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ON CONTEXT-FREE TREES

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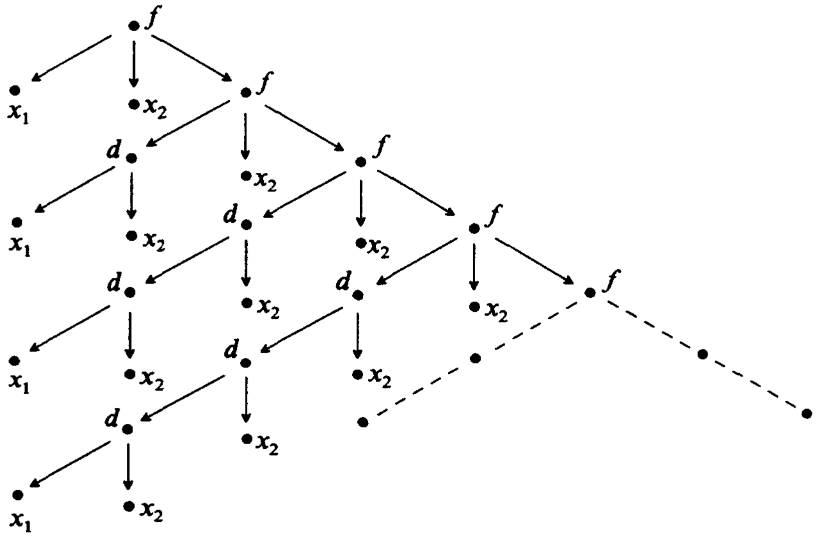
Introduction

In the first section we introduce the concept of algebraic theory with iterate (Definition 1.4). This concept is more general than the concept of rationally closed algebraic theory of the ADJ-group and the concept of generalized iterative algebraic theory of Ésik [3]. Although our concept arises from Căzănescu and Ungureanu's study concerning flowcharts [1], we now use it as a tool to study context-free trees. In this paper we study different methods to obtain the subtheory with iterate generated by a given set of morphisms only.

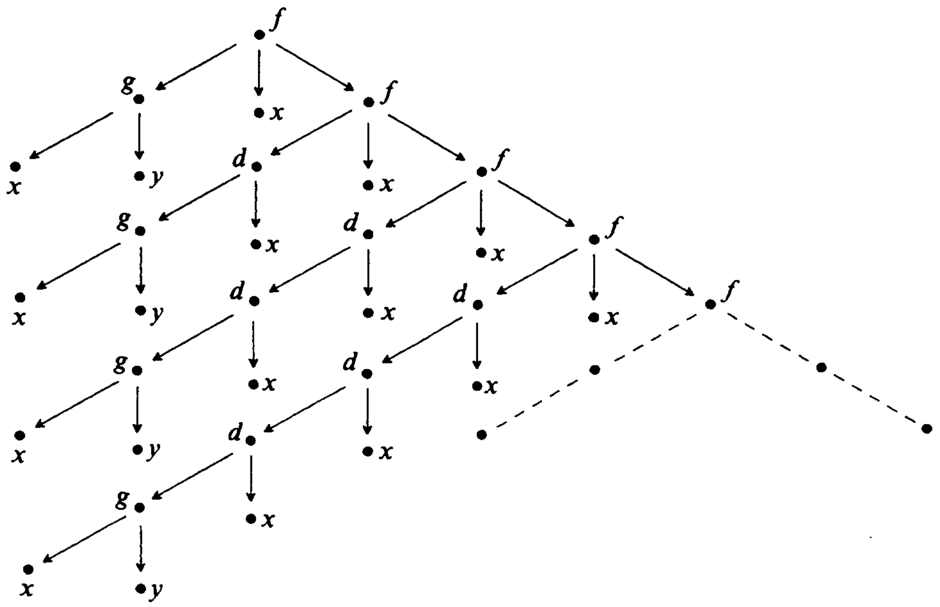
The context-free trees are a very useful tool to study recursive programs. For motivation, examples and semantics, see [6] or [5].

Let V be the set of variables. Let V' be the set of all the trees $\sigma(x_1, x_2, \ldots, x_n)$, where σ is an operation symbol and the x_i are distinct variables. As above, finite trees are written as expressions.

By the iteration of the operation symbol σ of arity n in the tree t we understand the least solution of the context-free equation $\sigma(x_1, x_2, \ldots, x_n) = t$. For example, the iteration of σ in the tree $f(x_1, x_2, \sigma(d(x_1, x_2), x_2))$ is the following tree:



Let us now substitute $g(x, y)$ for x_1 and x for x_2. The result is the following tree:



The latter tree is a substitution of trees for the variables in the iteration of σ in the tree $f(x_1, x_2, \sigma(d(x_1, x_2), x_2))$.

By substitution of the tree t' for the operation symbol σ of arity n in the tree t we understand that we use $(\sigma(x_1, x_2, \dots, x_n), t')$ as a rewriting rule for all the occurrences of σ in t. For example, the substitution of the tree $g(u(x_2), v(x_1))$ for f in the tree $f(g(x, y), f(x, z))$ is $g(u(g(u(z), v(x))), v(g(x, y)))$.

Theorem 2.1 gives the following four different ways to generate all context-free trees.

(A) The set of generators is $V \cup V'$. The trees are generated by two operations:
- substitution of trees for the variables of another tree,
- iteration of one operation symbol in one tree.

(B) The set of generators is V. The trees are generated by two operations:
- application of the usual operations of the algebra of trees,
- substitution of trees for the variables in the iteration of one operation symbol in one tree.

(C) The set of generators is V. The trees are generated by three operations:
- application of the usual operations of the algebra of trees,
- iteration of one operation symbol in one tree,
- substitution of a tree for one operation symbol in another tree.

(D) The set of generators is $V \cup V'$. The trees are generated by one operation:
- substitution of trees for the variables in the iteration of one operation symbol in one tree.

Let us say something on the applications of these generation methods.

Every generation method gives a structural method to prove properties of context-free trees. The classical structural induction says that if a set of trees includes V and is closed under the usual operations of the algebra of trees, then the set contains all total finite trees. The classical structural induction may be used to prove properties of total finite trees but it is not sufficient to prove properties of context-free trees because the structure of context-free trees is not the same as the structure of total finite trees. Then, the classical structural induction must be adapted to context-free trees. The adaptation is obtained by the generation methods.

On the other hand, these generation methods give the theoretical ground for defining finite expressions for context-free trees. Perhaps it is not uninteresting to study the computation rules for such expressions.

1. Algebraic theories with iterate

For each nonnegative integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let S be the set of sorts. Let S^* be the free monoid generated by S. The length of $u \in S^*$ is denoted by $|u|$ and u itself will be denoted by

$$u = u_1 u_2 \ldots u_{|u|},$$

where $u_i \in S$ for all $i \in [|u|]$.

Let T be an *S-sorted algebraic theory*, i.e., T is a category whose class of objects, denoted $|T|$, is the set S^* and in which every $u \in S^*$ is the coproduct of u_i for $i \in [|u|]$.

The empty word of S^*, denoted λ, is an initial object of T. For each $u \in S^*$, we set

$$0_u : \lambda \to u$$

the unique morphism from λ to u. For each $u \in S^*$, let

$$x_1^u, x_2^u, \ldots, x_{|u|}^u$$

be the *distinguished morphisms* of the coproduct u. If $|u| = 1$, then $x_1^u = 1_u$ where 1_u is the identity morphism of u. If $v \in S^*$ and $f_i : u_i \to v$ for every $i \in [|u|]$, then

$$\langle f_1, f_2, \ldots, f_{|u|}\rangle : u \to v,$$

the tupling of $f_1, f_2, \ldots, f_{|u|}$, is the unique morphism such that

$$x_i^u \langle f_1, f_2, \ldots, f_{|u|}\rangle = f_i$$

for all $i \in [|u|]$. If $u, v \in S^*$, we denote

$$1_u + 0_v = \langle x_1^{uv}, x_2^{uv}, \ldots, x_{|u|}^{uv}\rangle$$

and

$$0_u + 1_v = \langle x_{|u|+1}^{uv}, x_{|u|+2}^{uv}, \ldots, x_{|uv|}^{uv}\rangle.$$

If $u, v, w \in S^*$, $f \in T(u, w)$, and $g \in T(v, w)$, then the source pairing of f and g,

$$\langle f, g\rangle : uv \to w,$$

is the unique morphism with the following properties:

$$(1_u + 0_v)\langle f, g\rangle = f \quad \text{and} \quad (0_u + 1_v)\langle f, g\rangle = g.$$

If $f: u \to v$ and $g: p \to q$, then the sum of f and g,

$$f+g: up \to vq,$$

is defined by $f+g = \langle f(1_v + 0_q), g(0_v + 1_q)\rangle$.

A subcategory of T is called an *algebraic subtheory* of T if it has the same class of objects, contains all the distinguished morphisms, and is closed under tupling. The morphisms from the least subtheory of T are called base morphisms.

A family $G = \{G(s, u) \mid s \in S, u \in S^*\}$ is called a *genus* if $G(s, u) \subseteq T(s, u)$ for every $s \in S$ and $u \in S^*$. For a genus G let us define

$$G^0(u, v) = \{\langle f_1, f_2, \ldots, f_{|u|}\rangle \mid f_i \in G(u_i, v) \text{ for all } i \in [|u|]\}$$

for every $u, v \in S^*$. It is easy to see that G^0 includes G and is closed under source pairing and under left composition with base morphisms.

If G is an algebraic subtheory or T, then its genus $\{G(s, u) \mid (s, u) \in S \times S^*\}$ has the following properties:

(S1) $x_i^u \in G(u_i, u)$ for all $u \in S^*$ and $i \in [|u|]$,

(S2) if $s \in S$, $u, v \in S^*$, $f \in G(s, u)$ and $g \in G^0(u, v)$, then $fg \in G(s, v)$.

Conversely, if the genus G has properties (S1) and (S2), then G^0 is an algebraic subtheory of T.

An *S-sorted algebraic theory* is said to be *ordered* if, for each $u, v \in S^*$, $T(u, v)$ is a partially ordered set with least element $\perp_{uv}$ such that

(a) the composition of morphisms is monotonic,

(b) $\perp_{uv} f = \perp_{uw}$ for all $u, v, w \in S^*$ and all $f \in T(v, w)$,

(c) the tupling operation is monotonic.

A *rationally closed S-sorted algebraic theory* T is an ordered S-sorted algebraic theory equipped with a function

$$^\dagger: T(u, uv) \to T(u, v)$$

for all $u, v \in S^*$, where $f^\dagger$ is called the *iterate* of $f: u \to uv$ and must satisfy the following conditions for all $g: u \to v$ and $h: v \to w$:

(R1) $f\langle f^\dagger, 1_v\rangle = f^\dagger$,

(R2) $(f(1_u + h))^\dagger = f^\dagger h$,

(R3) if $f\langle g, 1_v\rangle \leqslant g$, then $f^\dagger \leqslant g$.

Note that $f^\dagger$ is the least solution for x in the equation

$$f\langle x, 1_v\rangle = x.$$

Proposition 1.1. *Let $f: u \to uv$ and $g: v \to w$ be morphisms in a rationally closed algebraic theory. The morphism $f^\dagger g$ is the least solution for x in the equation*

$$f\langle x, g\rangle = x.$$

Proof. The above equation is equivalent to

$$f(1_u+g)\langle x, 1_w\rangle = x.$$

Therefore, the least solution is $(f(1_u+g))^\dagger = f^\dagger g$. □

Proposition 1.2. *For all $f: u \to uvw$ and $g: v \to uvw$ in a rationally closed algebraic theory,*

$$\langle f, g\rangle^\dagger = \langle f^\dagger\langle h^\dagger, 1_w\rangle, h^\dagger\rangle,$$

where

$$h = g\langle f^\dagger, 1_{vw}\rangle.$$

Proof. Let $i = \langle f^\dagger\langle h^\dagger, 1_w\rangle, h^\dagger\rangle$. It follows that

$$\langle i, 1_w\rangle = \langle f^\dagger\langle h^\dagger, 1_w\rangle, \langle h^\dagger, 1_w\rangle\rangle = \langle f^\dagger, 1_{vw}\rangle\langle h^\dagger, 1_w\rangle.$$

The equalities

$$\langle f, g\rangle\langle i, 1_w\rangle = \langle f\langle f^\dagger, 1_{vw}\rangle\langle h^\dagger, 1_w\rangle, g\langle f^\dagger, 1_{vw}\rangle\langle h^\dagger, 1_w\rangle\rangle$$
$$= \langle f^\dagger\langle h^\dagger, 1_w\rangle, h\langle h^\dagger, 1_w\rangle\rangle = i$$

show that $\langle f, g\rangle^\dagger \leq i$.

Let $x = (1_u + 0_v)\langle f, g\rangle^\dagger$ and $y = (0_u + 1_v)\langle f, g\rangle^\dagger$. It follows from $\langle f, g\rangle^\dagger = \langle f, g\rangle\langle\langle f, g\rangle^\dagger, 1_w\rangle$ that

$$\langle x, y\rangle = \langle f\langle x, y, 1_w\rangle, g\langle x, y, 1_w\rangle\rangle.$$

Therefore,

$$x = f\langle x, \langle y, 1_w\rangle\rangle \quad\text{and}\quad y = g\langle x, \langle y, 1_w\rangle\rangle.$$

From Proposition 1.1 we deduce that $f^\dagger\langle y, 1_w\rangle \leq x$. From

$$h\langle y, 1_w\rangle = g\langle f^\dagger, 1_{vw}\rangle\langle y, 1_w\rangle = g\langle f^\dagger\langle y, 1_w\rangle, \langle y, 1_w\rangle\rangle$$
$$\leq g\langle x, \langle y, 1_w\rangle\rangle = y$$

it follows that

$$h^\dagger \leq y \quad\text{and}\quad f^\dagger\langle h^\dagger, 1_w\rangle \leq f^\dagger\langle y, 1_w\rangle \leq x.$$

Therefore, $i \leq \langle x, y\rangle = \langle f, g\rangle^\dagger$.

From $i = \langle f, g\rangle^\dagger$ we obtain

$$\langle f, g\rangle^\dagger = \langle f^\dagger\langle h^\dagger, 1_w\rangle, h^\dagger\rangle. \qquad \square$$

Proposition 1.3. *Let $f: u \to uv$ and $i: w \to u$ be morphisms in a rationally closed algebraic theory.*

(a) *If $j: u \to w$ and $ji \leq 1_w$, then $(if(j+1_v))^\dagger \leq if^\dagger$.*

(b) *If i is an isomorphism, then*

$$(if(i^{-1}+1_v))^\dagger = if^\dagger.$$

Proof. (a) It follows from

$$if(j+1_v)\langle if^\dagger, 1_v\rangle = if\langle jif^\dagger, 1_v\rangle \leq if\langle f^\dagger, 1_v\rangle = if^\dagger$$

that

$$(if(j+1_v))^\dagger \leq if^\dagger.$$

(b) From (a) we infer

$$(if(i^{-1}+1_v))^\dagger \leq if^\dagger$$

and

$$(i^{-1}(if(i^{-1}+1_v))(i+1_v))^\dagger \leq i^{-1}(if(i^{-1}+1_v))^\dagger.$$

Therefore,

$$if^\dagger = i(i^{-1}(if(i^{-1}+1_v))(i+1_v))^\dagger \leq (if(i^{-1}+1_v))^\dagger. \qquad \square$$

For every $u, v \in S^*$ we define the morphism

$$S_u^v : uv \to vu$$

by $S_u^v = \langle 0_v + 1_u, 1_v + 0_u \rangle$. Notice that $S_u^v S_v^u = 1_{uv}$. Therefore, S_v^u is an isomorphism.

Definition 1.4. An S-sorted algebraic theory is said to be *with iterate* if, for every $u, v \in S^*$, a mapping

$$^\dagger : T(u, uv) \to T(u, v)$$

is given, called iterate, and satisfying the following axioms:

(I1) $f\langle f^\dagger, 1_v\rangle = f^\dagger$ for every $f \in T(u, uv)$,

(I2) $(f(1_u + g))^\dagger = f^\dagger g$ for every $f \in T(u, uv)$ and $g \in T(v, w)$,

(I3) $\langle f, g\rangle^\dagger = \langle f^\dagger\langle h, 1_w\rangle, h\rangle$, where $h = (g\langle f^\dagger, 1_{vw}\rangle)^\dagger$ for every $f \in T(u, uvw)$ and $g \in T(v, uvw)$,

(I4) $\langle f, g\rangle^\dagger = S_u^v(\langle g, f\rangle(S_u^v + 1_w))^\dagger$ for $f \in T(u, uvw)$ and $g \in T(v, uvw)$.

Let us give some intuitive explanations. A morphism $f \in T(u, uv)$ may be thought of as a system of equations where u represents the unknowns and v represents some parameters. One may refer to the systems associate to context-free grammars, to the systems whose solutions are rational trees, or to the systems whose solutions are context-free trees as in Section 2. The iterate $f^\dagger$ represents the least solution of the system f. Axiom (I1) says that $f^\dagger$ is a solution of the system f. The morphism $g \in T(v, w)$ from axiom (I2) represents a substitution of new parameters represented by w for the old parameters. Axiom (I2) is a commutative law: we may replace the parameters first and then solve the system or vice versa. Axiom (I3) permits to split the system $\langle f, g\rangle$ into two subsystems f and g. To obtain the solution $\langle f, g\rangle^\dagger$ we have in axiom (I3) the following algorithm: introduce the solution $f^\dagger$ of the first subsystem f in the second subsystem g and get the solution h of this new system $g\langle f^\dagger, 1_{vw}\rangle$; the solution of $\langle f, g\rangle$ is given by the substitution $f^\dagger\langle h, 1_w\rangle$ of h in the solution of the first subsystem and by h itself. Axiom (I4) is another commutative law. It says that the solution of a system does not depend on the order of the equations into the system. The identities (1.1), (1.2), (1.3), and (1.4) below may be interpreted in a similar way.

We mention that, in the above definition, axiom (I2) may be replaced by its particular case: $(f+0_w)^\dagger = f^\dagger + 0_w$ for every $f \in T(u, uv)$ and $w \in S^*$.

The proof that every generalized iterative theory is an algebraic theory with iterate may be found in [3].

Theorem 1.5. *Every rationally closed algebraic theory is an algebraic theory with iterate.*

Proof. Proposition 1.2 proves axiom (I3). Axiom (I4) is an easy consequence of Proposition 1.3(b). □

In the sequel, T will be an S-sorted algebraic theory with iterate.

If $f \in T(u, v)$, then

$$(0_u + f)^\dagger = f. \tag{1.1}$$

Indeed, we deduce from axiom (I1) that

$$(0_u + f)^\dagger = (0_u + f)\langle (0_u + f)^\dagger, 1_v\rangle = f.$$

If $f \in T(u, uw)$ and $g \in T(v, vw)$, then

$$\langle f(1_u + 0_v + 1_w), 0_u + g\rangle^\dagger = \langle f^\dagger, g^\dagger\rangle. \tag{1.2}$$

Indeed, as $(0_u + g)\langle (f(1_u + 0_v + 1_w))^\dagger, 1_{vw}\rangle = g$, we deduce from axioms (I3) and (I2) that

$$\begin{aligned}\langle f(1_u + 0_v + 1_w), 0_u + g\rangle^\dagger &= \langle (f(1_u + 0_v + 1_w))^\dagger \langle g^\dagger, 1_w\rangle, g^\dagger\rangle \\ &= \langle f^\dagger(0_v + 1_w)\langle g^\dagger, 1_w\rangle, g^\dagger\rangle = \langle f^\dagger, g^\dagger\rangle.\end{aligned}$$

If $g \in T(v, u)$ and $f \in T(u, uvw)$, then

$$(gf^\dagger)^\dagger = g(f\langle 1_u + 0_w, g + 1_w\rangle)^\dagger. \tag{1.3}$$

Indeed, using in turn axiom (I3), axiom (I4), axiom (I3), and identity (1.1) we deduce

$$\begin{aligned}(gf^\dagger)^\dagger &= ((g + 0_{vw})\langle f^\dagger, 1_{vw}\rangle)^\dagger = (0_u + 1_v)\langle f, g + 0_{vw}\rangle^\dagger \\ &= (0_u + 1_v)S^v_u(\langle g + 0_{vw}, f\rangle(S^v_u + 1_w))^\dagger \\ &= (1_v + 0_u)\langle 0_v + g + 0_w, f(S^v_u + 1_w)\rangle^\dagger \\ &= (0_v + g + 0_w)^\dagger\langle (f(S^v_u + 1_w)\langle (0_v + g + 0_w)^\dagger, 1_{uw}\rangle)^\dagger, 1_w\rangle \\ &= (g + 0_w)\langle (f(S^v_u + 1_w)\langle g + 0_w, 1_u + 0_w, 0_u + 1_w\rangle)^\dagger, 1_w\rangle \\ &= g(f\langle 1_u + 0_w, g + 0_w, 0_u + 1_w\rangle)^\dagger = g(f\langle 1_u + 0_w, g + 1_w\rangle)^\dagger.\end{aligned}$$

If $f \in T(u, uv)$ and $i \in T(w, u)$ is an isomorphism, then

$$if^\dagger = (if(i^{-1} + 1_v))^\dagger. \tag{1.4}$$

Indeed, using (1.1) and (1.3) we deduce

$$\begin{aligned}&(if(i^{-1}+1_v))^\dagger\\ &\quad = (i(0_u + f(i^{-1}+1_v))^\dagger)^\dagger\\ &\quad = i((0_u + f(i^{-1}+1_v))\langle 1_u + 0_v, i + 1_v\rangle)^\dagger = i(f(i^{-1}+1_v)(i+1_v))^\dagger = if^\dagger.\end{aligned}$$

Lemma 1.6. *Let G be a genus such that*

(S3) *for all $(s, v) \in S \times S^*$, if $f \in G(s, sv)$, then $f^\dagger \in G(s, v)$,*

and

(S4) *$g \in G(s, tv)$ and $f \in G(t, tv)$ imply $g\langle f^\dagger, 1_v\rangle \in G(s, v)$ for all $s, t \in S$ and all $v \in S^*$.*

If $g \in G^0(u, uv)$, then
(a) *$g^\dagger \in G^0(u, v)$,*
(b) *if $s \in S$ and $h \in G(s, uv)$, then $h\langle g^\dagger, 1_v\rangle \in G(s, v)$.*

Proof. The lemma follows by induction on $|u|$. For $|u| = 1$, the conclusion immediately follows from the hypothesis.

Let $g \in G^0(su, suv)$ where $s \in S$, and $u, v \in S^*$. Since $g = \langle x_1^{su} g, (0_s + 1_u)g\rangle$, axiom (I3) implies

$$g^\dagger = \langle (x_1^{su} g)^\dagger \langle h^\dagger, 1_v\rangle, h^\dagger\rangle,$$

where $h = (0_s + 1_u) g \langle (x_1^{su} g)^\dagger, 1_{uv}\rangle$. It follows from (S3) that $(x_1^{su} g)^\dagger \in G(s, uv)$ and from (S4) that $h \in G^0(u, uv)$. We deduce from the inductive hypothesis that

$$h^\dagger \in G^0(u, v) \quad \text{and} \quad (x_1^{su} g)^\dagger \langle h^\dagger, 1_v\rangle \in G(s, v).$$

Therefore, $g^\dagger \in G^0(su, v)$.

If $t \in S$ and $i \in G(t, suv)$, then

$$i\langle g^\dagger, 1_v\rangle = i\langle (x_1^{su} g)^\dagger, 1_{uv}\rangle\langle h^\dagger, 1_v\rangle \in G(t, v). \qquad \square$$

An algebraic subtheory of T is said to be an *algebraic subtheory with iterate* of T if it is closed under iterate.

If G is the genus of an algebraic subtheory with iterate, then G has properties (S1), (S2), (S3), (S4),

(S5) if $s, t \in S$, $u \in S^*$, $f \in G(t, su)$ and $g \in G(s, u)$, then $f\langle g, 1_u\rangle \in G(t, u)$

and

(S6) if $s \in S$, $u, v, w \in S^*$ and $f \in G(s, uv)$, then $f(1_u + 0_w + 1_v) \in G(s, uwv)$.

Theorem 1.7. *Let G be a genus in an algebraic theory with iterate. Each of the following conditions:*

(SA): (S1), (S2), *and* (S3),

(SB): (S1), (S2), *and* (S4),

(SC): (S1), (S3), (S5), *and* (S6),

(SD): (S1), (S4), and (S6)

is sufficient for G^0 *to be an algebraic subtheory with iterate.*

Proof. We first prove the equivalence of conditions (SA), (SB), (SC), and (SD).

Case [(S1) and (S2) imply (S5)]: With the same notation as in (S5) we remark that $f\langle g, 1_u\rangle = f\langle g, x_1^u, x_2^u, \ldots, x_{|u|}^u\rangle$.

Case [(S1) and (S2) imply (S6)]: With the same notation as in (S6) we remark that

$$f(1_u + 0_w + 1_v) = f\langle x_1^{uwv}, \ldots, x_{|u|}^{uwv}, x_{|uw|+1}^{uwv}, \ldots, x_{|uw|+|v|}^{uwv}\rangle.$$

Case [(S1) and (S4) imply (S3)]: With the same notation as in (S3) we remark that $f^\dagger = x_1^{sv}\langle f^\dagger, 1_v\rangle$.

Case [(S3) and (S5) imply (S4)]: Obvious.

Case [(S4) and (S6) imply (S5)]: With the same notation as in (S5) we remark that $f\langle g, 1_u\rangle = f\langle (g(0_s + 1_u))^\dagger, 1_u\rangle$.

Case [(S5) and (S6) imply (S2)]: Let $f \in G(s, u)$ and $g \in G^0(u, v)$. From (S6), $f(1_u + 0_v) \in G(s, uv)$. For every $i \in [|u|]$, let $wi = u_{i+1}u_{i+2} \ldots u_{|u|}$ and $h_i = x_i^u g(0_{wi} + 1_v)$. As $h_i \in G(u_i, wiv)$, by (S6) the conclusion follows from (S5) and the equality

$$fg = f(1_u + 0_v)\langle h_1, 1_{w1v}\rangle\langle h_2, 1_{w2v}\rangle \ldots \langle h_{|u|}, 1_v\rangle.$$

The conclusion follows from (S1), (S2), (S3), (S4), and Lemma 1.6. □

The latter theorem may be used to obtain the least algebraic subtheory with iterate which includes a given set of morphisms. Of course, each morphism $f: u \to v$ is replaced by the set of its components $\{x_i^u f \mid i \in [|u|]\}$. Then, one of the four conditions of Theorem 1.7 may be used.

Condition (SD) seems to be preferable because if G has properties (S1) and (S6), then the least genus which contains G and has property (S4) will have properties (S1) and (S6), too. For (S6), the proof is by induction on the number of applications of rule (S4). Indeed, for $s, t \in S$, $u, v, w \in S^*$, $f: t \to tuv$, and $g: s \to tuv$ we notice that

$$\begin{aligned} g\langle f^\dagger, 1_{uv}\rangle(1_u + 0_w + 1_v) &= g\langle f^\dagger(1_u + 0_w + 1_v), 1_u + 0_w + 1_v\rangle \\ &= (g(1_{tu} + 0_w + 1_v))\langle (f(1_{tu} + 0_w + 1_v))^\dagger, 1_{uwv}\rangle. \end{aligned}$$

This remark may be used when G is the genus of an algebraic subtheory because in this case G has properties (S1) and (S6).

Another way to obtain the least algebraic subtheory with iterate which includes a given set of morphisms is given in the following theorems which are similar to the main result in [2].

Elgot [2] introduced the concept of iterative algebraic theory, where the iterate is only a partial function, but if $f: u \to uv$ has an iterate, then it is the unique solution of the equation $f\langle x, 1_v\rangle = x$. We mention that the iterate of a pointed iterative algebraic theory may be extended to obtain an algebraic theory with iterate. The main result of Elgot [2] gives, in an iterative algebraic theory, two methods to obtain the least algebraic subtheory which is closed under the iterate and includes a given set of morphisms.

If G is a genus, let us denote, for every $u, v \in S^*$,

$$\mathrm{IG}(u, v) = \{(1_u + 0_w)f^\dagger \mid f \in G^0(uw, uwv),\ w \in S^*\}.$$

Theorem 1.8. *If the genus G is closed under right composition with base morphisms, then* IG *includes G and is closed under composition, source pairing, and iterate. If, moreover, G has property* (S1), *then* IG *is the least algebraic subtheory with iterate which includes G.*

Proof. As G is closed under right composition with base morphisms, we deduce that G^0 is closed under right composition with base morphisms and under sum.

Let $f \in G^0(u, v)$. It follows from (1.1) that

$$f = (0_u + f)^\dagger = (1_u + 0_\lambda)(f(0_u + 1_v))^\dagger \in \mathrm{IG}(u, v).$$

Therefore, IG includes G^0, hence IG includes G.

If $f = (1_u + 0_p)g^\dagger$ with $g \in G^0(up, upv)$ and $h = (1_v + 0_q)i^\dagger$ with $i \in G^0(vq, vqw)$, then

$$\begin{aligned} fh &= (1_u + 0_p)g^\dagger(1_v + 0_{qw})\langle i^\dagger, 1_w\rangle && \text{(by (I2))}\\ &= (1_u + 0_{pvq})\langle (g + 0_{qw})^\dagger\langle i^\dagger, 1_w\rangle, i^\dagger\rangle && \text{(by (I3))}\\ &= (1_u + 0_{pvq})\langle g(1_{upv} + 0_{qw}), i(0_{up} + 1_{vqw})\rangle^\dagger \in \mathrm{IG}(u, v). \end{aligned}$$

If $f = (1_u + 0_p)g^\dagger$ with $g \in G^0(up, upw)$ and $h = (1_v + 0_q)i^\dagger$ with $i \in G^0(vq, vqw)$, then

$$\begin{aligned} \langle f, h\rangle &= (1_u + 0_p + 1_v + 0_q)\langle g^\dagger, i^\dagger\rangle && \text{(by (1.2))}\\ &= (1_{uv} + 0_{pq})(1_u + S^p_v + 1_q)\langle g(1_{up} + 0_{vq} + 1_w), 0_{up} + i\rangle^\dagger && \text{(by (1.4))}\\ &= (1_{uv} + 0_{pq})((1_u + S^p_v + 1_q)\langle g(1_u + 0_v + 1_p + 0_q + 1_w),\\ &\qquad i(0_u + 1_v + 0_p + 1_{qw})\rangle)^\dagger. \end{aligned}$$

Therefore, $\langle f, h\rangle \in \mathrm{IG}(uv, w)$.

If $f = (1_u + 0_v)g^\dagger$ with $g \in G^0(uv, uvuw)$, then we deduce

$$f^\dagger = (1_u + 0_v)(g\langle 1_{uv} + 0_w, 1_u + 0_v + 1_w\rangle)^\dagger \quad \text{(by (1.3))}.$$

Therefore, $f^\dagger \in \mathrm{IG}(u, w)$.

If, moreover, G has property (S1), then G^0 contains all base morphisms. Therefore, IG contains all base morphisms. □

Theorem 1.9. *Let G be a genus which is closed under right composition with base morphisms. If, for all $(s, u) \in S \times S^*$,*

$$R(s, u) = \{x_1^{sv} f^\dagger \mid f \in G^0(sv, svu),\ v \in S^*\},$$

then $R^0 = \mathrm{IG}$.

Proof. It follows from the definitions that $R(s, u) = \mathrm{IG}(s, u)$ for every $(s, u) \in S \times S^*$. As IG is closed under source pairing, we deduce that IG includes R^0.

We prove by induction on $|u|$ that $\mathrm{IG}(u, v) \subseteq R^0(u, v)$ for all $u, v \in S^*$. Let $f \in \mathrm{IG}(us, v)$ with $s \in S$. Then, $f = (1_{us} + 0_w)g^\dagger$, where $g \in G^0(usw, uswv)$. By inductive hypothesis, $(1_u + 0_{sw})g^\dagger \in R^0(u, v)$. Therefore, $(1_u + 0_s)f \in R^0(u, v)$. As

$$(0_u + 1_s)f = (0_u + 1_s + 0_w)g^\dagger = (1_s + 0_{uw})(S_s^u + 1_w)g^\dagger \quad \text{(by (1.4))}$$

$$= (1_s + 0_{uw})((S_s^u + 1_w)g(S_u^s + 1_{wv}))^\dagger,$$

we deduce that $(0_u + 1_s)f \in R(s, v)$, and therefore $f \in R^0(us, v)$. □

2. The theory with iterate of context-free trees

Let T and T' be two S-sorted algebraic theories. A *theory morphism* $F: T \to T'$ is a functor satisfying the following conditions:

(a) $F(u) = u$ for all $u \in S^*$,

(b) $F(x_i^u) = x_i^u$ for all $u \in S^*$ and $i \in [|u|]$.

We notice that if $f: u \to v$ and $g: w \to v$ are morphisms in T, then $F(\langle f, g\rangle) = \langle F(f), F(g)\rangle$.

Let Σ be a set and let $a: \Sigma \to S \times S^*$ be a function. If T is an S-sorted algebraic theory, then $h: \Sigma \to T$ is an *interpretation* of Σ iff $h(\sigma) \in T(a(\sigma))$ for all $\sigma \in \Sigma$. Let T_Σ denote the free S-sorted algebraic theory generated by Σ and let $I_\Sigma: \Sigma \to T_\Sigma$ be its standard interpretation. Then, for each S-sorted algebraic theory T and for each interpretation $h: \Sigma \to T$ there exists a unique theory morphism $h': T_\Sigma \to T$ such that $I_\Sigma h' = h$, i.e., $h'(I_\Sigma(\sigma)) = h(\sigma)$ for all $\sigma \in \Sigma$.

Let T and T' be ordered S-ordered algebraic theories. An *ordered theory morphism* $F: T \to T'$ is a theory morphism such that for all $u, v \in S^*$ the restriction of F to $T(u, v)$ is a monotonic and strict ($f(\perp_{uv}) = \perp_{uv}$) function.

An *ω-continuous S-sorted algebraic theory* T is an ordered S-sorted algebraic theory satisfying the following conditions:

(a) for each $u, v \in S^*$, $T(u, v)$ is ω-complete, i.e., each ω-chain has a least upper bound,

(b) the composition of morphisms is ω-continuous, i.e., the composition preserves least upper bounds of ω-chains. Every ω-continuous S-sorted algebraic theory is a rationally closed S-sorted algebraic theory.

Let T and T' be ω-continuous S-sorted algebraic theories. An *ω-continuous theory morphism* $F: T \to T'$ is an ordered theory morphism such that for all $u, v \in S^*$ the restriction of F to $T(u, v)$ is an ω-continuous function. We order the set of all ω-continuous theory morphisms from T to T' by the natural pointwise ordering: for $F, G: T \to T'$, $F \leqslant G$ iff, for all $f \in T(u, v)$, $F(f) \leqslant G(f)$. This ordering is ω-complete. Indeed, if $\{F_n\}_{n \in \omega}$ is an ω-chain and, for $u, v \in S^*$ and $f: u \to v$,

$$F(f) = \bigvee \{F_n(f) \mid n \in \omega\},$$

then F is an ω-continuous theory morphism. Every ω-continuous theory morphism $F: T \to T'$ preserves the iterate, i.e., $F(f^\dagger) = F(f)^\dagger$ for each $f \in T(u, uv)$.

Let CT_Σ denote the free ω-continuous theory generated by Σ and let $J_\Sigma : \Sigma \to \mathrm{CT}_\Sigma$ be its standard interpretation. Without loss of generality we assume that T_Σ is an algebraic subtheory of CT_Σ and I_Σ is the co-restriction of J_Σ. If T is an ω-continuous S-sorted algebraic theory and $f: \Sigma \to T$ an interpretation, then we denote by $f^{\#}: \mathrm{CT}_\Sigma \to T$ the unique ω-continuous theory morphism such that $J_\Sigma f^{\#} = f$. If we pointwise order the set of all the interpretations from Σ to T, then the applications "#" is an isomorphism of partially ordered sets.

In the sequel, we denote by T the ω-continuous $(S \times S^*)$-sorted algebraic theory which is used for solving systems of context-free equations. We give its definition. Letters p, q, and r will denote elements of $(S \times S^*)^*$. For all p, let

$$\Sigma_p = \{\sigma_1^p, \sigma_2^p, \ldots, \sigma_{|p|}^p\}$$

and let

$$a_p : \Sigma_p \to S \times S^*$$

be the function defined for all $i \in [|p|]$ by

$$a_p(\sigma_i^p) = p_i.$$

By definition, $T(p, q)$ is the set of all interpretations of Σ_p in CT_{Σ_q}. The set $T(p, q)$ is ordered pointwise: if $f, g \in T(p, q)$, then $f \leqslant g$ iff $f(\sigma_i^p) \leqslant g(\sigma_i^p)$ in $\mathrm{CT}_{\Sigma_q}(p_i)$ for all $i \in [|p|]$. This ordering is ω-complete and $T(p, q)$ has a least element $\perp_{pq}$. The composition of morphisms is defined by

$$fg = fg^{\#},$$

where $f \in T(p, q)$ and $g \in T(q, r)$. The composition is associative and ω-continuous; for each p, the morphism $1_p = J_{\Sigma_p}$ is an identity and $\perp_{pq} g = \perp_{pr}$ for all $g \in T(q, r)$. For all p and $i \in [|p|]$ the distinguished morphism

$$y_i^p : p_i \to p$$

is defined by

$$y_i^p(\sigma_1^{p_i}) = J_{\Sigma_p}(\sigma_i^p).$$

If $f_i : p_i \to q$ for all $i \in [|p|]$, then their tupling is defined by

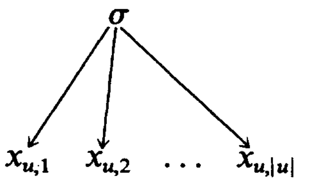
$$\langle f_1, f_2, \ldots, f_{|p|}\rangle(\sigma_i^p) = f_i(\sigma_1^{p_i})$$

for all $i \in [|p|]$. The tupling is monotonic.

As in the right-hand side of a context-free equation we usually have a finite total tree, we need a subtheory of T. For all p, q, let $T'(p, q)$ be the set of all the interpretations f of Σ_p in CT_{Σ_q} such that $f(\sigma_i^p) \in T_{\Sigma_q}(p_i)$ for all $i \in [|p|]$. It is easy to see that T' is an algebraic subtheory of T.

Let CFT be the least algebraic subtheory with iterate of T which contains T'. In fact, CFT is the algebraic theory of context-free trees. We shall give in this special case a better construction of CFT than the general one presented in Section 1. We shall begin with some intuitive explanations and some notations.

For some $s \in S$ and $u \in S^*$, the elements of $\mathrm{CT}_\Sigma(s, u)$ are Σ-trees (trees with symbols of operations from Σ) of sort s and with $|u|$ variables $x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}$ of sorts $u_1, u_2, \ldots, u_{|u|}$. The subset $T_\Sigma(s, u)$ contains all total finite trees of the same kind. If $u \in S^*$ and $i \in [|u|]$, then the Σ-tree $x_i^u \in T_\Sigma(u_i, u)$ is equal to the variable $x_{u,i}$. If $\sigma \in \Sigma$ and $a(\sigma) = (s, u)$, then $I_\Sigma(\sigma) = J_\Sigma(\sigma)$ is the following Σ-tree:

$\sigma(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}) =$ 

If $(s, u) \in S \times S^*$, an element of $T((s, u), p)$ may be identified with an element a' of $\mathrm{CT}_{\Sigma_p}(s, u)$, i.e., with a Σ_p-tree of sort s and with variables $x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}$, but it is perhaps better to think of it as the equality

$$\sigma_1^{(s,u)}(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|}) = a',$$

which gives a definition of the operation symbol $\sigma_1^{(s,u)}$ with the aid of the Σ_p-tree a', i.e., in terms of the operation symbols of Σ_p.

A morphism $f \in T(p, q)$ then gives a definition of the operation symbols of Σ_p in terms of the operation symbols of Σ_q. In this way, a system of context-free equations is a morphism $f \in T(p, pq)$ where $\sigma_1^{pq}, \sigma_2^{pq}, \ldots, \sigma_{|p|}^{pq}$ are unknown operation symbols and $\sigma_{|p|+1}^{pq}, \ldots, \sigma_{|p|+|q|}^{pq}$ are known operation symbols, because when we compute $f^\dagger$, the operation symbols $\sigma_1^{pq}, \ldots, \sigma_{|p|}^{pq}$ are identified with operation symbols $\sigma_1^p, \ldots, \sigma_{|p|}^p$.

For each p, each $u \in S^*$, and each $i \in [|u|]$, let

$$x_i^{p,u} : (u_i, u) \to p$$

be the morphism of T' defined by

$$x_i^{p,u}(\sigma_1^{(u_i,u)}) = x_i^u.$$

The morphism $x_i^{p,u}$ shows that the operation $\sigma_1^{(u_i,u)}$ is the ith projection.

For each $u \in S^*$ and each $t = (s, s_1 s_2 \ldots s_n) \in S \times S^*$ let

$$r = (s_1, u)(s_2, u) \ldots (s_n, u)t$$

and let $M_t^u : (s, u) \to r$ be a morphism of T' defined by

$$M_t^u(\sigma_1^{(s,u)}) = J_{\Sigma_r}(\sigma_{n+1}^r)\langle J_{\Sigma_r}(\sigma_1^r), \ldots, J_{\Sigma_r}(\sigma_n^r)\rangle.$$

The above equality shows that

$$\sigma_1^{(s,u)}(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|})$$
$$= \sigma_{n+1}^r(\sigma_1^r(x_{u,1}, \ldots, x_{u,|u|}), \ldots, \sigma_n^r(x_{u,1}, \ldots, x_{u,|u|})).$$

For each $t \in S \times S^*$ and each p we assume $G(t, p) \subseteq T(t, p)$ and we list some conditions on the genus G:

(1) $x_i^{p,u} \in G((u_i, u), p)$ for all $u \in S^*$, $i \in [|u|]$ and p,

(2) $y_i^p \in G(p_i, p)$ for all p and $i \in [|p|]$,

(3) for each $u \in S^*$, $t = (s, s_1 s_2 \ldots s_n) \in S \times S^*$, and p, if $f \in G(t, p)$ and $f_i \in G((s_i, u), p)$ for all $i \in [n]$, then $M_t^u\langle f_1, f_2, \ldots, f_n, f\rangle \in G((s, u), p)$,

(4) for each $u \in S^*$, $t = (s, s_1 s_2 \ldots s_n) \in S \times S^*$, and p, if $f \in G(t, tp)$ and $f_i \in G((s_i, u), p)$ for all $i \in [n]$, then $M_t^u\langle f_1, f_2, \ldots, f_n, f^\dagger\rangle \in G((s, u), p)$,

(5) for each $u \in S^*$, p, and $i \in [|p|]$, if $p_i = (s, s_1 s_2 \ldots s_n)$ and if $f_j \in G((s_j, u), p)$ for all $j \in [n]$, then $M_{p_i}^u\langle f_1, f_2, \ldots, f_n, y_i^p\rangle \in G((s, u), p)$,

(6) for each p and each $t \in S \times S^*$, if $f \in G(t, tp)$, then $f^\dagger \in G(t, p)$,

(7) for each p and each $t, t' \in S \times S^*$, if $f \in G(t', tp)$ and $g \in G(t, p)$, then $f\langle g, 1_p\rangle \in G(t', p)$,

(8) for each $t \in S \times S^*$ and each p, q, if $f \in G(t, p)$, then $0_q + f \in G(t, qp)$.

We first notice that condition (2) equals (S1), (6) equals (S3), (7) equals (S5) and that (S6) implies condition (8). Then we give some intuitive explanations, where the elements of $G(t, p)$ are thought of as Σ_p-trees.

Condition (1) says that G contains the variables.

Condition (2) says that for each p and $i \in [|p|]$, if $p_i = (s, u)$, then $G(p_i, p)$ contains $\sigma_i^p(x_{u,1}, x_{u,2}, \ldots, x_{u,|u|})$.

Condition (3) says that G is closed under substitution for variables.

Condition (4) says that G is closed under substitution for variables in an iteration.

Condition (5) says that G is closed under the algebraic operations.

Condition (7) says that G is closed under substitution for an operation symbol.

Theorem 2.1. *The genus of* CFT *is equal to the least genus satisfying any* (*hence each*) *of the following conditions*:

(A) *conditions* (1), (2), (3), *and* (6),

(B) *conditions* (1), (4), *and* (5),

(C) *conditions* (1), (5), (6), *and* (7),

(D) *conditions* (1), (2), *and* (4).

Proof. The proof is divided into five parts.

Part I. For each $s \in S$, $u \in S^*$, p, and $f \in T((s, u), p)$,

$$M^u_{(s,u)}\langle x_1^{p,u}, x_2^{p,u}, \ldots, x_{|u|}^{p,u}, f\rangle = f.$$

Indeed, if $r = (u_1, u)(u_2, u) \ldots (u_{|u|}, u)(s, u)$, then

$$\begin{aligned}
&(M^u_{(s,u)}\langle x_1^{p,u}, x_2^{p,u}, \ldots, x_{|u|}^{p,u}, f\rangle)(\sigma_1^{(s,u)}) \\
&= \langle x_1^{p,u}, x_2^{p,u}, \ldots, x_{|u|}^{p,u}, f\rangle^{\#}(M^u_{(s,u)}(\sigma_1^{(s,u)})) \\
&= \langle x_1^{p,u}, \ldots, x_{|u|}^{p,u}, f\rangle^{\#}(J_{\Sigma_r}(\sigma^r_{|u|+1})\langle J_{\Sigma_r}(\sigma_1^r), \ldots, J_{\Sigma_r}(\sigma^r_{|u|})\rangle) \\
&= f(\sigma_1^{(s,u)})\langle x_1^{p,u}(\sigma_1^{(u_1,u)}), \ldots, x_{|u|}^{p,u}(\sigma_1^{(u_{|u|},u)})\rangle \\
&= f(\sigma_1^{(s,u)})\langle x_1^u, x_2^u, \ldots, x_{|u|}^u\rangle = f(\sigma_1^{(s,u)}).
\end{aligned}$$

Part II. We prove some implications between the previous eight conditions.

(a) [(1) and (5) imply (2)]: Follows from Part I with y_i^p for f.

(b) [(2) and (4) imply (5)]: Since $y_i^p = (y_{i+1}^{p,p})^\dagger$.

(c) [(2) and (3) imply (5)]: Obvious.

(d) [(3) and (6) imply (4)]: Obvious.

(e) [(1) and (4) imply (6)]: With the same notation as in (6) and with $t = (s, u)$ it follows from Part I that

$$f^\dagger = M^u_{(s,u)}\langle x_1^{p,u}, x_2^{p,u}, \ldots, x_{|u|}^{p,u}, f^\dagger\rangle.$$

(f) [(4) and (8) imply (3)]: With the same notation as in (3) it follows from (1.1) that

$$M^u_t\langle f_1, f_2, \ldots, f_n, f\rangle = M^u_t\langle f_1, f_2, \ldots, f_n, (0_t + f)^\dagger\rangle.$$

(g) [(5), (7), and (8) imply (3)]: With the same notation as in (3) it suffices to notice the equality

$$M^u_t\langle f_1, \ldots, f_n, f\rangle = M^u_t\langle 0_t + f_1, \ldots, 0_t + f_n, y_1^{tp}\rangle\langle f, 1_p\rangle.$$

Part III. It follows from implications (a) and (b) that conditions (B) and (D) are equivalent.

Let GA, GB, and GC be the least genera satisfying conditions (A), (B), and (C), respectively.

It follows from (d) that (A) implies (D). Therefore, (A) implies (B), hence GA includes GB.

It is easy to show that the genus of CFT fulfills conditions (C). Therefore, the genus of CFT includes GC.

Part IV. We prove that GB includes the genus of CFT.

It is known that, for each $u \in S^*$, the Σ_p-algebra

$$(\{T_{\Sigma_p}(s, u)\}_{s \in S}, \{\sigma_i\}_{i \in [|p|]})$$

is freely generated by $\{x_i^u \mid i \in [|u|]\}$, where, for all $i \in [|p|]$, if $p_i = (s, s_1 s_2 \ldots s_n)$ and $h_j \in T_{\Sigma_p}(s_j, u)$ for every $j \in [n]$, then

$$\sigma_i(h_1, h_2, \ldots, h_n) = I_{\Sigma_p}(\sigma_i^p)\langle h_1, h_2, \ldots, h_n\rangle.$$

Since there exists a natural bijection between $T_{\Sigma_p}(s,u)$ and $T'((s,u),p)$, it follows that the Σ_p-algebra

$$(\{T'((s,u),p)\}_{s\in S},\{\theta_i\}_{i\in[|p|]})$$

is freely generated by $\{x_i^{p,u} \mid i\in[|u|]\}$, where, for all $i\in[|p|]$, if $p_i=(s,s_1s_2\ldots s_n)$ and $f_j\in T'((s_j,u),p)$ for every $j\in[n]$, then

$$\theta_i(f_1,f_2,\ldots,f_n)=f$$

if and only if

$$\sigma_i(f_i(\sigma_1^{(s_1,u)}),f_2(\sigma_1^{(s_2,u)}),\ldots,f_n(\sigma_1^{(s_n,u)}))=f(\sigma_1^{(s,u)}).$$

The following calculation (where $r=(s_1,u)(s_2,u)\ldots(s_n,u)p_i$):

$$\begin{aligned}
&(M^u_{p_i}\langle f_1,f_2,\ldots,f_n,y_i^p\rangle)(\sigma_1^{(s,u)})\\
&\quad=\langle f_1,f_2,\ldots,f_n,y_i^p\rangle^{\#}(M^u_{p_i}(\sigma_1^{(s,u)}))\\
&\quad=\langle f_1,f_2,\ldots,f_n,y_i^p\rangle^{\#}(J_{\Sigma_r}(\sigma^r_{n+1})\langle J_{\Sigma_r}(\sigma^r_1),\ldots,J_{\Sigma_r}(\sigma^r_n)\rangle)\\
&\quad=y_i^p(\sigma_1^{p_i})\langle f_1(\sigma_1^{(s_1,u)}),\ldots,f_n(\sigma_1^{(s_n,u)})\rangle\\
&\quad=I_{\Sigma_p}(\sigma_i^p)\langle f_1(\sigma_1^{(s_1,u)}),\ldots,f_n(\sigma_1^{(s_n,u)})\rangle\\
&\quad=\sigma_i(f_1(\sigma_1^{(s_1,u)}),f_2(\sigma_1^{(s_2,u)}),\ldots,f_n(\sigma_1^{(s_n,u)}))
\end{aligned}$$

shows that $\theta_i(f_1,f_2,\ldots,f_n)=M^u_{p_i}\langle f_1,f_2,\ldots,f_n,y_i^p\rangle$.

As the genus GB fulfills conditions (1) and (5), it follows that GB includes the genus of T'.

It follows from implications (a) and (e) that GB fulfills conditions (2) and (6), i.e., (S1) and (S3). We shall prove that GB fulfills condition (S2) as well.

We first prove by induction that GB fulfills conditions (S6), i.e., with $y=1_q+0_{p'}+1_r$,

$$h\in \mathrm{GB}(t',qr)\ \text{implies}\ hy\in \mathrm{GB}(t',qp'r)$$

for all $t'\in S\times S^*$ and $q,p',r\in(S\times S^*)^*$.

If $h=x_i^{qr,u}$, where $u\in S^*$, $i\in[|u|]$, and $t'=(u_i,u)$, then

$$hy=x_i^{qp'r,u}\in \mathrm{GB}(t',qp'r).$$

With the same notation as in (4), where $p=qr$ and $t'=(s,u)$, if

$$h=M^u_t\langle f_1,f_2,\ldots,f_n,f^\dagger\rangle,$$

then

$$\begin{aligned}
hy&=M^u_t\langle f_1y,f_2y,\ldots,f_ny,(f(1_t+y))^\dagger\rangle\\
&=M^u_t\langle f_1y,f_2y,\ldots,f_ny,(f(1_{tq}+0_{p'}+1_r))^\dagger\rangle.
\end{aligned}$$

Therefore, since by the inductive hypothesis,

$$f_i y \in \mathrm{GB}((s_i, u), qp'r) \quad \text{and} \quad f(1_{tq} + 0_{p'} + 1_r) \in \mathrm{GB}(t, tqp'r),$$

it follows that $hy \in \mathrm{GB}(t', qp'r)$.

With the same notation as in (5), where $t' = (s, u)$ and $p = qr$, if

$$h = M^u_{p_i}\langle f_1, f_2, \ldots, f_n, y^p_i\rangle,$$

then

$$hy = \begin{cases} M^u_{q_i}\langle f_1 y, f_2 y, \ldots, f_n y, y^{qp'r}_i\rangle & \text{if } i \in [|q|], \\ M^u_{r_{i-|q|}}\langle f_1 y, f_2 y, \ldots, f_n y, y^{qp'r}_{|p'|+i}\rangle & \text{if } |q| < i \leq |p|. \end{cases}$$

Therefore, it follows from the inductive hypothesis that

$$hy \in \mathrm{GB}(t', qp'r).$$

As (S6) implies (8), it follows from (f) that GB fulfills condition (3).

For technical reasons, we shall prove by induction on h that GB fulfills a stronger condition than (S2), i.e., for each $t' \in S \times S^*$ and each $q, p', r \in (S \times S^*)^*$, if

$$h \in \mathrm{GB}(t', qr) \quad \text{and} \quad h_i \in \mathrm{GB}(r_i, p') \text{ for } i \in [|r|],$$

then

$$h(1_q + \langle h_1, h_2, \ldots, h_{|r|}\rangle) \in \mathrm{GB}(t', qp').$$

Let $z = 1_q + \langle h_1, h_2, \ldots, h_{|r|}\rangle$. If $h = x^{qr,u}_i$, where $u \in S^*$, $i \in [|u|]$, and $t' = (u_i, u)$, then

$$hz = x^{qp',u}_i \in \mathrm{GB}(t', qp').$$

With the same notation as in (4), where $p = qr$ and $t' = (s, u)$, if

$$h = M^u_t\langle f_1, f_2, \ldots, f_n, f^\dagger\rangle$$

then

$$\begin{aligned} hz &= M^u_t\langle f_1 z, f_2 z, \ldots, f_n z, (f(1_t + z))^\dagger\rangle \\ &= M^u_t\langle f_1 z, f_2 z, \ldots, f_n z, (f(1_{tq} + \langle h_1, h_2, \ldots, h_{|r|}\rangle))^\dagger\rangle. \end{aligned}$$

Therefore, it follows from the inductive hypothesis that $hz \in \mathrm{GB}(t', qp')$.

With the same notation as in (5), where $p = qr$ and $t' = (s, u)$, if

$$h = M^u_{p_i}\langle f_1, f_2, \ldots, f_n, y^p_i\rangle$$

then

$$\begin{aligned} hz &= M^u_{p_i}\langle f_1 z, f_2 z, \ldots, f_n z, y^p_i z\rangle \\ &= \begin{cases} M^u_{q_i}\langle f_1 z, f_2 z, \ldots, f_n z, y^{qp'}_i\rangle & \text{if } i \in [|q|], \\ M^u_{r_{i-|q|}}\langle f_1 z, f_2 z, \ldots, f_n z, 0_q + h_{i-|q|}\rangle & \text{if } |q| < i \leq |p|. \end{cases} \end{aligned}$$

If $i \in [|q|]$, the inductive hypothesis and (5) imply that $hz \in \mathrm{GB}(t', qp')$. If $|q| < i \leq |p|$, then $hz \in \mathrm{GB}(t', qp')$ by the inductive hypothesis, and conditions (8) and (3).

Since GB fulfills conditions (S1), (S2), and (S3), it follows from Theorem 1.7 that GB includes the genus of CGT.

Part V. We still have to show that GC includes GA. We shall prove that GC fulfills condition (A).

It follows from (a) that GC fulfills (2).

We shall prove by induction that GC fulfills (S6). We shall use the same notation as in the similar proof for GB and omit the identical cases.

If $h = f^\dagger$, where $f: t' \to t'qr$, then

$$hy = (f(1_{t'} + y))^\dagger = (f(1_{t'q} + 0_{p'} + 1_r))^\dagger.$$

Therefore, it follows from the inductive hypothesis that $hy \in \mathrm{GC}(t', qp'r)$.

With the same notation as in (7), where $p = qr$, if $h = f\langle g, 1_p \rangle$, then

$$hy = f\langle gy, y \rangle = (f(1_{tq} + 0_{p'} + 1_r))\langle gy, 1_{qp'r} \rangle.$$

Therefore, it follows from the inductive hypothesis that $hy \in \mathrm{GC}(t', qp'r)$.

Since (S6) implies (8), it follows from (g) that GC fulfills (3). □

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