Example of a Quadratic System with Two Cycles Appearing in a Homoclinic Loop Bifurcation

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Received November 4, 1985; revised December 12, 1985

We give here a planar quadratic differential system depending on two parameters, \( \lambda, \beta \). There is a curve in the \( \lambda-\beta \) space corresponding to a homoclinic loop bifurcation (HLB). The bifurcation is degenerate at one point of the curve and we get a narrow tongue in which we have two limit cycles. This is the first example of such a bifurcation in planar quadratic differential systems. We propose also a model for the bifurcation diagram of a system with two limit cycles appearing at a singular point from a degenerate Hopf bifurcation, and dying in a degenerate HLB. This model shows a deep duality between degenerate Hopf bifurcations and degenerate HLBs. We give a bound for the maximal number of cycles that can appear in certain simultaneous Hopf and homoclinic loop bifurcations. We also give an example of quadratic system depending on three parameters which has at one place a degenerate Hopf bifurcation of order 3, and at another place a Hopf bifurcation of order 2 together with a HLB. We characterize the planar quadratic systems which are integrable in the neighbourhood of a homoclinic loop.

1. INTRODUCTION

The question answered here is part of Hilbert's 16th problem: Find the maximal number and positions of limit cycles for a differential system:

\[
\begin{align*}
\dot{x} &= P(x, y) \\
\dot{y} &= Q(x, y)
\end{align*}
\]  

(1.1)

with \( P \) and \( Q \) polynomials in \( x \) and \( y \). We are interested here in the case where \( P \) and \( Q \) are quadratic. It is known that a singular point cannot give rise to more than three limit cycles in a degenerate Hopf bifurcation \([1]\). The maximal number of limit cycles appearing in a degenerate homoclinic loop bifurcation (HLB) in a quadratic system has not yet been determined.

* This research was supported by the NSERC and the Ministry of Education of Quebec.
The following example given by Bogdanov [2]:
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -1 - \mu y + x^2 + xy
\end{align*}
\]  
(1.2)
is an example of a quadratic system, in which a limit cycle rises from the singular point \((-1, 0)\), in a Hopf bifurcation, at \(\mu = -1\), and grows bigger and bigger, until it dies in a HLB starting at the saddle point \((1, 0)\). The HLB occurs at \(\mu = -5/7\).

We consider here the following system:
\[
\begin{align*}
\dot{x} &= (\lambda - \delta - 1)x + y - x^2 + (1 + \delta)xy - y^2 \\
\dot{y} &= 2x + 2/3 x^2 - xy.
\end{align*}
\]  
(1.3)

This system has two singular points: \((0, 0)\), which is a saddle, and \((0, 1)\), which is a source. The system has a HLB on a curve in the \((\lambda, \delta)\)-plane. We show that this curve cuts the line \(\lambda = \delta - 1 = 0\) at one point. Since the trace of the saddle point is zero there, the HLB at this point is degenerate. Numerical simulation gives that this bifurcation is of codimension 2, since the bifurcation diagram has a narrow tongue starting from this point in which the system has two limit cycles.

Our system has two other bifurcations of codimension 2: we have a degenerate Hopf bifurcation of order 2 at \(\lambda = \delta = 0\), and we have the intersection of a Hopf bifurcation and a HLB at \(\lambda = 0, \delta \approx -0.4\). This is the first example of two cycles appearing simultaneously around a singular point in a quadratic system, one from a Hopf bifurcation, one from a HLB. Numerical simulation gives that the three bifurcations are organized as in Fig. 1. The curve \((H)\) corresponds to the Hopf bifurcation, the curve \((BH)\) to the HLB, and the curve \((2C)\) to a saddle-node bifurcation for cycles, namely the fusion of an attractive limit cycle with a repulsive one before they both disappear. The codimension 2 bifurcations mentioned above are located at the three vertices of the triangle. We pretend that the bifurcation diagram of Fig. 1 appears generically (at least locally) in many two-parameter families of planar differential systems. In Fig. 1 the local bifurcation diagrams around the vertex with the Hopf bifurcation of order two, and around the vertex with the HLB of order two, are similar. This shows a deep duality between Hopf bifurcations of order \(k\) and HLB of order \(k\), as was remarked by Joyal in [3]. We obtain that “cycles that appear in a Hopf bifurcation die in a HLB.”

The bonus of our construction of a quadratic system with a HLB of order two is that it gives an example of a quadratic system with the simultaneous birth of two limit cycles around a singular point, one from a Hopf bifurcation, one from a HLB. It also gives a positive answer to an
open problem mentioned in [4]: there exists a quadratic system with a limit cycle inside a homoclinic loop. The natural question to ask is then: What is the maximal number of limit cycles that can appear in simultaneous Hopf bifurcation and HLB? We give a bound for that number in Section 4, except in the case where the Hopf bifurcation is non-degenerate. The proof is dual to the proof that it is not possible to give birth to more than two limit cycles around two different singular points in two simultaneous Hopf bifurcations [5]. This enforces still more the principle that Hopf bifurcation at a focus and HLB at a saddle point are dual situations.

In the same way we consider the case of codimension 3 bifurcations. We expect a tetrahedral region, in which the system would have three limit cycles and the four vertices would correspond to:

- Hopf bifurcation of order 3
- Hopf bifurcation of order 2 together with nondegenerate HLB
- Hopf bifurcation of order 1 together with degenerate HLB of order 2
- Degenerate HLB of order 3.

We show that this is not possible, since the third kind of vertex cannot exist. The following example of a vector field depending on three parameters has bifurcations of the two first kinds.
\[ \dot{x} = (\lambda - \delta - 5) x + y - x^2 + (5 + \delta) xy - y^2 \]
\[ \dot{y} = (9 - \varepsilon + 2\delta) x + x^2 + (-8 + \varepsilon - 2\delta) xy. \]  
(1.4)

Equation (1.4) has a Hopf bifurcation of order 3 at \( \lambda = \delta = \varepsilon = 0 \), and a Hopf bifurcation of order 2 at \( \lambda = \varepsilon = 0, \delta \approx 1.785 \).

2. HOPF BIFURCATION AND POINCARÉ NORMAL FORM

We recall briefly how the Hopf bifurcation works. If the system (1.1) has a singular point at the origin and the linear part has pure imaginary eigenvalues \( \pm i\omega \), then the system can be brought, in complex coordinates, to the form:

\[ \dot{z} = i\omega z + F(z, \bar{z}) \]  
(2.1)

with

\[ F(z, \bar{z}) = \sum_{2 < i + j < r} a_{ij} z^i \bar{z}^j + O(|z|^{r+1}). \]  
(2.2)

There exists a polynomial change of variables:

\[ z = w + \sum_{2 \leq i + j \leq r} b_{ij} w^i \bar{w}^j \]  
(2.3)

such that (2.1) becomes:

\[ \dot{w} = i\omega w + c_1 w^2 \bar{w} + c_2 w^3 \bar{w} + \ldots + c_k w^{k+1} \bar{w}^k + O(|w|^{2k+3}) \]  
(2.4)

Equation (2.4) is called the Poincaré normal form of (2.1). Equation (1.1) has a Hopf bifurcation of order \( k \) at the origin if:

\[ \text{Re}(c_1) = \text{Re}(c_2) = \ldots = \text{Re}(c_{k-1}) = 0 \quad \text{and} \quad \text{Re}(c_k) \neq 0. \]  
(2.5)

Then any perturbation of (2.1) has at most \( k \) limit cycles and, for any \( i \leq k \), there exists a perturbation with exactly \( i \) limit cycles (Refs. [5–7]).

3. ANALYSIS OF THE SYSTEM (1.3)

**Theorem 3.1.** The system (1.3) has a homoclinic loop surrounding a limit cycle for some values of \( \lambda, \delta \).

**Proof:** We consider the system on the line \( \lambda = k\delta + k, k \in (-\infty, -3) \).
(i) Analysis of the phase portrait of (1.3) at $\lambda = 0$, $\delta = -1$ (cf. Fig. 2(a)). Let $W^s$ (resp. $W^u$) be the stable (unstable) manifold of the origin. They start with slopes $-\sqrt{2}$ (resp. $\sqrt{2}$). If $v(x, y) = (v_1(x, y), v_2(x, y))$ is our vector field, then

$$v_1(-x, y) = v_1(x, y)$$  \hspace{1cm} (3.1)

$$v_2(-x, y) + v_2(x, y) = 4/3 x^2 \Rightarrow |v_2(x, y)| > |v_2(-x, y)| \text{ for } x > 0.$$  

In Fig. 2 we represent the different regions where $\dot{x} > 0$, $\dot{x} < 0$, $\dot{y} > 0$, $\dot{y} < 0$. It is evident that $W^s$ must cut the $y$-axis at $1 < y^* < 2$. Suppose that $W^u$ cuts the $y$-axis at $1 < y^{**} < y^*$. Since $(0, 1)$ is weakly repulsive ($\lambda = 0$ gives a Hopf bifurcation and Re($c_1$) = $-\delta/4 > 0$), there must exist a limit cycle $\gamma$. Equation (3.1) shows that any limit cycle must be wider to the left than to the right of the $y$-axis. We must have:

$$\int_y v_1 dy - v_2 dx = 0 = \iint -3x \, dx \, dy.$$  \hspace{1cm} (3.2)

![Fig. 2. Phase portraits of the system (1.3).](image-url)
But we necessarily have $\int -3x \, dx \, dy > 0$, since the cycle is more to the left than to the right. A similar argument shows that $y^{**} \neq y^*$. Therefore if $W^u$ cuts the $y$-axis necessarily $y^{**} > y^*$.

(ii) Analysis of the phase portrait on $\lambda = k\delta + k$, $\lambda \to -\infty$, $k \in (-\infty, -3)$. $\dot{x}$ behaves like $k \delta x + \delta xy = \delta x(k + y)$. $k$ is chosen less than $-3$ for the system not to have any interference with the line $y = -k$. Then $\dot{x} < 0$ in the first quadrant and is positive in the second one. Also the minimum slope of $W^s$ at the origin is $-\sqrt{2}$. If we think of this as a limiting case, which means that $W^u$ and $W^s$ cannot cross the $y$-axis between $y = 0$ and $y = 1$, we get Fig. 2(b).

(iii) So there is a point on this line where a HLB occurs. At this point $\lambda - \delta - 1 = (k - 1)(\delta + 1) < 0$, so the HLB is attractive. On the other hand $\lambda < 0$, so the point $(0, 1)$ is attractive. Therefore there must exist a limit cycle.

COROLLARY 3.2. There is a region above the HLB in which there are at least two limit cycles inside the domain $\lambda - \delta - 1 < 0$. We have an attractive limit cycle surrounding a repulsive one.

Proof. The HLB is attractive, so it will give birth to an attractive limit cycle around the repulsive limit cycle found in Theorem 3.1.

THEOREM 3.3. The system (1.3) has a degenerate HLB at one point of the line $\lambda - \delta - 1 = 0$, in the interval $\delta \in (-\infty, -1)$.

Proof. We analyse the system on the line $\lambda = \delta + 1$. We know that the relative positions of $W^s$ and $W^u$ are given in Fig. 2(a) for $(\lambda, \delta) = (0, -1)$ and we find that they are given by Fig. 2(b) for $\lambda, \delta \to -\infty$, since $\dot{x}$ behaves like $xy$.

PROPOSITION 3.4. (i) The HLB curve crosses each line $\delta = \text{cons}$.

(ii) The HLB curve crosses the $\delta$-axis for a $\delta^* \in (-1, -\infty)$.

Proof. (i) We consider the system at $\delta = \text{cons}$, $\lambda \to -\infty$ and $\lambda \to +\infty$. We obtain figures similar to Fig. 2(b) and its symmetrical image for a symmetry with respect to the $y$-axis.

(ii) We consider the system at $\lambda = 0$ and $\delta \to +\infty$. $W^u$ and $W^s$ have relative positions as in Fig. 2(b).

PROPOSITION 3.5. System (1.3) has a Hopf bifurcation at the point $(0, 1)$ for $\lambda = 0$. It is non-degenerate for $\delta \neq 0$ and degenerate of order 2 for $\delta = 0$. There is a tongue starting at $(\lambda, \delta) = (0, 0)$ in which there are two limit cycles. The tongue is in the region $\lambda, \delta < 0$. 


Proof. Using notation of Section 2, \( \text{Re}(c_1) = -\delta/4 \neq 0 \) for \( \delta \neq 0 \). At \( \lambda = \delta = 0 \) we get \( \text{Re}(c_2) = -49/108 \) (calculation done on Macsyma). The two cycles are in the region \( \lambda < 0, \text{Re}(c_1) > 0, \text{Re}(c_2) < 0 \). (Note that \( \lambda \) is the real part of the eigenvalues of (1.3) at the point \((0, 1)\).)

**Theorem 3.6.** The system (1.3) has a bifurcation of codimension 2 which is the intersection of a Hopf bifurcation and a HLB on the line \( \lambda = 0 \). It happens for a \( \delta \in (-1, +\infty) \). (Numerical simulation gives \( \delta \cong -0.4 \).) So there is an angular region starting from this point in which the system has two limit cycles. The region is inside the larger region \( \lambda < 0, \lambda - \delta - 1 < 0 \).

Proof. Follows from Proposition 3.4(ii).

A global bifurcation diagram obtained numerically is given in Fig. 3. We find a triangular region as expected in Fig. 1. The parabola to the lower right corresponds to a saddle-node bifurcation (curve \((SN)\) of Fig. 3).

**Fig. 3. Bifurcation diagram of the system (1.3).**
Below the parabola two new singular points appear. Since they are both on the line \( y = 2 + 2/3 \times x \), they are of no interest in our present analysis. It appears that the region where there are two limit cycles has a corner at the degenerate HLB. So we conjecture that the HLB is of codimension 2. One can remark that the local bifurcation diagrams around the degenerate Hopf bifurcation at \((\lambda, \delta) = (0, 0)\) and around the degenerate HLB approximately at \((\lambda, \delta) = (-0.0651, -1.0651)\) are dual to each other. This situation was first remarked in [3]. Fig. 3 is an illustration of the principle: limit cycles that appear in Hopf bifurcations die in HLB and vice versa. This illustration is verified in many examples, one of which is the example of Bogdanov in equation (1.2) [2].

We also remark that the region where there are two limit cycles is very thin. This is frequent in quadratic systems: The more limit cycles we have, the thinner the region of the parameter-space where they can be seen. In the example of three limit cycles around the origin of Shi Songling [7], one of the parameters is as small as \(10^{-200}\).

4. Simultaneous Birth of Cycles in Hopf Bifurcations and HLB

The question we are asking here is the following: what is the maximal number of limit cycles that can appear in simultaneous Hopf bifurcations and HLB in a planar quadratic system? Theorem 3.6 says that this number is at least 2. We limit ourselves to consider two singular points, since it is shown that, for a quadratic system, any cycle contains a unique singular point [9].

**Theorem 4.1.** The maximal number of limit cycles that can appear in simultaneous degenerate Hopf bifurcation and HLB of order 1 is 4. It is not possible to have simultaneous degenerate Hopf bifurcation and degenerate HLB.

**Proof.** We can suppose that the system has \((0, 1)\) as a focus and \((0, 0)\) as a saddle point. Around \((0, 1)\), we can assume it has the form [5]:

\[
\begin{align*}
\dot{x} &= -y + ax^2 + bx y - y^2 \\
\dot{y} &= x + cx^2 + dx y
\end{align*}
\]

with \((0, 0)\) as a focus and \((0, 1)\) as a saddle point (cf. [5]). The system localized around \((0, 0)\) becomes (with \(1 + d > 0\):

\[
\begin{align*}
\dot{x} &= -bx + y + ax^2 + bxy - y^2 \\
\dot{y} &= (1 + d) x + cx^2 + dxy
\end{align*}
\]
For the HLB to be degenerate we need \( b = 0 \), and for the degeneracy of the Hopf bifurcation we need:

\[
\text{Re}(c_1) = 0 = (b(a - 1) - c(d + 2a))/8. \tag{4.3}
\]

In either case we obtain \( b = c = 0 \) or \( b = d + 2a = 0 \), which both correspond to integrable systems (cf. [5]). We need then to use Bautin's theorem [1]: no more than three limit cycles can appear in a degenerate Hopf bifurcation.

**Remark.** The proof of the proposition is exactly dual to the proof that no more than two limit cycles can appear in simultaneous Hopf bifurcations [5].

5. **BIFURCATIONS OF CODIMENSION 3 INVOLVING HLB**

**Theorem 5.1.** It is not possible to have a quadratic system with a Hopf bifurcation together with a degenerate HLB, unless the system is integrable.

**Proof.** We can suppose that the system is in form (4.1) around \((0, 1)\). If the system has a degenerate HLB, then \( b = 0 \). The system localized at \((0, 0)\) then becomes:

\[
\begin{align*}
x &= y + ax^2 - y^2 \\
y &= (1 + d)x + cx^2 + dxy.
\end{align*}
\]  

Then:

\[
v_1(-x, y) = v_1(x, y) \\
v_2(-x, y) + v_2(x, y) = 2cx^2 \Rightarrow |v_2(x, y)| > |v_2(-x, y)| \quad \text{for } x > 0 \text{ if } c > 0 \\
|v_2(x, y)| < |v_2(-x, y)| \quad \text{for } x > 0 \text{ if } c < 0.
\]  

Then if we suppose we have a homoclinic loop we get a contradiction, as in Theorem 3.1(i).

**Proposition 5.2.** The system (1.4) has a Hopf bifurcation of codimension 3 at the point \((0, 1)\) for the parameter values \( \lambda = \delta = \epsilon = 0 \). We have three cycles in a region \( \lambda > 0, \delta > 0, \epsilon > 0 \).

**Proof.** The system (1.4) localized at \((0, 1)\) gives, for \( \lambda = \delta = \epsilon = 0 \):

\[
\begin{align*}
\dot{x} &= -Y - x^2 + 5xY - Y^2 \\
\dot{Y} &= x + x^2 - 8xY.
\end{align*}
\]  

\tag{5.3}
For this system we get \( \text{Re}(c_1) = \text{Re}(c_2) = 0, \text{Re}(c_3) < 0 \). We have three limit cycles in a region where \( \lambda > 0, \text{Re}(c_1) < 0, \text{Re}(c_2) > 0 \) (cf. [5] for calculations of conditions).

Discussion of System (1.4). We obtain numerically a Hopf bifurcation of order 2 together with a HLB. The Hopf bifurcation of order 2 occurs on the line \( \lambda = \varepsilon = 0 \). If we follow the invariant manifolds of \((0, 0)\), when \( \delta \) varies we find that they cross each other for \( \delta \geq 1.785 \).

6. INTEGRABLE SYSTEMS IN THE NEIGHBOURHOOD OF A HOMOCLINIC LOOP

We now derive conditions for a quadratic system to be integrable in the neighbourhood of a homoclinic loop. For this we remark that a quadratic system integrable in the neighbourhood of the loop is necessarily integrable everywhere inside.

Theorem 6.1. A quadratic system integrable in the neighbourhood of a homoclinic loop is integrable everywhere inside the loop. It happens if either the system is Hamiltonian or it can be brought in suitable coordinates to the form:

\[
\begin{align*}
\dot{x} &= y + ax^2 - y^2 \\
\dot{y} &= x + dxy
\end{align*}
\] (6.1)

Proof. We use the following properties of a quadratic system:

— A homoclinic loop is necessarily convex.

— A homoclinic loop contains a unique singular point, which is necessarily a focus or a center.

These facts are proved in [9]. We can suppose that the saddle point is \( p = (0, 0) \) and the singular point inside the loop is \( q = (0, 1) \). The segment joining these two points is without contact with the curve [9], and we can consider a return map along this segment: in fact all paths cross the segment \([p, q]\) in one direction and cross the line \([q, \infty]\) in the other direction [9]. If a path does not return to the segment \([p, q]\) then it has to go to \( q \), but this is impossible, since \( q \) is not a node. The return map is thus defined everywhere on \([p, q]\), it is analytic on the open segment, and is equal to the identity in a neighbourhood of \( p \). Therefore it is the identity map everywhere on the segment, and \( q \) is a center. We suppose that the system localized at \( q \) is:

\[
\begin{align*}
\dot{x} &= -Y + ax^2 + bxY - Y^2 \\
\dot{Y} &= x + cx^2 + dxY.
\end{align*}
\] (6.2)
Since the HLB at $p$ is degenerate, we must have $b = 0$. Then (6.2) is integrable in the neighbourhood of $q$ iff one of the following conditions is satisfied ([5] or [9]):

I. $c = 0$ 

II. $d + 2a = 0$. 

(6.3)

We must also have the condition $1 - d > 0$ for $p$ to be a saddle point. The case II is the case of a Hamiltonian system with Hamiltonian:

$$H(x,y) = -y^2/2 - ax^2y + y^3/3 + (1-d)x^2/2 + cx^3/3.$$ 

(6.4)

 Scaling in case I brings the system to the form (6.1). An algebraic integrand can be found in the neighbourhood of $q$ ([9]) and extended inside the whole loop.

In both cases the vector field is symmetric under the $y$-axis.

**REFERENCES**


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