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Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings

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ABSTRACT

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : C \to H,\ i = 1, 2, \dots, N$, be a finite family of generalized asymptotically nonexpansive mappings. It is our purpose, in this paper to prove strong convergence of Mann's type method to a common fixed point of $\{T_i : i = 1, 2, \dots, N\}$ provided that the interior of common fixed points is nonempty. No compactness assumption is imposed either on T or on C. As a consequence, it is proved that Mann's method converges for a fixed point of nonexpansive mapping provided that interior of $F(T) \neq \emptyset$. The results obtained in this paper improve most of the results that have been proved for this class of nonlinear mappings.

1. Introduction and preliminaries

Let C be a nonempty subset of a real Hilbert space H; a mapping $T : C \to C$ is a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in C$ we have $\|Tx - Ty\| \leq k\|x - y\|$. It is said to be *nonexpansive* if for all $x, y \in C$ we have $\|Tx - Ty\| \leq \|x - y\|$. T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all integers $n \geq 1$ and all $x, y \in C$. Clearly, every contraction mapping is nonexpansive and every nonexpansive mapping is *asymptotically nonexpansive* with sequence $k_n = 1,\ \forall n \geq 1$. There are however, asymptotically nonexpansive mappings which are not nonexpansive (see e.g., [1]).

As a generalization of the class of nonexpansive mappings, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972 and has been studied by several authors (see e.g., [3–6]). Goebel and Kirk proved that if C is a nonempty closed convex and bounded subset of a uniformly convex Banach space (more general than a Hilbert space) then every asymptotically nonexpansive self-mapping of C has a fixed point.

The weak and strong convergence problems to a fixed points of nonexpansive and asymptotically nonexpansive mappings have been studied by many authors (for example, see [7,8,2,3,9–11] and the references therein).

Let C be a closed subset of a Hilbert space H and T be a self-mapping contraction, the classical *Picard iteration method*,

$$x_0 \in C, \qquad x_{n+1} = Tx_n, \quad n \geq 1 \tag{1.1}$$

converges to the unique fixed point of T. Unfortunately, the Picard iteration method does not always converge to a fixed point of nonexpansive mappings. It suffices to take, for example, T to be the anticlockwise rotation of the unit disk in $\mathbb{R}^2$ (with the usual Euclidean norm) about the origin of coordinate of an angle, say, θ.

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In 1953, Mann [12] introduced the iteration sequence $\{x_n\}_{n\in\mathbb{N}}$ which is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.2}$$

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in [0, 1]. Construction of fixed points of nonexpansive mappings via Mann's algorithm [12] has extensively been investigated recently in the literature (see, e.g., [13,14] and references therein). Related works can also be found in [15,16,14,17,18]. If T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.2) *converges weakly* to a fixed point of T (this is indeed true in a uniformly convex Banach space with a Frechét differentiable norm [14]). However, this convergence is in general not strong (see the counterexample in [19]; see also [20]). Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [22] (see also [21]) proposed the following modification of the Mann iteration method (1.2):

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \quad n \ge 0, \end{cases} \tag{1.3}$$

and proved the following.

Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \ne \emptyset$. Then $F(T)$ is closed and convex.

Theorem BC. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \ne \emptyset$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (1.3) converges strongly to $z = P_{F(T)}(x_0)$, where P_C is the metric projection mapping from a Hilbert space H onto a nonempty, closed and convex subset C of H.*

It is worth mentioning that Scheme (1.3) involves computation of intersection of closed convex subsets C_n and Q_n for each $n \ge 1$ and hence is not easy for computation.

In [23,24], Schu introduced a Mann type process to approximate fixed points of asymptotically nonexpansive mappings defined on nonempty closed convex and bounded subsets of a Hilbert space H. More precisely, he proved the following theorem.

Theorem JS1 (*[23, Theorem 1.5, p. 409]*). *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of H. Let $T : C \to C$ be completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1,\infty)$ for all $n \ge 1$, $\lim k_n = 1$, and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0, 1]$ satisfying the condition $\epsilon \le \alpha_n \le 1-\epsilon$ for all $n \ge 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \ge 1, \tag{1.4}$$

converges strongly to some fixed point of T.

In [25], Rhoades extended Theorem JS1 to uniformly convex Banach spaces using a modified Ishikawa iteration method given in [26]. In [10], Osilike and Aniagbosor proved that the theorems of Schu and Rhoades remain true without the boundedness condition imposed on C, provided that $F(T) = \{x \in C : Tx = x\} \ne \emptyset$.

Recently, Chang et al. [7] have proved *weak convergence* theorem for asymptotically nonexpansive mappings and nonexpansive mappings. In fact, he proved that, if T is an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to \infty$ with a fixed point in C and $\{\alpha_n\}$ is a sequence in [0; 1] satisfying the following conditions: (i) There exist positive a integer n_0 and $\epsilon > 0$ such that $0 < \epsilon \le \alpha_n \le 1-\epsilon$, $n \ge n_0$, (ii) $\sum_{n=0}^{\infty}(k_n - 1) < \infty$. Then the Mann type iterative sequence $\{x_n\}$ defined by (1.4) converges weakly to some fixed point x^* in C.

But it is worth mentioning that, in all the above results for asymptotically nonexpansive mappings, either *compactness* assumption is imposed on the map T or the convergence is *weak convergence*. Our concern now is the following:

Is it possible to obtain strong convergence of Mann's type scheme (1.4) to a fixed point of asymptotically nonexpansive mappings without any compactness assumption on T?

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $P : H \to C$ be the projection mapping of H onto C. A non-self-mapping $T : C \to H$ is called *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$||T(PT)^{n-1}(x) - T(PT)^{n-1}(y) \le (1+\mu_n)\|x - y\|, \quad \forall x, y \in C,\ n \ge 1.$$

The concept of non-self asymptotically nonexpansive mappings as an important generalization of asymptotically nonexpansive self-mappings was introduced by Chidume et al. [8] and studied by many other authors (see, e.g., [27,28]).

In [8], Chidume et al. proved that, if $T : C \to H$ is a *completely continuous* and asymptotically nonexpansive mapping with a sequence $\{\mu_n\} \subset [0,\infty)$ such that $\sum \mu_n < \infty$, and $F(T) \ne \emptyset$ and $\{\alpha_n\} \subset (0, 1)$ is a sequence such that $\epsilon \le 1-\alpha_n \le 1-\epsilon$ for all $n \ge 1$ and some $\epsilon > 0$. Then for an arbitrary point $x_1 \in C$, the sequence $\{x_n\}$ defined by

$$x_1 \in C, \qquad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad \forall n \ge 1$$

converges strongly to some fixed point of T.

Recently, Zhou et al. [29] have introduced the following: a mapping $T : C \to H$ is called *asymptotically nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$||(PT)^n(x) - (PT)^n(y) \leq (1 + \mu_n)\|x - y\|, \quad \forall x, y \in C, \ n \geq 1. \tag{1.5}$$

In [29], Zhou et al. proved that, if $T_1, T_2 : C \to H$ are two weakly inward and asymptotically nonexpansive mappings with sequences $\{\mu_n^{(1)}\}, \{\mu_n^{(2)}\} \subset [1, \infty)$, $\sum_{n=1}^{\infty} \mu_n^{(1)} < \infty$, $\sum_{n=1}^{\infty} \mu_n^{(2)} < \infty$, respectively and $\{x_n\} \subset C$ is a sequence defined by

$$x_1 \in C, \qquad x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad \forall n \geq 1$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[\epsilon, 1-\epsilon]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for some $\epsilon > 0$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2 provided that one of the T_1 or T_2 is *completely continuous* and $F(T_1) \cap F(T_2) \neq \emptyset$.

But it is worth mentioning that all convergence results for non-self asymptotically nonexpansive mappings and hence non-self nonexpansive mappings requires *compactness* type assumption, completely continuous, on T or on C. Our second concern is the following: *Is it possible to obtain convergence of Mann's iteration scheme to a fixed point of class of asymptotically nonexpansive and hence nonexpansive mappings without the requirement of compactness type assumption*?

Let C be a closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be *generalized asymptotically nonexpansive* if there exist $\{\mu_n\}, \{\nu_n\} \subset [0, \infty)$ as $n \to \infty$ such that $\mu_n, \nu_n \to 0$ satisfying the following inequality:

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \|x - y\| + \nu_n, \quad \forall x, y \in C, \tag{1.6}$$

and T is said to be *generalized asymptotically quasi-nonexpansive* if there exist $\{\mu_n\}, \{\nu_n\} \subset [0, \infty)$ such that $\mu_n, \nu_n \to 0$ as $n \to \infty$ satisfying the following inequality:

$$\|(PT)^n x - x^*\| \leq \|x - x^*\| + \mu_n \|x - x^*\| + \nu_n, \quad \forall x \in C, \ x^* \in F(T).$$

The class of generalized asymptotically nonexpansive mappings was introduced by Shahzad and Zegeye [11]. It is clear from the definition that a generalized asymptotically nonexpansive mappings include the class asymptotically nonexpansive mappings. In [11], Shahzad and Zegeye proved that if T_i for $i \in I = \{1, 2, \ldots, N\}$ are uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive self mappings of C with $F := \cap_{i=1}^{N} F(T_i) \neq \emptyset$, then implicit Mann's type scheme given by:

$$x_0 \in C, \qquad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad \forall n \geq 1,$$

where $n = (k-1)N + i$, $T_n = T_{n(modN)} = T_i$, $i \in I$ and $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta > 0$, converges strongly to a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, N\}$ provided that one of the mappings T_i is *semi-compact*.

It is our purpose, in this paper to prove strong convergence of Mann's type scheme to a common fixed point of finite family of generalized asymptotically nonexpansive mappings provided that the interior of common fixed points is nonempty. No compactness assumption is imposed either on at least one of the mappings or on C. Moreover, computation of closed and convex set C_n for each $n \geq 1$ is not required. Consequently, the above concerns are answered in the affirmative in Hilbert space setting. The results obtained in this paper improve the results of Theorem IS, Theorem MC and improve the results of Osilike and Aniagbosor [10], Chang et al. [7], Chidume et al. [8], Zhou et al. [29] in a Hilbert space settings.

In the sequel we shall need the following definition and lemmas.

Let H be a real Hilbert space. The function $\phi : H \times H \to \mathbb{R}$ defined by

$$\phi(x, y) := \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \quad \text{for } x, y \in E, \tag{1.7}$$

is studied by Alber [30], Kamimura and Takahashi [31] and Reich [32].

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E. \tag{1.8}$$

The function ϕ has also the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, x - z\rangle \quad \text{for all } x, y, z \in E. \tag{1.9}$$

Let K be a nonempty subset of a Hilbert space H. For $x \in K$, the *inward set* of x, $I_K(x)$, is defined by $I_K(x) := \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}$. A mapping $T : K \to H$ is called *weakly inward* if $Tx \in \text{cl}[I_K(x)]$ for all $x \in K$, where $\text{cl}[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

In what follows we shall make use of the following.

Lemma 1.1. *Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha_i, \in [0, 1]$ for $i = 1, 2, \ldots, n$ such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$ the following equality holds:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \leq i, j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Proof.

$$\begin{aligned}\|\alpha_0x_0+\cdots+\alpha_nx_n\|^2 &= \langle\alpha_0x_0+\cdots+\alpha_nx_n, \alpha_0x_0+\cdots+\alpha_nx_n\rangle\\ &= \alpha_0^2\|x_0\|^2+\cdots+\alpha_n^2\|x_n\|^2\\ &\quad+2\alpha_0\alpha_1\mathrm{Re}\langle x_0,x_1\rangle+2\alpha_0\alpha_2\mathrm{Re}\langle x_0,x_2\rangle+\cdots+2\alpha_0\alpha_n\mathrm{Re}\langle x_0,x_n\rangle\\ &\quad+2\alpha_1\alpha_2\mathrm{Re}\langle x_1,x_2\rangle+2\alpha_1\alpha_3\mathrm{Re}\langle x_1,x_3\rangle+\cdots+2\alpha_1\alpha_n\mathrm{Re}\langle x_1,x_n\rangle\\ &\quad\cdots\\ &\quad+2\alpha_{n-1}\alpha_n\mathrm{Re}\langle x_{n-1},x_n\rangle\\ &= \alpha_0^2\|x_0\|^2+\cdots+\alpha_n^2\|x_n\|^2\\ &\quad+\alpha_0\alpha_1\Big[\|x_0\|^2+\|x_1\|^2-\|x_0-x_1\|^2\Big]+\cdots+\alpha_0\alpha_n\Big[\|x_0\|^2+\|x_n\|^2-\|x_0-x_n\|^2\Big]\\ &\quad+\alpha_1\alpha_2\Big[\|x_1\|^2+\|x_2\|^2-\|x_1-x_2\|^2\Big]+\cdots+\alpha_1\alpha_n\Big[\|x_1\|^2+\|x_n\|^2-\|x_1-x_n\|^2\Big]\\ &\quad\cdots\\ &\quad+\alpha_{n-1}\alpha_n\Big[\|x_{n-1}\|^2+\|x_n\|^2-\|x_{n-1}-x_n\|^2\Big]\\ &= \alpha_0\|x_0\|^2+\cdots+\alpha_n\|x_n\|^2\\ &\quad-\alpha_0\alpha_1\|x_0-x_1\|^2-\cdots-\alpha_0\alpha_n\|x_0-x_n\|^2\\ &\quad-\alpha_1\alpha_2\|x_1-x_2\|^2-\cdots-\alpha_1\alpha_n\|x_1-x_n\|^2\\ &\quad\cdots\\ &\quad-\alpha_{n-1}\alpha_n\|x_{n-1}-x_n\|^2\\ &= \sum_{i=0}^{n}\alpha_i\|x_i\|^2-\sum_{0\le i,j\le n}\alpha_i\alpha_j\|x_i-x_j\|^2.\end{aligned}$$

Hence the conclusion holds. □

Lemma 1.2 ([33]). *Let $\{a_n\}$, $\{\beta_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real numbers satisfying the following relation:*

$$a_{n+1}\le(1+\beta_n)a_n+\sigma_n,\quad n\ge n_0,$$

where, n_0 is some nonnegative integer. If $\sum\beta_n<\infty$ and $\sum\sigma_n<\infty$. Then, $\lim_{n\to\infty}a_n$ exists.

Lemma 1.3 ([34]). *Let H be a real Hilbert space and C be a nonempty closed convex subset of H with P as a metric projection. Let $T : C\to E$ be a mapping satisfying weakly inward condition. Then $F(PT)=F(T)$.*

2. Main results

Theorem 2.1. *Let H be a real Hilbert space; C a closed convex nonempty subset of H. Let $T_i : C\to H$, for $i=1,2,\ldots,N$, be weakly inward continuous generalized asymptotically nonexpansive mappings with sequences $\{\mu_n^{(i)}\}$ and $\{\nu_n^{(i)}\}$ such that $\sum_{n\ge1}\mu_n^{(i)}<\infty$, $\sum_{n\ge1}\nu_n^{(i)}<\infty$ for $i=1,2,\ldots,N$ defined as in (1.6) and the interior of $F:=\bigcap_{i=1}^{N}F(T_i)\neq\emptyset$. Let $\{\alpha_i\}_{i=0}^{N}$ be such that $\epsilon\le\alpha_i\le1-\epsilon$ for some $\epsilon>0$ such that $\sum_{i=0}^{N}\alpha_i=1$. Starting from an arbitrary $x_0\in C$, define $\{x_n\}$ by*

$$x_{n+1}=\alpha_0x_n+\alpha_1(PT_1)^nx_n+\cdots+\alpha_N(PT_N)^nx_n,\quad\forall n\ge1. \tag{2.1}$$

Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1,T_2,\ldots,T_N\}$.

Proof. Let $x^*\in F$. Let $\mu_n:=\max\{\mu_n^{(i)}, i=1,2,\ldots,N\}$ and $\nu_n:=\min\{\nu_n^{(i)} : i=1,2,\ldots,N\}$. Then, from (2.1) and Lemma 1.1 we have that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_0x_n+\alpha_1(PT_1)^nx_n+\cdots+\alpha_N(PT_N)^nx_n-x^*\|^2\\ &= \|\alpha_0(x_n-x^*)+\alpha_1((PT_1)^nx_n-(PT_1)^nx^*)+\cdots+\alpha_N((PT_N)^nx_n-(PT_N)^nx^*)\|^2\\ &\le \alpha_0\|x_n-x^*\|^2+\alpha_1\|(PT_1)^nx_n-(PT_1)^nx^*\|^2+\cdots+\alpha_N\|(PT_N)^nx_n-(PT_N)^nx^*\|^2\\ &\quad-\alpha_0\alpha_1\|x_n-(PT_1)^nx_n\|^2-\alpha_0\alpha_2\|x_n-(PT_2)^nx_n\|^2-\cdots-\alpha_0\alpha_N\|x_n-(PT_N)^nx_n\|^2\\ &\le \alpha_0\|x_n-x^*\|^2+\alpha_1(1+\mu_n)\|x_n-x^*\|^2+\cdots+\alpha_N(1+\mu_n)\|x_n-x^*\|^2+N\nu_n\\ &\quad-\alpha_0\alpha_1\|x_n-(PT_1)^nx_n\|^2-\alpha_0\alpha_2\|x_n-(PT_2)^nx_n\|^2-\cdots-\alpha_0\alpha_N\|x_n-(PT_1)^nx_n\|^2\\ &\le (1+\mu_n)\|x_n-x^*\|+N\nu_n-\alpha_0\alpha_1\|x_n-(PT_1)^nx_n\|^2\\ &\quad-\alpha_0\alpha_2\|x_n-(PT_2)^nx_n\|^2-\cdots-\alpha_0\alpha_N\|x_n-(PT_1)^nx_n\|^2 &(2.2)\\ &\le (1+\mu_n)\|x_n-x^*\|+N\nu_n. &(2.3)\end{aligned}$$

So by Lemma 1.2 we conclude that $\lim_{n\to\infty}\|x_n - x^*\|$ exists and hence $\{x_n\}$, $\{(PT_1)^n x_n\}, \{(PT_2)^n\}, \ldots, \{(PT_N)^n x_n\}$ are bounded.

Furthermore, from (1.9) we also have that,

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle.$$

This implies that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}\Big(\phi(p, x_n) - \phi(p, x_{n+1})\Big). \tag{2.4}$$

Moreover, since the interior of F is nonempty, there exists $p^* \in F$ and $r > 0$ such that $(p^* + rh) \in F$, whenever $\|h\| \le 1$. Thus, from (2.3) and (2.4) we get that

$$0 \le \langle x_{n+1} - (p^* + rh), x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}(M\mu_n + N\nu_n), \tag{2.5}$$

for some $M > 0$. Then from (2.5) and (2.4) we obtain that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\le \langle x_{n+1} - p^*, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}(M\mu_n + N\nu_n) \\ &= \frac{1}{2}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2}(M\mu_n + N\nu_n), \end{aligned}$$

and hence

$$\langle h, x_n - x_{n+1}\rangle \le \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}(M\mu_n + N\nu_n).$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \le \frac{1}{2r}(\phi(p^*, x_n) - \phi(p^*, x_{n+1})) + \frac{1}{2r}(M\mu_n + N\nu_n).$$

So, if $n > m$, then we get that

$$\begin{aligned} \|x_m - x_n\| &= \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\| \\ &\le \sum_{i=m}^{n-1}\|x_i - x_{i+1}\| \\ &\le \frac{1}{2r}\sum_{i=m}^{n-1}(\phi(p^*, x_i) - \phi(p^*, x_{i+1})) + \frac{1}{2r}\sum_{i=m}^{n-1}(\mu_i + N\nu_i) \\ &= \frac{1}{2r}(\phi(p^*, x_m) - \phi(p^*, x_n)) + \frac{1}{2r}\sum_{i=m}^{n-1}(M\mu_i + N\nu_i). \end{aligned} \tag{2.6}$$

But we know that $\{\phi(p^*, x_n)\}$converges, $\sum\mu_n < \infty$ and $\sum\nu_n < \infty$. Therefore, we obtain from (2.6) that $\{x_n\}$ is a Cauchy sequence. Since H is complete there exists $x^* \in H$ such that

$$x_n \to x^* \in H. \tag{2.7}$$

Thus, since $\{x_n\}$ is subset of C, where C is closed and convex we have that $x^* \in C$.

Moreover, from (2.2) and the fact that $\epsilon \le \alpha_i$ for each $i \in \{= 1, 2, \ldots, N\}$, we get that

$$\epsilon^2\|x_n - (PT_1)^n x_n\| + \cdots + \epsilon^2\|x_n - (PT_N)^n x_n\| \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \mu_n M + N\nu_n$$

so that

$$\epsilon^2\sum_{n\ge1}\|x_n - (PT_1)^n x_n\| + \cdots + \epsilon^2\sum_{n\ge1}\|x_n - (PT_N)^n x_n\| \le \|x_1 - x^*\|^2 + \sum_{n\ge1}(M\mu_n + N\nu_n) < \infty. \tag{2.8}$$

Thus, we get that

$$\lim_{n\to\infty}\|x_n - (PT_i)^n x_n\| = 0 \quad \text{for each } i = 1, 2, \ldots, N. \tag{2.9}$$

Furthermore, we claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. In fact, from (2.1) we have that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_1((PT_1)^n x_n - x_n) + \cdots + \alpha_N((PT_N)^n x_n - x_n)\| \\ &\le \alpha_1\|(PT_1)^n x_n - x_n\| + \cdots + \alpha_N\|(PT_N)^n x_n - x_n\|. \end{aligned} \tag{2.10}$$

Hence, it follows from (2.9) that

$$\|x_{n+1}-x_n\| \to 0 \quad \text{as } n\to\infty. \tag{2.11}$$

Then from the fact that T_i is generalized asymptotically nonexpansive, for each $i \in \{1, 2, \ldots, N\}$, we get that

$$\begin{aligned}\|x_n-(PT_i)x_n\| &\le \|x_n-x_{n+1}\|+\|x_{n+1}-(PT_i)^{n+1}x_{n+1}\|+\|(PT_i)^{n+1}x_{n+1}-(PT_i)^{n+1}x_n\|+\|T_i^{n+1}x_n-(PT_i)x_n\|\\ &\le \|x_{n+1}-x_n\|+\|x_{n+1}-(PT_i)^{n+1}x_{n+1}\|\\ &\quad +(1+\mu_{n+1})\|x_{n+1}-x_n\|+\nu_{n+1}+\|(PT_i)^{n+1}x_n-(PT_i)x_n\|.\end{aligned} \tag{2.12}$$

It follows from (2.9), (2.11), (2.12) and continuity of T_i and hence PT_i that

$$\lim_{n\to\infty}\|x_n-(PT_i)x_n\|=0. \tag{2.13}$$

Thus, since $x_n \to x^*$ as $n \to \infty$ and PT_i is continuous we obtain that $x^* \in F(PT_i)$. Furthermore, by Lemma 1.3 we get that $x^* \in F(T_i)$ for each $i \in \{1, 2, \ldots, N\}$. Therefore, we get that $x^* \in \bigcap_{i=1}^N F(T_i)$. The proof is complete. □

We now give an example of generalized asymptotically nonexpansive mappings with interior of $F(T) \neq \emptyset$.

Example. Let $C := \left[\frac{-1}{\pi}, \frac{1}{\pi}\right]$ and define $T : C \to C$ by

$$T(x)=\begin{cases}\frac{x}{2}\left|\sin\left(\frac{1}{x}\right)\right|, & x\in\left(0,\frac{1}{\pi}\right]\\ x, & x\in\left[\frac{-1}{\pi},0\right].\end{cases}$$

Then clearly, T is continuous and $F(T) = \left[\frac{-1}{\pi}, 0\right]$. Moreover, following the method in [35] we obtain that $T^n x \to 0$, uniformly, for each $x \in \left(0, \frac{1}{\pi}\right]$. Furthermore, we observe that $T^n x = x$ for each $x \in \left[\frac{-1}{\pi}, 0\right]$. Now, for each fixed n, define $f_n(x, y) = |T^n x - T^n y| - |x - y|$, for $x, y \in C$. Set $\nu_n := \sup_{x,y\in C} f_n(x, y) \vee 0$. Then compactness of C gives that for each $n \in N$ there exists $x_n, y_n \in C$ such that $\nu_n = \sup_{x,y\in C} f_n(x, y) \vee 0 = f_n(x_n, y_n) \vee 0 = (|T^n x_n - T^n y_n| - |x_n - y_n|) \vee 0$. Then, since

$$\nu_n=\begin{cases}(T^nx_n-T^ny_n)-(x_n-y_n)=T^nx_n-x_n, & \text{if } x_n\in\left(0,\frac{1}{\pi}\right], y_n\in\left[\frac{-1}{\pi},0\right],\\ |T^nx_n-T^ny_n|-|x_n-y_n|\le|T^nx_n|+|T^ny_n|, & \text{if } x_n,y_n\in\left(0,\frac{1}{\pi}\right],\\ |x_n-y_n|-|x_n-y_n|, & \text{if } x_n,y_n\in\left[\frac{-1}{\pi},0\right],\end{cases}$$

we obtain that $\lim_{n\to\infty} \nu_n = 0$ and $|T^n x - T^n y| \le |x - y| + \nu_n$ which implies that $|T^n x - T^n y| \le |x - y| + \mu_n|x - y| + \nu_n$ for $\mu_n \to 0$ as $n \to \infty$. This shows that T is continuous generalized asymptotically nonexpansive mapping with *interior of* $F(T) = \left(\frac{-1}{\pi}, 0\right) \neq \emptyset$.

If in Theorem 2.1, we assume that $N = 1$ we obtain the following corollary.

Corollary 2.2. *Let H be a real Hilbert pace; C a closed convex nonempty subset of H. Let $T : C \to H$, be weakly inward continuous generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\}$ and $\{\nu_n\}$ such that $\sum_{n\ge1} \mu_n < \infty$, $\sum_{n\ge1} \nu_n < \infty$ and the interior of $F(T) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^2$ be such that $\epsilon \le \alpha_i \le 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^2 \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1}=\alpha_0x_n+\alpha_1(PT)^nx_n, \quad \forall n\ge1. \tag{2.14}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T_1.

Proof. Put $T := T_1 = T_2 = \cdots = T_N$. Then Eq. (2.1) reduces to Eq. (2.14) and hence the conclusion follows from Theorem 2.1. □

If in Theorem 2.1, we assume that T_i, for $i = 1, 2, \ldots, N$, are asymptotically nonexpansive we obtain the following corollary.

Corollary 2.3. *Let H be a real Hilbert pace; C a closed convex nonempty subset of H. Let $T_i : C \to H$, for $i = 1, 2, \ldots, N$, be weakly inward asymptotically nonexpansive mappings with sequences $\{\mu_n^{(i)}\}$ such that $\sum_{n\ge1} \mu_n^{(i)} < \infty$, for $i = 1, 2, \ldots, N$ and the interior of $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^N$ be such that $\epsilon \le \alpha_i \le 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^N \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1}=\alpha_0x_n+\alpha_1(PT_1)^nx_n+\alpha_2(PT_2)^nx_n+\cdots+\alpha_N(PT_N)^nx_n, \quad \forall n\ge1.$$

Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Since every asymptotically nonexpansive is continuous generalized asymptotically nonexpansive with $\nu_n := 0$ for all $n \geq 1$, the proof follows from Theorem 2.1. □

If in Theorem 2.1, we assume that T_i, for $i = 1, 2, \ldots, N$, are self mappings then we obtain the following corollary.

Corollary 2.4. *Let C a closed convex nonempty subset of a real Hilbert pace H. Let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be continuous generalized asymptotically nonexpansive mappings with sequences $\{\mu_n^{(i)}\}$ and $\{\nu_n^{(i)}\}$ such that $\sum_{n\geq 1} \mu_n^{(i)} < \infty$, $\sum_{n\geq 1} \nu_n^{(i)} < \infty$ for $i = 1, 2, \ldots, N$ and the interior of $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^{N}$ be such that $\epsilon \leq \alpha_i \leq 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^{N} \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T_1^n x_n + \alpha_2 T_2^n x_n + \cdots + \alpha_N T_N^n x_n, \quad \forall n \geq 1.$$

Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Theorem 2.5. *Let H be a real Hilbert pace; C a closed convex nonempty subset of H. Let $T_i : C \to H$, for $i = 1, 2, \ldots, N$, be weakly inward nonexpansive mappings with the interior of $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^{N}$ be such that $\epsilon \leq \alpha_i \leq 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^{N} \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n + \alpha_1 (PT_1) x_n + \alpha_2 (PT_2) x_n + \cdots + \alpha_N (PT_N) x_n, \quad \forall n \geq 1. \tag{2.15}$$

Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

Proof. Following the method of proof of Theorem 2.1 we obtain the required assertion. □

If in Theorem 2.5, we assume that T_i, for $i = 1, 2, \ldots, N$, are self mappings then we obtain the following corollary.

Corollary 2.6. *Let H be a real Hilbert pace; C a closed convex nonempty subset of H. Let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be nonexpansive mappings with the interior of $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^{N}$ be such that $\epsilon \leq \alpha_i \leq 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^{N} \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T_1 x_n + \alpha_2 T_2 x_n + \cdots + \alpha_N T_N x_n, \quad \forall n \geq 1. \tag{2.16}$$

Then, $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_N\}$.

If in Theorem 2.5, we assume that $N = 1$ we obtain the following corollary.

Corollary 2.7. *Let H be a real Hilbert pace; C a closed convex nonempty subset of H. Let $T : C \to H$, be weakly inward nonexpansive mappings with the interior of $F(T) \neq \emptyset$. Let $\{\alpha_i\}_{i=0}^{2}$ be such that $\epsilon \leq \alpha_i \leq 1 - \epsilon$ for some $\epsilon > 0$ such that $\sum_{i=0}^{2} \alpha_i = 1$. Starting from an arbitrary $x_0 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_0 x_n + \alpha_1 (PT_1) x_n, \quad \forall n \geq 1. \tag{2.17}$$

Then, $\{x_n\}$ converges strongly to a fixed point of T.

Remark 2.8. Our results extend and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 2.1 extends Theorem JS1 of Schu [23], Theorem MC of Nakajo and Takahashi [22]in the sense that our convergence does not require either T be completely continuous or computation of C_n for each $n \geq 1$. The compactness assumption imposed either on T or on C, to get strong convergence, in the results of Osilike and Aniagbosor [10], Chang et al. [7], Chidume et al. [8], Zhou et al. [29] are not required. Our results provide affirmative answer to the concerns raised above.

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