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Generating strings for bipartite Steinhaus graphs

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Abstract

Let $b(n)$ be the number of bipartite Steinhaus graphs with n vertices. We show that $b(n)$ satisfies the recurrence, $b(2)=2$, $b(3)=4$, and for $k \geqslant 2$, $b(2k+1)=2b(k+1)+1$, $b(2k)=b(k)+b(k+1)$. Thus $b(n) \leqslant \frac{3}{2}n - \frac{7}{2}$ with equality when n is one more than a power of two. To prove this recurrence, we describe the possible generating strings for these bipartite graphs.

1. Introduction

Let $T = a_{11}a_{12}\ldots a_{1n}$ be an n-long string of zeroes and ones. The *Steinhaus graph* generated by T has as its adjacency matrix the *Steinhaus matrix* $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 0 & \text{if } 1 \leqslant i = j \leqslant n; \\ (a_{i-1,j-1} + a_{i-1,j}) (\operatorname{mod} 2) & \text{if } 1 < i < j \leqslant n; \\ a_{ji} & \text{if } 1 \leqslant j < i \leqslant n. \end{cases}$$

The vertices of a Steinhaus graph are usually labelled by their row number. In Fig. 1, the graph generated by 011000 is pictured. A *Steinhaus triangle* is the upper-triangular part of a Steinhaus matrix (excluding the diagonal) and hence is generated by a string of length $n-1$.

Steinhaus in [11] asked if there were Steinhaus triangles containing the same number of zeroes and ones and Harborth [8] answered this in the affirmative by showing that for each n, $n \equiv 0, 1 \pmod 4$, there are at least four strings of length $n-1$ that generate such triangles. Wang [13] named these triangles after Steinhaus and Chang [4] investigated the possible number of ones in these triangles. Molluzzo [9] recognized that graphs could easily be made from Steinhaus triangles and proved several results on the complements of Steinhaus graphs. The complements of

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$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 \\ 6 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

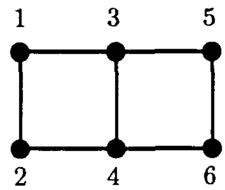

Fig. 1.

Steinhaus graphs were further studied in [5] and conditions and a conjecture on the existence of regular Steinhaus graphs were given in [1]. Brigham and Dutton [3] conjectured that almost all Steinhaus graphs have diameter two and this was proved in [2]. In [6] it was shown that a Steinhaus graph is bipartite if and only if the graph contains no triangles and that a bipartite Steinhaus graph has a perfect matching if and only if the sizes of the two sets in the partition are equal. In this paper, we characterize the binary strings that generate bipartite Steinhaus graphs and give a recurrence for the number of such strings.

It is not difficult to see that deleting the first row and column or the last row and column of a Steinhaus matrix results in another Steinhaus matrix and that the only disconnected Steinhaus graphs are those generated by the sequences that are all zeroes. Recall that any connected bipartite graph can be 2-colored in essentially only one way and that any subgraph of a bipartite graph is bipartite. We color our graphs with colors α and β and vertex 1 is always colored α. In this paper, $\mathbb{Z}$ is the set of integers and if $\mathbb{A} \subseteq \mathbb{Z}$, then $\mathbb{A}_i = \{x \in \mathbb{A}\colon\ x \geqslant i\}$. For example, $\mathbb{Z}_0$ is the set of non-negative integers, $\mathbb{N} = \mathbb{Z}_1$ is the set of positive integers, and if $\mathbb{O}$ is the set of odd positive integers, then $\mathbb{O}_3$ is the set of odd positive integers larger than 1. As is usual, $\lfloor x \rfloor$ is the floor of x and $\lceil x \rceil$ is the ceiling of x. We denote $\log_2(x)$ by $\lg(x)$ and if T is a string of zeroes and ones, then T^k is the string T concatenated with itself $k-1$ times. (For example, if $T = 01$, then $T^3 = 010101$.)

We now present some facts concerning Pascal's rectangle modulo two (see Fig. 2) that will be needed in Section 2. The rows of the rectangle are labelled $R_1^*, R_2^*, \ldots$ and so the kth element of R_n^* is 0 if $k > n$ and is $\binom{n-1}{k-1} (\bmod 2)$ if $1 \leqslant k \leqslant n$. We denote by $R_{n,k}$ the string formed by the first k elements of R_n^* and we set $R_n = R_{n,n}$. We start with Lucas's theorem.

Theorem (Lucas). *Let p be prime and $n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k$ and $m = m_0 + m_1 p + m_2 p^2 + \cdots + m_k p^k$ with $0 \leqslant n_i < p$ and $0 \leqslant m_i < p$ for $0 \leqslant i \leqslant k$. Then*

$$\binom{n}{m} \equiv \binom{n_0}{m_0}\binom{n_1}{m_1}\binom{n_2}{m_2}\cdots\binom{n_k}{m_k} (\bmod p),$$

where $\binom{a}{b} = 0$ if $b > a$.

$$\begin{array}{ll}
R_1^* \to & 1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad \ldots \\
R_2^* \to & 1\quad 1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad \ldots \\
R_3^* \to & 1\quad 0\quad 1\quad 0\quad 0\quad 0\quad 0\quad 0\quad \ldots \\
R_4 \to & \boxed{1\quad 1\quad 1\quad 1}\quad 0\quad 0\quad 0\quad 0\quad \ldots \\
R_{5,3} \to & \boxed{1\quad 0\quad 0}\quad 0\quad 1\quad 0\quad 0\quad 0\quad \ldots \\
 & 1\quad 1\quad 0\quad 0\quad 1\quad 1\quad 0\quad 0\quad \ldots \\
 & \vdots
\end{array}$$

Fig. 2. Pascal's rectangle modulo 2.

A short proof of this theorem can be found in [7] and for a visual version see [12]. Many references to this and related results can be found in [10] but for our purposes, we need only consider the case $p = 2$. In this case, for $r \geqslant 0$ and $0 \leqslant m \leqslant 2^r - 1$, $\binom{2^r-1}{m}$ is odd and for $0 < m < 2^r$, $\binom{2^r}{m}$ is even. This proves the following result.

Fact 1.1. *For $m \in \mathbb{Z}_0$ and $k \leqslant 2^m$, $R_{2^m,k} = 1^k$ and also, $R_{2^m+1} = 1(0^{2^m-1})1$.*

Facts 1.2 and 1.3 follow easily from the latter part of Fact 1.1.

Fact 1.2. *If $2^{m-1} < k \leqslant 2^m$, then $R_{2^m+j,k} = R_{j,k}$ and so, for k fixed, $R_{j,k}$ is periodic of period 2^m.*

Fact 1.3. *For $k < 2^m$, $R_{k+2^m} = R_k(0^{2^m-k})R_k$ and so $R_{k+2^m,2^{m+1}} = (R_k0^{2^m-k})^2$.*

Fact 1.4. *For $j, k \in \mathbb{N}$, $R_{2^k(2^j-1)+1,2^{k+j}} = (R_{1,2^k})^{2^j}$.*

Proof. We use induction on j. By Fact 1.1, $R_{2^k+1} = 10^{2^k-1}1$ and so $R_{2^k+1,2^{k+1}} = 10^{2^k-1}10^{2^k-1} = (R_{1,2^k})^2$. But this is just Fact 1.4 when $j = 1$.

Now assume that $j > 1$. Hence $R_{2^k(2^{j-1}-1)+1,2^{k+j-1}} = (R_{1,2^k})^{2^{j-1}}$ and therefore $R_{2^k(2^{j-1}-1)+1} = (R_{1,2^k})^{2^{j-1}-1}1$. Using this and Fact 1.3 gives

$$\begin{aligned}
R_{2^k(2^j-1)+1,2^{k+j}} &= R_{2^{k+j-1}+(2^{k+j-1}-2^k+1),2^{k+j}} \\
&= (R_{2^{k+j-1}-2^k+1}0^{2^k-1})^2 \\
&= ((R_{1,2^k})^{2^{j-1}-1}10^{2^k-1})^2 \\
&= ((R_{1,2^k})^{2^{j-1}-1}R_{1,2^k})^2 \\
&= ((R_{1,2^k})^{2^{j-1}})^2 \\
&= (R_{1,2^k})^{2^j}. \quad \square
\end{aligned}$$

2. Characterization of bipartite strings

A string that generates a bipartite Steinhaus graph is called a *bipartite string* and a bipartite string T is *maximal* if for both $\gamma = 0$ and $\gamma = 1$ the string T_γ obtained by concatenating γ to the right-hand side of T is not a bipartite string. We show in this section that all bipartite strings are prefixes of those of the form $0^k T^j 0^m$, where T is the first $2^{\lceil \lg(k) \rceil}$-elements of row $2^{\lceil \lg(k) \rceil} - k + 1$ of Pascal's rectangle. Note that this implies that a bipartite string with one leading zero is of the form $01^j 0^m$. It is also the case that m depends on j for if j is not a power of two, then m is at most the largest power of two dividing j and if j is a power of two, then m is arbitrary. We start with the following lemma.

Lemma 2.1. *For* k, $m \in \mathbb{Z}_0$ *and* $j \in \mathbb{O}_3$, $0^{2^k}(R_{1,2^k})^{j \cdot 2^m} 0^{2^{k+m}}$ *is a maximal bipartite string.*

Proof. In Fig. 3, excluding the last column, such a string is illustrated for $k = 2, m = 1$, and $j = 3$. We show that such a string is bipartite by showing that the vertices can be colored in the following way:

$$\text{color } \alpha\text{: } \{1, \ldots, 2^k, 2^k + j \cdot 2^{k+m} + 1, \ldots, 2^k + j \cdot 2^{k+m} + 2^{k+m}\},$$

$$\text{color } \beta\text{: } \{2^k + 1, \ldots, 2^k + j \cdot 2^{k+m}\}.$$

It is cear that the vertices $\{1, \ldots, 2^k\}$ are adjacent to only those vertices colored β. By Fact 1.1, row 2^k of the matrix consists of 1's in columns $2^k + 1$ to $2^k + j \cdot 2^{k+m}$. Hence none of the vertices colored β are adjacent to each other. In the matrix, columns $2^k + j \cdot 2^{k+m} + 1$ to $2^k + j \cdot 2^{k+m} + 2^{k+m}$ and rows $2^k + 1$ to $2^k + j \cdot 2^{k+m}$ are simply the first 2^{k+m} columns and $j \cdot 2^{k+m}$ rows of Pascal's rectangle. By Fact 1.2, row $2^k + j \cdot 2^{k+m} + 1$ is zero to the right of the main diagonal and therefore, none of the vertices $\{2^k + j \cdot 2^{k+m} + 1, \ldots, 2^k + j \cdot 2^{k+m} + 2^{k+m}\}$ are adjacent to each other.

If a 1 were added to such a string, then the resulting vertex, say v, is one vertex on the triangle $(2^k, 2^k + 1, v)$ and so the string is not bipartite. If a 0 were added to the string, then the resulting vertex is adjacent to exactly the vertices $\{2^k + (2r - 1) \cdot 2^{k+m} + i\text{: } 1 \leqslant r \leqslant (j + 1)/2,\ 1 \leqslant i \leqslant 2^{k+m}\}$. Since the vertices $\{2^k + 1, \ldots, 2^k + j \cdot 2^{k+m}\}$ are all adjacent to vertex $(2^k + j \cdot 2^{k+m} + 1)$, we have the triangle $(v, 2^k + j \cdot 2^{k+m} + 1, 2^k + 2^{k+m} + 1)$. Therefore, the original string is maximal. $\square$

Corollary 2.2. *For* $k \in \mathbb{N}$, $m \in \mathbb{Z}_0$, $j \in \mathbb{O}_3$, *and* $0 \leqslant r < 2^k$, *the string* $0^{2^k - r}(R_{r+1,2^k})^{j \cdot 2^m} 0^{2^{k+m}}$ *is a maximal bipartite string.*

Proof. Delete the first r rows and columns from the matrix generated by the string in Lemma 2.1. Subtracting r from each vertex in the proof of Lemma 2.1 will give the corresponding proof. $\square$

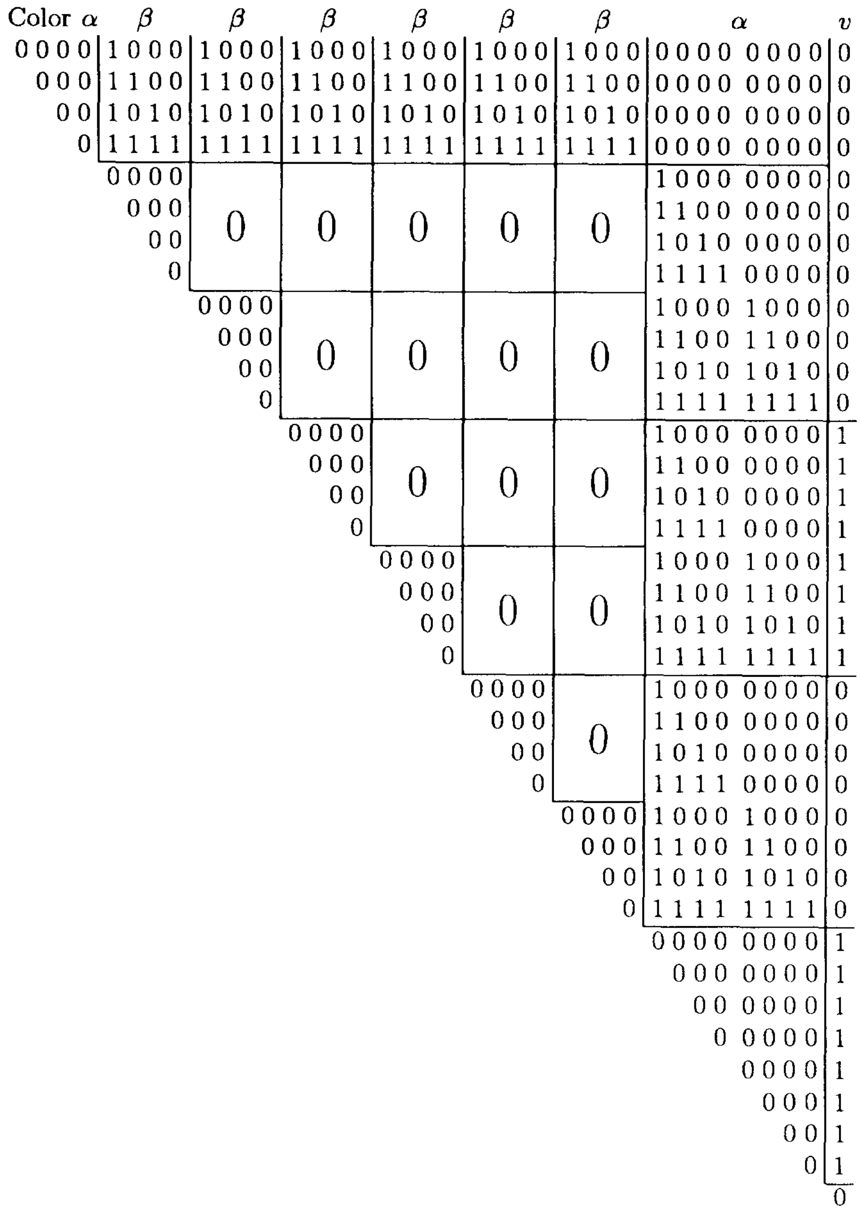

Fig. 3.

Lemma 2.3. *For $k, m \in \mathbb{Z}_0$, the string $0^{2^k}(R_{1,2^k})0^m$ is a bipartite string.*

Proof. Fig. 4(a) illustrates the matrix produced by such a string with $k = 2$ and $m = 5$. The outlined blocks are the $(2^k \times 2^k)$-upper-left square of Pascal's rectangle. It is clear that coloring the blocks of vertices with alternate colors gives a 2-coloring of the graph. $\square$

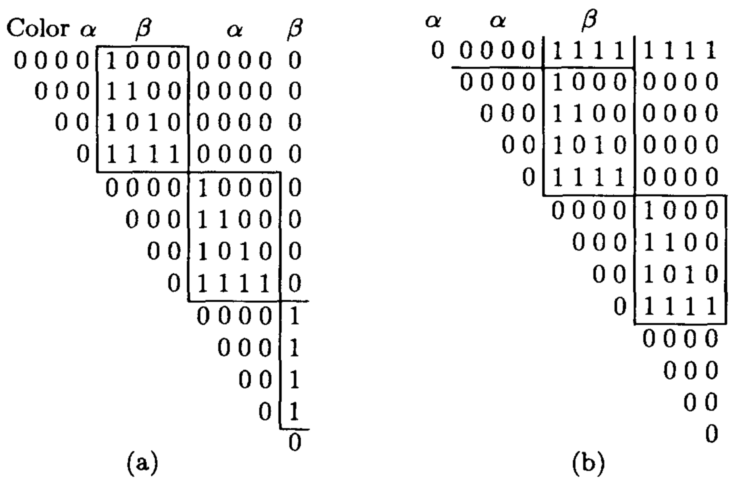

Fig. 4.

Corollary 2.4. *For $k \in \mathbb{N}$, $m \in \mathbb{Z}_0$, and for $0 \leqslant r < 2^k$, $0^{2^k - r}(R_{r+1,\,2^k})0^m$ is a bipartite string.*

Proof. The proof is the same as that of the Corollary 2.2. □

We are now ready to state the two lemmas that characterize most bipartite Steinhaus graphs. For Lemmas 2.5 and 2.6, if $k \in \mathbb{N}$, then let $K = 2^{\lceil \lg(k) \rceil}$ and $T = R_{K-k+1,\,K}$.

Lemma 2.5. *A string of the form $0^k T^{j \cdot 2^m} 0^{K \cdot 2^m}$ is a maximal bipartite stirng for $j \in \mathbb{O}_3$ and $m \in \mathbb{Z}_0$.*

Proof. This is just Corollary 2.2. □

Lemma 2.6. *Any string of the form $0^k T^{2^j} 0^m$ is a bipartite string for any j, $m \in \mathbb{Z}_0$.*

Proof. By Fact 4, $0^{2^k}(R_{1,\,2^k})^{2^j}0^m$ is the same as $0^{2^k}(R_{2^{k+j}-2^k+1,\,2^{k+j}})0^m$. If $\lambda = k + j$ and $r = 2^{k+j} - 2^k$, then this string is $0^{2^\lambda - r}(R_{r+1,\,2^\lambda})0^m$ and hence by Corollary 2.4, the string is bipartite. Now the string $0^{2^k - r}(R_{r+1,\,2^k})^{2^j}0^m$ is the result of deleting r rows from the matrix generated by the string $0^{2^k}(R_{1,\,2^k})^{2^j}0^m$ and hence is bipartite. This concludes the proof since all strings of the form $0^k T^{2^j} 0^m$ can be written as $0^{2^\lambda - r}(R_{r+1,\,2^\lambda})^{2^j}0^m$. □

We next show that the strings described in Lemmas 2.5 and 2.6 are essentially the only bipartite strings.

Lemma 2.7. *If T is a bipartite string with one leading zero, then T is a prefix of a string described in the statements of Lemmas 2.5 and 2.6.*

Proof. We use induction on the number of vertices, n. Suppose that T is an n-long bipartite string with one leading zero. Then the string T', resulting from deleting the right-most element of T, must also be a bipartite string with one leading zero and hence must be a prefix of a string described in Lemmas 2.5 and 2.6. From both lemmas, it is easy to see that a bipartite string with one leading zero is of the form

$$\underbrace{01\cdots1}_{j\cdot 2^m}\underbrace{0\cdots0}_{M},$$

where $j \in \mathbb{O}$, $m \in \mathbb{Z}_0$, $M \geqslant 0$, and if $j > 1$, then $M \leqslant 2^m$. Thus if $M = 0$, then T generates either $K_{1,n-1}$ (if the right-most element of T is an 1) or $K_{2,n-2}$ (if the right-most element of T is a 0). If $M > 0$, then the right-most element of T must be a 0, for otherwise T generates a graph with the triangle $(1, 2, n)$. By Lemma 2.5, if $j > 1$ and $M = 2^m$, then T' is a maximal bipartite string which contradicts T being a bipartite string. So if $j > 1$, then $M < 2^m$ and so $M + 1 \leqslant 2^m$. □

Theorem 2.8. *A Steinhaus graph is bipartite if and only if it is generated by a prefix of a string described in Lemmas 2.5 and 2.6.*

Proof. We proceed by induction on the number of leading zeroes, k, in the bipartite strings. By inspection this is true for graphs with less than six vertices. The case $k = 1$ is Lemma 2.7. Assume now that T is a bipartite string with $k > 1$ leading zeroes. Let G be the graph generated by T, let G' be the graph G with vertex 1 deleted, and let T' be the generating string of G'. There are two cases depending on whether or not $k - 1$ is a power of two.

Case 1: $k - 1$ is not a power of two. Since T has k leading zeroes, T' has $\lambda = k - 1$ leading zeroes. If $K = 2^{\lceil \lg(\lambda) \rceil}$, then by the inductive hypothesis, T' is a prefix of a sequence of the form $0^\lambda (R_{K-\lambda+1,K})^{j\cdot 2^m} 0^M$ where $j \in \mathbb{O}$, $m \in \mathbb{Z}_0$, and M is arbitrary if $j = 1$ and $M \leqslant K \cdot 2^m$ otherwise. Now let $R' = R_{K-\lambda+1,K}$ and so the row preceding R' in Pascal's Rectangle is $R = R_{K-k+1,K}$. Hence the first two rows of the matrix of G must be prefixes of

$$\begin{array}{l} 00\cdots0R\cdots R0\cdots0 \\ \ \,\underbrace{0\cdots0}_{k-1}\underbrace{R'\cdots R'}_{j\cdot 2^m}\underbrace{0\cdots0}_{M.} \end{array}$$

Hence $T = 0^k (R_{K-k+1,K})^{j\cdot 2^m} 0^M$ as desired.

Case 2: $k-1$ is a power of two. Let $2^\lambda = k-1$ and so G' must be generated by a prefix of a sequence of the form

$$\overbrace{0\cdots0}^{2^\lambda}\ \overbrace{\overbrace{10\cdots0}^{2^\lambda}\cdots\overbrace{10\cdots0}^{2^\lambda}}^{j\cdot 2^m}\ \underbrace{0\cdots0}_{M},$$

where $j \in \mathbb{O}$, $m \in \mathbb{Z}_0$, and $0 \leqslant M \leqslant 2^{\lambda+m}$ if $j > 1$ and M is arbitrary otherwise. There are two subcases. The first subcase is if $m = 0$ and the second subcase is if $m > 0$ or if T' is a prefix of the sequence with $M = 0$. The first subcase is illustrated in Fig. 4(b) with $k = 5$ and $j = 1$. In this case, the graph is not bipartite since the last 2^λ vertices must be be colored with both colors, α and β. In the second subcase, G must be generated by a prefix of $0^k\,(R_{2^\lambda,2^{\lambda+1}})^{j\cdot 2^{m-1}}0^M$ which is $0^k\,(R_{K-k+1,K})^{j\cdot 2^{m-1}}0^M$ where $K = 2^{\lambda+1}$. Now if $j > 1$, then $0 \leqslant M \leqslant 2^{\lambda+m} = K\cdot 2^{m-1}$. Hence the string is as described in either Lemma 2.5 or Lemma 2.6.

3. The number of bipartite strings

Let $b(n,\lambda)$ be the number of bipartite strings of length n with exactly λ leading zeroes and let

$$b(n) = \sum_{\lambda=1}^{n} b(n,\lambda).$$

In this section we give a tight upper bound and a recurrence for $b(n)$. First, the recurrence.

Theorem 3.1. *If $b(n)$ is the number of bipartite strings of length n, then $b(2) = 2$, $b(3) = 4$, and for $k \geqslant 2$,*

(a) $b(2k+1) = 2b(k+1) + 1$,

(b) $b(2k) = b(k) + b(k+1)$.

The proof of this theorem divides naturaly into several lemmas, each of which further divides into three cases. For the string $0^\lambda T^j 0^m$, the cases are:

Case 1. $m > 0$ and j is a power of two;

Case 2. $m > 0$ and j is not a power of two;

Case 3. $m = 0$.

In Case 3, the string T will appear $(j-1)$-times but it is not necessary for the entire string to appear the jth time. Note too that the string may end in zeroes because T may end in zeroes.

For $\gamma \in \{1,2,3\}$, let $b_\gamma(n,\lambda)$ be the number of bipartite strings of length n with λ leading zeroes of the form given in Case γ. By Theorem 2.8, for $1 \leqslant \lambda \leqslant n$,

$$b_3(n,\lambda) = 1. \tag{1}$$

Throughout the proof of Theorem 3.1 we need the following result.

Fact 3.2. *For $k \in \mathbb{Z}_2$,*

$$\lceil \lg(2k-1) \rceil = \lceil \lg(2k) \rceil,$$

$$\lceil \lg(2k+1) \rceil = \lceil \lg(2k) \rceil.$$

Lemma 3.3. *For $k \in \mathbb{Z}_2$, $b(2k+1,1) = b(k+1,1) + 2$.*

Proof. The strings counted by $b_2(n,1)$ are of the form $01^{j\cdot 2^m}0^s$ where $m \in \mathbb{Z}_0, j \in \mathbb{O}_3$, and $s \leqslant 2^m$. So the string $01^{j\cdot 2^m}0^s$ is counted by $b_2(k+1,1)$ if and only if $j\cdot 2^m + s + 1 = k+1$, or $j \cdot 2^{m+1} + 2s + 1 = 2k+1$. The latter occurs if and only if the string $01^{j\cdot 2^{m+1}}0^{2s}$ is counted by $b_2(2k+1,1)$. Since $01^{2k-1}0$ is also counted by $b_2(2k+1,1)$, we have

$$b_2(2k+1,1) = b_2(k+1,1) + 1. \tag{2}$$

By Theorem 2.8, all bipartite strings with 1 leading zero are of the form 01^j0^m. Hence, $b_1(n,1)$ is the number of powers of two not exceeding $n-2$, namely

$$b_1(n,1) = 1 + \lfloor \lg(n-2) \rfloor. \tag{3}$$

Using (3) and Fact 3.2 gives

$$\begin{aligned} b_1(k+1,1) &= 1 + \lfloor \lg(k-1) \rfloor \\ &= \lfloor \lg(2k-2) \rfloor \\ &= \lfloor \lg(2k-1) \rfloor \\ &= b_1(2k+1,1) - 1. \end{aligned} \tag{4}$$

To finish the proof, just add the appropriate form of (1) to the sum of (2) and (4). □

Lemma 3.4. *For $k \in \mathbb{Z}_2$, $b(2k,1) = b(k+1,1) + 1$.*

Proof. Using (3) and Fact 3.2, we have

$$\begin{aligned} b_1(2k,1) &= 1 + \lfloor \lg(2k-2) \rfloor \\ &= 1 + 1 + \lfloor \lg(k-1) \rfloor \\ &= 1 + b_1(k+1,1). \end{aligned} \tag{5}$$

Now suppose that $01^{j \cdot 2^m}0^s$ ($j \in \mathbb{O}_3$, $s \leqslant 2^m$) is counted by $b_2(2k, 1)$. Note that $m \neq 0$ for if $m = 0$, $s = 1$ and $1 + j + s = 2k$, a contradiction. To show that $01^{j \cdot 2^m}0^s$ is counted by $b_2(2k, 1)$ if and only if $01^{j \cdot 2^{m-1}}0^t$ ($t = (s + 1)/2$) is counted by $b_2(k + 1, 1)$, we must check to see that $1 + j \cdot 2^{m-1} + t = k + 1$ and that $t \leqslant 2^{m-1}$. Since $1 + j \cdot 2^m + s = 2k$, $s + 1 = 2(k - j \cdot 2^{m-1})$, and so $1 + j \cdot 2^{m-1} + t = k + 1$. Now $t = (s + 1)/2 \leqslant (2^m + 1)/2 = 2^{m-1} + \frac{1}{2}$ and hence $t \leqslant 2^{m-1}$. Thus

$$b_2(2k, 1) = b_2(k + 1, 1). \tag{6}$$

To finish the proof, just add the appropriate form of (1) to the sum of (5) and (6). □

Lemma 3.5. *For* $k, \lambda \in \mathbb{Z}_2$, $b(2k + 1, \lambda) = b(k + 1, \lceil \lambda/2 \rceil)$.

Proof. By Theorem 2.8, a string is counted by $b_1(n, \lambda)$ if and only if it has the form $0^\lambda T^{2^j} 0^m$ with the length of T being $K = 2^{\lceil \lg(\lambda) \rceil}$. Thus $n = \lambda + K \cdot 2^j + m$ with $m \geqslant 1$ and $j \geqslant 0$. Any j betwen 0 and $\lg((n - \lambda - 1)/K)$ determines exactly one string Ncounted by $b_1(n, \lambda)$ and all strings counted by $b_1(n, \lambda)$ have j in this range. Hence

$$\begin{aligned} b_1(n, \lambda) &= 1 + \left\lfloor \lg\left(\frac{n - \lambda - 1}{K}\right) \right\rfloor \\ &= 1 + \lfloor \lg(n - \lambda - 1) - \lceil \lg(\lambda) \rceil \rfloor \\ &= \lfloor \lg(n - \lambda - 1) \rfloor - \lceil \lg(\lambda) \rceil + 1. \end{aligned} \tag{7}$$

For $n = 2k + 1$ and $\lambda = 2r$, $K/2 = 2^{\lceil \lg(r) \rceil}$ and so by (7),

$$\begin{aligned} b_1(2k + 1, 2r) &= \lfloor \lg((2k + 1) - 2r - 1) \rfloor - \lceil \lg(2r) \rceil + 1 \\ &= \lfloor \lg(k - r) \rfloor + 1 - \lceil \lg(r) \rceil - 1 + 1 \\ &= \lfloor \lg((k + 1) - r - 1) \rfloor - \lceil \lg(r) \rceil + 1 \\ &= b_1(k + 1, r). \end{aligned}$$

Similarly, $b_1(2k + 1, 2r + 1) = b_1(k + 1, r + 1)$ and so

$$b_1(2k + 1, \lambda) = b_1(k + 1, \lceil \lambda/2 \rceil). \tag{8}$$

Now suppose that $0^\lambda T^{j \cdot 2^m} 0^s$ is counted by $b_2(n, \lambda)$. Thus $j \in \mathbb{O}_3$, $m \in \mathbb{Z}_0$, the length of T is $K = 2^{\lceil \lg(\lambda) \rceil}$, $s \leqslant K \cdot 2^m$, and hence

$$n - \lambda - K \cdot 2^m \leqslant K \cdot j \cdot 2^m \leqslant n - \lambda - 1. \tag{9}$$

Fixing n and λ (and thus fixing K), $b_2(n, \lambda)$ is just the number of pairs (j, m) that satisfy the inequalities in (9). To show that $b_2(k + 1, \lambda) = b_2(2k + 1, 2\lambda)$, we need to show that the pair (j, m) is a solution to

$$(k + 1) - \lambda - K \cdot 2^m \leqslant K \cdot j \cdot 2^m \leqslant (k + 1) - \lambda - 1, \tag{10}$$

if and only if the pair (j, m) is also a solution to

$$(2k+1) - 2\lambda - K' \cdot 2^m \leqslant K' \cdot j \cdot 2^m \leqslant (2k+1) - 2\lambda - 1, \tag{11}$$

where $K' = 2^{\lceil \lg(2\lambda) \rceil} = 2K$. It is easy to manipulate (10) to

$$(2k+1) - 2\lambda - K' \cdot 2^m + 1 \leqslant K' \cdot j \cdot 2^m \leqslant (2k+1) - 2\lambda - 1. \tag{12}$$

But $K' \cdot j \cdot 2^m$ is even and since the left-hand side of (12) is even and one greater than the left-hand side of (11), (j, m) is a solution to (11) if and only if (j, m) is a solution to (12). Thus

$$b_2(k+1, \lambda) = b_2(2k+1, 2\lambda). \tag{13}$$

If $K = 2^{\lceil \lg(\lambda) \rceil}$, then $2K = 2^{\lceil \lg(2\lambda) \rceil}$ and by Fact 3.2, $2K = 2^{\lceil \lg(2\lambda - 1) \rceil}$. Using this, it is not difficult to show that $b_2(k+1, \lambda) = b_2(2k+1, 2\lambda - 1)$. This, along with (13), (8), and (1), proves that $b(2k+1, \lambda) = b(k+1, \lceil \lambda/2 \rceil)$. □

Proof of Theorem 3.1(a). In the second step of the following we use Lemmas 3.3 and 3.5. For $k \in \mathbb{Z}_2$,

$$\begin{aligned} b(2k+1) &= b(2k+1, 2k+1) + \sum_{i=2}^{k} (b(2k+1, 2i-1) + b(2k+1, 2i)) \\ &\quad + b(2k+1, 2) + b(2k+1, 1) \\ &= b(k+1, k+1) + \sum_{i=2}^{k} (b(k+1, i) + b(k+1, i)) \\ &\quad + b(k+1, 1) + (b(k+1, 1) + 2) \\ &= 2 - b(k+1, k+1) + 2\sum_{i=1}^{k} b(k+1, i) \\ &= 2b(k+1) + 1. \quad \square \end{aligned}$$

Lemma 3.6. *For $k, \lambda \in \mathbb{Z}_2$, $b(2k, 2\lambda) = b(k, \lambda)$.*

Proof. Using (7) and Fact 3.2, we have

$$\begin{aligned} b_1(2k, 2\lambda) &= \lfloor \lg(2k - 2\lambda - 1) \rfloor - \lceil \lg(2\lambda) \rceil + 1 \\ &= \lfloor \lg(2k - 2\lambda - 2) \rfloor - \lceil \lg(\lambda) \rceil \\ &= \lfloor \lg(k - \lambda - 1) \rfloor - \lceil \lg(\lambda) \rceil + 1 \\ &= b_1(k, \lambda). \end{aligned}$$

By the comments following (9), $b_2(k, \lambda)$ is the number of pairs (j, m) that are solutions to

$$k - \lambda - K \cdot 2^m \leqslant K' \cdot j \cdot 2^m \leqslant k - \lambda - 1 \tag{14}$$

and $b_2(2k, 2\lambda)$ is the number of pairs (j, m) that are solutions to

$$2k - 2\lambda - K' \cdot 2^m \leqslant K' \cdot j \cdot 2^m \leqslant 2k - 2\lambda - 1, \tag{15}$$

where $K = 2^{\lceil \lg(\lambda) \rceil}$ and $K' = 2^{\lceil \lg(2\lambda) \rceil} = 2K$. Multiplying (14) through by 2 almost yields (15) except that the right-hand side is $2k - 2\lambda - 2$. But since $K' \cdot j \cdot 2^m$ must be even, replacing $2k - 2\lambda - 2$ by $2k - 2\lambda - 1$ does not change the number of solutions. Hence $b_2(k, \lambda) = b_2(2k, 2\lambda)$. $\square$

Lemma 3.7. *For* $1 < \lambda < k$, $b(2k, 2\lambda + 1) = b(k + 1, \lambda + 1)$.

Proof. The proof of this lemma is similar to the proof of Lemma 3.6 and is omitted. $\square$

Proof of Theorem 3.1(b). The second step of the following uses Lemmas 3.4, 3.6 and 3.7. For $k \in \mathbb{Z}_2$,

$$\begin{aligned} b(2k) &= b(2k, 1) + \sum_{i=1}^{k} b(2k, 2i) + \sum_{i=1}^{k-1} b(2k, 2i + 1) \\ &= (b(k + 1, 1) + 1) + \sum_{i=1}^{k} b(k, i) + \sum_{i=1}^{k-1} b(k + 1, i + 1) \\ &= b(k) + b(k + 1) - b(k + 1, k + 1) + 1 \\ &= b(k) + b(k + 1). \quad \square \end{aligned}$$

We end this paper by exhibiting a tight upper bound for $b(n)$.

Theorem 3.8. *For* $n \geqslant 2$, $b(n) \leqslant \frac{5}{2}n - \frac{7}{2}$ *and for* $n = 2^k + 1$ $(k \in \mathbb{N})$, $b(n) = \frac{5}{2}n - \frac{7}{2}$.

Proof. This is true for $n = 3$ and $n = 4$ and we proceed by induction using Theorem 3.1. For $k \geqslant 3$,

$$\begin{aligned} b(2k) &= b(k) + b(k + 1) \\ &\leqslant \tfrac{5}{2}k - \tfrac{7}{2} + \tfrac{5}{2}(k + 1) - \tfrac{7}{2} \\ &\leqslant \tfrac{5}{2}(2k) - \tfrac{7}{2} - 1 \\ &\leqslant \tfrac{5}{2}(2k) - \tfrac{7}{2}. \end{aligned}$$

For $k \geqslant 2$,

$$\begin{aligned} b(2k + 1) &= 2b(k + 1) + 1 \\ &\leqslant 2(\tfrac{5}{2}(k + 1) - \tfrac{7}{2}) + 1 \\ &= 5k - 1 \\ &= \tfrac{5}{2}(2k + 1) - \tfrac{7}{2}. \end{aligned}$$

Note that if $n = 2^k + 1$, then the one inequality in the above is really an equality. This gives the tight upper bound. □

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