathrm{Tor}_i^S\big(K[\Delta], K\big)\big]_F \otimes_K S \xrightarrow{\bar\partial} \bigoplus_{F \subset [n],\ \sharp F = i-1+l} \big[\mathrm{Tor}_{i-1}^S\big(K[\Delta], K\big)\big]_F \otimes_K S,$$

where $\bar\partial$ is induced by ∂. In the sequel, $-\{i\}$ denotes the subset $[n] \setminus \{i\}$ of $[n]$. Then we may identify the sequence (5.3) with

$$0 \to \big[\mathrm{Tor}_{n-2}^S\big(K[\Delta], K\big)\big]_{[n]} \otimes_K S \xrightarrow{\bar\partial} \bigoplus_{i \in [n]} \big[\mathrm{Tor}_{n-3}^S\big(K[\Delta], K\big)\big]_{-\{i\}} \otimes_K S$$

and hence, by the isomorphism (5.1), with

$$0 \to \tilde{H}^1(\Delta; K) \otimes_K S \xrightarrow{\bar\varepsilon} \bigoplus_{i \in [n]} \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S. \tag{5.4}$$

Here $\bar\varepsilon$ is composed by $\bar\varepsilon_i : \tilde{H}^1(\Delta; K) \otimes_K S \to \tilde{H}^1(\Delta_{-\{i\}}; K) \otimes_K S$ which is induced by the chain map

$$\varepsilon_i : \tilde{\mathcal{C}}^\bullet(\Delta; K) \otimes_K S \to \tilde{\mathcal{C}}^\bullet(\Delta_{-\{i\}}; K) \otimes_K S,$$
$$\varepsilon_i\big(e_G^* \otimes 1\big) = \begin{cases} (-1)^{\alpha(i,G)} e_G^* \otimes x_i & \text{if } i \notin G; \\ 0 & \text{otherwise.} \end{cases}$$

Well, let C be a cycle in Δ of the form $(v_1, v_2), (v_2, v_3), \dots, (v_t, v_1)$ with distinct vertices $v_1, \dots, v_t$. We say C has a *chord* if there exists an edge (v_i, v_j) of G such that $j \not\equiv i+1 \pmod t$, and C is said to be *minimal* if it has no chord. It is easy to see that the 1st homology of Δ is generated by those of minimal cycles contained in Δ, that is, we have the surjective map:

$$\bigoplus_{\substack{C\subset\Delta \\ C:\,\text{minimal cycle}}} \tilde{H}_1(C;K) \to \tilde{H}_1(\Delta;K).$$

Now by our assumption that Δ contains a cycle of length $\leqslant n-1$ (that is, Δ itself is not a minimal cycle), we have the surjective map

$$\bigoplus_{i\in[n]} \tilde{H}_1(\Delta_{-\{i\}};K) \xrightarrow{\bar{\eta}} \tilde{H}_1(\Delta;K) \tag{5.5}$$

where $\bar{\eta}$ is induced by the chain map $\eta : \bigoplus \tilde{\mathcal{C}}_\bullet(\Delta_{-\{i\}};K) \to \tilde{\mathcal{C}}_\bullet(\Delta;K)$, and η is the sum of

$$\eta_i : \tilde{\mathcal{C}}_\bullet(\Delta_{-\{i\}};K) \ni e_G \mapsto (-1)^{\alpha(i,G)} e_G \in \tilde{\mathcal{C}}_\bullet(\Delta;K).$$

Taking the K-dual of (5.5), we have the injective map

$$\tilde{H}^1(\Delta;K) \xrightarrow{\bar{\eta}^*} \bigoplus_{i\in[n]} \tilde{H}^1(\Delta_{-\{i\}};K),$$

where $\bar{\eta}^*$ is the K-dual map of $\bar{\eta}$, and composed by the K-dual

$$\bar{\eta}_i^* : \tilde{H}^1(\Delta;K) \to \tilde{H}^1(\Delta_{-\{i\}};K)$$

of $\bar{\eta}_i$. Then for all $0 \neq z \in \tilde{H}^1(\Delta;K)$, we have $\bar{\eta}_i^*(z) \neq 0$ for some i. Recalling the map $\bar{\varepsilon} : \tilde{H}^1(\Delta;K) \otimes_K S \to \bigoplus \tilde{H}^1(\Delta_{-\{i\}};K) \otimes_K S$ in (5.4) and its construction, we know for $z \in \tilde{H}^1(\Delta;K)$,

$$\bar{\varepsilon}(z\otimes y) = \sum_{i=1}^{n} \bar{\eta}_i^*(z) \otimes x_i y,$$

and hence $\bar{\varepsilon}$ is injective. □

Remark 5.3. (1) If Δ is an n-gon, then $\Delta^\vee$ is an $(n-3)$-dimensional Buchsbaum complex with $\tilde{H}_{n-4}(\Delta^\vee;K) = K$. If $n=5$, then $\Delta^\vee$ is a triangulation of the Möbius band. But, for $n \geqslant 6$, $\Delta^\vee$ is not a homology manifold. In fact, let $\{1,2\}, \{2,3\}, \dots, \{n-1,n\}, \{n,1\}$ be the facets of Δ, then if $F = [n] \setminus \{1,3,5\}$, easy computation shows that $\operatorname{lk}_{\Delta^\vee} F$ is a 0-dimensional complex with 3 vertices, and hence $\tilde{H}_0(\operatorname{lk}_{\Delta^\vee} F;K) = K^2$.

(2) If $\operatorname{indeg}\Delta \geqslant 3$, then the simplicial complexes given in Example 4.7 are not the only examples which attain the equality $\operatorname{ld}(\Delta) = n - \operatorname{indeg}(\Delta)$. We shall give two examples of such complexes.

Let Δ be the triangulation of the real projective plane $\mathbb{P}^2\mathbb{R}$ with 6 vertices which is given in [1, Fig. 5.8, p. 236]. Since $\mathbb{P}^2\mathbb{R}$ is a manifold, $K[\Delta]$ is Buchsbaum. Hence we have

$$H^2_{\mathfrak{m}}\big(K[\Delta]\big) = \big[H^2_{\mathfrak{m}}\big(K[\Delta]\big)\big]_0 \cong \tilde{H}_1(\Delta;K).$$

So, if $\mathrm{char}(K)=2$, then we have $\mathrm{depth}_S(\mathrm{Ext}_S^4(K[\Delta],\omega_S))=0$. Note that we have $\Delta=\Delta^\vee$ in this case. Therefore, easy computation shows that

$$\mathrm{ld}(\Delta^\vee)=\mathrm{ld}(\Delta)=3=6-3=6-\mathrm{indeg}(\Delta).$$

Next, as is well known, there is a triangulation of the torus with 7 vertices. Let Δ be the triangulation. Since $\dim\Delta=2$, we have $\mathrm{indeg}(\Delta^\vee)=7-\dim\Delta-1=4$. Observing that $K[\Delta]$ is Buchsbaum, we have, by easy computation, that

$$\mathrm{ld}(\Delta^\vee)=3=7-4=7-\mathrm{indeg}(\Delta^\vee).$$

Thus $\Delta^\vee$ attains the equality, but is not a simplicial complex given in Example 4.7, since it follows, from Alexander's duality, that

$$\dim_K \tilde{H}_i(\Delta^\vee;K)=\dim_K \tilde{H}_{4-i}(\Delta;K)=\begin{cases}2\neq 1 & \text{for } i=3;\\ 0 & \text{for } i\geqslant 4.\end{cases}$$

More generally, the dual complexes of d-dimensional Buchsbaum complexes Δ with $\tilde{H}_{d-1}(\Delta;K)\neq 0$ satisfy the equality

$$\mathrm{ld}(\Delta^\vee)=n-\mathrm{indeg}(\Delta^\vee),$$

but many of them differ from the examples in Example 4.7, and we can construct such complexes more easily as $\mathrm{indeg}(\Delta^\vee)$ is larger.

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