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Fuzzy Uniform Spaces

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Introduction

In this paper we introduce the notion of fuzzy uniform spaces. In Section 2 we give some basic results and show how a fuzzy topology is derived from a fuzzy uniformity. In Sections 3 and 4 we prove that the notions which are introduced are good extensions and we prove that the category of uniform spaces is nicely injected in the category of fuzzy uniform spaces. Finally in Section 5 we prove some basic results about the associated fuzzy topology.

1. Preliminaries

The unit interval is denoted I, $]0, 1]$ is denoted I_0 and $[0, 1[$ is denoted I_1.

If X is a set and $Y \subset X$, then the characteristic function of Y is denoted 1_Y. Also if X is a set, then the set of all finite subsets of X is denoted $2^{(X)}$.

If f is a function from a set X to a set X', then we denote $f \times f$ the function $X \times X \to X' \times X'$ defined by $f \times f(x, y) = (f(x), f(y))$ for all $(x, y) \in X \times X$.

A fuzzy closure on X [3] is a map $^{-}: I^X \to I^X$ which fulfills the properties:

(FC1) For all α constant, $\bar{\alpha} = \alpha$.

(FC2) For all $\mu \in I^X$, $\bar{\mu} \geqslant \mu$.

(FC3) For all $\mu, \xi \in I^X$, $\overline{\mu \vee \xi} = \bar{\mu} \vee \bar{\xi}$.

(FC4) For all $\mu \in I^X$, $\bar{\bar{\mu}} = \bar{\mu}$.

If X is a set, the diagonal of $X \times X$ is denoted $D(X)$.

If $U \subset X \times X$, the symmetric of U is defined by $\{(y, x): (x, y) \in U\}$ and is denoted by U^{-1}.

If $U, V \subset X \times X$, then their composition $U \circ V$ is defined as the set $\{(x, y) \in X \times X$: there exists $z \in X$, $(x, z) \in V$, $(z, y) \in U\}$.

A uniformity on X is a subset $\mathscr{U}$ of $I^{X \times X}$ fulfilling the following properties

(U1) $\mathscr{U}$ is a filter.

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(U2) For all $U \in \mathscr{U}$, $D(X) \subset U$.

(U3) For all $U \in \mathscr{U}$, $U^{-1} \in \mathscr{U}$.

(U4) For all $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V \circ V \subset U$.

If $V \subset X \times X$ and $n \in \mathbb{N}$, we denote by $\overset{n}{V}$ the set $\overset{n}{V} = V \circ \overset{n-1}{V}$ inductively defined from $\overset{2}{V} = V \circ V$. Clearly then (U4) is equivalent to saying that for all $n \in \mathbb{N}$ and for all $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $\overset{n}{V} \subset U$.

If $V \subset X \times X$ and $A \subset X$, then the section of V over A is the subset of X defined by $V(A) = \{x \in X$: there exists $y \in A$ such that $(y, x) \in V\}$. If $A = \{x\}$, we write $V(x)$.

For a good account of the most important properties of uniform spaces we refer the reader to [2].

For definitions and results on prefilters and on fuzzy convergence we refer to [5].

Recall that if $\mathscr{F}$ is a filter on X, $\omega_{I_1}(\mathscr{F}) = \{\mu \in I^X$: for all $\varepsilon \in I_1$, $\overset{-1}{\mu}\,]\varepsilon, 1] \in \mathscr{F}\}$ and if $\mathfrak{F}$ is a prefilter on X, $\iota_{I_1}(\mathfrak{F}) = \{\overset{-1}{\mu}\,]\varepsilon, 1]$: $\mu \in \mathfrak{F}$, $\varepsilon \in I_1\}$. Also if $\mathfrak{F}$ and $\mathfrak{G}$ are prefilters such that for all $\mu \in \mathfrak{F}$ and $\xi \in \mathfrak{G}$, $\mu \wedge \xi \neq 0$, then we put $\mathfrak{F} \vee \mathfrak{G}$ the prefilter $\{\mu \wedge \xi : \mu \in \mathfrak{F}, \xi \in \mathfrak{G}\}$. If $\mathfrak{B}$ is a prefilterbasis, then the prefilter generated by it is denoted by $[\mathfrak{B}]$, i.e.,

$$[\mathfrak{B}] = \{\mu \in I^X \text{: there exists } \beta \in \mathfrak{B}, \beta \leqslant \mu\}.$$

Also if $\mathfrak{B}$ is a prefilterbasis we define $\hat{\mathfrak{B}}$ by

$$\hat{\mathfrak{B}} = \{\sup_{\epsilon \in I_0} (\beta_\epsilon - \varepsilon) \colon (\beta_\epsilon)_{\epsilon \in I_0} \in \mathfrak{B}^{I_0}\}.$$

PROPOSITION 1.1. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are prefilterbases, then*

(i) $\mathfrak{B} \subset \hat{\mathfrak{B}}$,

(ii) $\hat{\hat{\mathfrak{B}}} \subset [\hat{\mathfrak{B}}]$,

(iii) $\mathfrak{B} \subset \mathfrak{B}' \Rightarrow \hat{\mathfrak{B}} \subset \hat{\mathfrak{B}}'$.

Proof. (i) and (iii) are trivial. To prove (ii) let for all $\varepsilon \in I_0$, $\mu_\epsilon \in \hat{\mathfrak{B}}$, i.e., $\mu_\epsilon = \sup_{\delta \in I_0}(\beta_\epsilon^\delta - \delta)$ where for all $\delta \in I_0$ and $\varepsilon \in I_0$, $\beta_\epsilon^\delta \in \mathfrak{B}$. Then

$$\begin{aligned}\sup_{\epsilon \in I_0} (\mu_\epsilon - \varepsilon) &= \sup_{\epsilon \in I_0} (\sup_{\delta \in I_0} (\beta_\epsilon^\delta - \delta - \varepsilon)) \\ &= \sup_{\alpha \in I_0} \sup_{\substack{\epsilon, \delta \in I_0 \\ \epsilon + \delta = \alpha}} (\beta_\epsilon^\delta - \alpha) \\ &= \sup_{\alpha \in I_0} (\nu_\alpha - \alpha),\end{aligned}$$

where $\nu_\alpha = \sup_{\epsilon, \delta \in I_0, \epsilon + \delta = \alpha} \beta_\epsilon^\delta \in [\mathfrak{B}]$. Consequently, $\sup_{\epsilon \in I_0}(\mu_\epsilon - \varepsilon) \in [\hat{\mathfrak{B}}]$, which proves the assertion.

PROPOSITION 1.2. *If $\mathfrak{B}$ is a prefilterbasis, then* $[\hat{\mathfrak{B}}] = \widehat{[\mathfrak{B}]}$.

Proof. Let $\mu \geqslant \sup_{\varepsilon \in I_0}(\beta_\varepsilon - \varepsilon)$ where for all $\varepsilon \in I_0$, $\beta_\varepsilon \in \mathfrak{B}$. Then for all $\varepsilon \in I_0$, $\mu_\varepsilon = (\mu + \varepsilon) \wedge 1 \geqslant \beta_\varepsilon$. Consequently, $\mu_\varepsilon \in [\mathfrak{B}]$. Now it is easily checked that $\mu = \sup_{\varepsilon \in I_0}(\mu_\varepsilon - \varepsilon)$ which proves that $[\hat{\mathfrak{B}}] \subset \widehat{[\mathfrak{B}]}$. To prove the converse inclusion let $\mu = \sup_{\varepsilon \in I_0}(\mu_\varepsilon - \varepsilon)$ where for all $\varepsilon \in I_0$, $\mu_\varepsilon \in [\mathfrak{B}]$, then for all $\varepsilon \in I_0$ there exists $\beta_\varepsilon \in \mathfrak{B}$ such that $\beta_\varepsilon \leqslant \mu_\varepsilon$ and consequently $\sup_{\varepsilon \in I_0}(\beta_\varepsilon - \varepsilon) \leqslant \mu$ which shows $\mu \in [\hat{\mathfrak{B}}]$.

We shall denote by $\tilde{\mathfrak{B}}$ the prefilter $[\hat{\mathfrak{B}}] = \widehat{[\mathfrak{B}]}$.

PROPOSITION 1.3. *If $\mathfrak{B}$ and $\mathfrak{B}'$ are prefilterbases, then*

(i) $\mathfrak{B} \subset \tilde{\mathfrak{B}}$,

(ii) $\tilde{\tilde{\mathfrak{B}}} = \tilde{\mathfrak{B}}$,

(iii) $\mathfrak{B} \subset \mathfrak{B}' \Rightarrow \tilde{\mathfrak{B}} \subset \tilde{\mathfrak{B}}'$.

Proof. This follows at once from the previous propositions.

PROPOSITION 1.4. *If $\mathfrak{F}$ is a prefilter, then* $\tilde{\mathfrak{F}} = \hat{\mathfrak{F}}$.

Proof. This follows at once from Proposition 1.2.

2. DEFINITIONS AND FUNDAMENTAL PROPERTIES

If X is any set, $\mu \in I^X$ and $\nu \in I^{X \times X}$, then we define the section of ν over μ by

$$\nu\langle\mu\rangle(x) = \sup_{y \in X} \mu(y) \wedge \nu(y, x) \qquad \text{for all} \quad x \in X.$$

If $A \subset X$ and $U \subset X \times X$, then $1_U\langle 1_A\rangle = 1_{U(A)}$.

If $\mu = 1_x$ for some $x \in X$ we shall denote the section of ν over μ by $\nu\langle x\rangle$ or ν_x instead of $\nu\langle 1_x\rangle$.

If $\nu, \xi \in I^{X \times X}$ we define their composition as

$$\nu \circ \xi(x, y) = \sup_{z \in X} \xi(x, z) \wedge \nu(z, y) \qquad \text{for all} \quad (x, y) \in X \times X.$$

Again if $U, V \subset X \times X$ it is clear that $1_U \circ 1_V = 1_{U \circ V}$.

If $\nu \in I^{X \times X}$ and $n \in \mathbb{N}$, we denote by $\overset{n}{\nu}$ the fuzzy set $\overset{n}{\nu} = \nu \circ \overset{n-1}{\nu}$ inductively defined from $\overset{2}{\nu} = \nu \circ \nu$.

If $\nu \in I^{X \times X}$, we define its symmetric by

$$_s\nu(x, y) = \nu(y, x) \qquad \text{for all} \quad (x, y) \in X \times X.$$

If $U \subset X \times X$, then $_s 1_U = 1_{U^{-1}}$.

DEFINITION 2.1. A fuzzy uniformity on X is a subset $\mathfrak{U} \subset I^{X \times X}$ which fulfills the following properties:

(FU1) $\mathfrak{U}$ is a prefilter.

(FU2) $\hat{\mathfrak{U}} = \mathfrak{U}$, i.e., for every family $(\nu_\varepsilon)_{\varepsilon \in I_0} \in \mathfrak{U}^{I_0} \Rightarrow \sup_{\varepsilon \in I_0}(\nu_\varepsilon - \varepsilon) \in \mathfrak{U}$.

(FU3) For all $\nu \in \mathfrak{U}$ and for all $x \in X$, $\nu(x, x) = 1$.

(FU4) For all $\nu \in \mathfrak{U}$, ${}_s\nu \in \mathfrak{U}$.

(FU5) For all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_0$, there exists $\nu_\varepsilon \in \mathfrak{U}$ such that $\nu_\varepsilon \circ \nu_\varepsilon - \varepsilon \leqslant \nu$.

The pair $(X, \mathfrak{U})$ will be called a fuzzy uniform space. The members of $\mathfrak{U}$ are called fuzzy entourages.

DEFINITION 2.2. A subset $\mathfrak{B} \subset I^{X \times X}$ is called a fuzzy uniform basis if and only if the following conditions are fulfilled.

(FUB1) $\mathfrak{B}$ is a prefilterbasis.

(FUB2) For all $\beta \in \mathfrak{B}$ and for all $x \in X$, $\beta(x, x) = 1$.

(FUB3) For all $\beta \in \mathfrak{B}$ and for all $\varepsilon \in I_0$, there exists $\beta_\varepsilon \in \mathfrak{B}$ such that $\beta_\varepsilon - \varepsilon \leqslant {}_s\beta$.

(FUB4) For all $\beta \in \mathfrak{B}$ and for all $\varepsilon \in I_0$, there exists $\beta_\varepsilon \in \mathfrak{B}$ such that $\beta_\varepsilon \circ \beta_\varepsilon - \varepsilon \leqslant \beta$.

Obviously (FUB3) and (FUB4) can be replaced by the single condition

(FUB3′) For all $\beta \in \mathfrak{B}$ and for all $\varepsilon \in I_0$ there exists $\beta_\varepsilon \in \mathfrak{B}$ such that $\beta_\varepsilon \circ \beta_\varepsilon - \varepsilon \leqslant {}_s\beta$.

DEFINITION 2.3. If $\mathfrak{U}$ is a fuzzy uniformity on X, then we shall say that $\mathfrak{B}$ is a basis for $\mathfrak{U}$ if and only if $\mathfrak{B}$ is a prefilterbasis and $\tilde{\mathfrak{B}} = \mathfrak{U}$.

The following propositions are immediate consequences of the definitions, and the proofs are left to the reader.

PROPOSITION 2.1. *If $\mathfrak{B}$ is a fuzzy uniform basis, then $\tilde{\mathfrak{B}}$ is a fuzzy uniformity with $\mathfrak{B}$ as basis. Conversely, if $\mathfrak{B}$ is a basis fo: a fuzzy uniformity, then $\mathfrak{B}$ is a fuzzy uniform basis.*

PROPOSITION 2.2. *If $\mathfrak{U}$ is a fuzzy uniformity on X, then the family of symmetric fuzzy entourages*

$$ {}_s\mathfrak{U} = \{\nu \in \mathfrak{U} : {}_s\nu = \nu\} $$

is a basis for $\mathfrak{U}$. In particular, $[{}_s\mathfrak{U}] = \mathfrak{U}$.

PROPOSITION 2.3. *The map* $\bar{\ }: I^X \to I^X$ *defined by* $\bar{\mu} = \inf_{\nu \in \mathfrak{U}} \nu\langle\mu\rangle$ *is a fuzzy closure operator.*

Proof. (FC1) follows at once from the obvious fact that for any α constant and any $\nu \in \mathfrak{U}$, $\nu\langle\alpha\rangle = \alpha$. (FC2) follows from (FU3). To prove (FC3) let $\mu, \xi \in I^X$, then

$$\begin{aligned}\overline{\mu \vee \xi} &= \inf_{\nu \in \mathfrak{U}} \nu\langle\mu \vee \xi\rangle \\ &= \inf_{\nu \in \mathfrak{U}} \nu\langle\mu\rangle \vee \nu\langle\xi\rangle \\ &\geqslant \inf_{\nu \in \mathfrak{U}} \nu\langle\mu\rangle \vee \inf_{\nu' \in \mathfrak{U}} \nu'\langle\xi\rangle = \bar{\mu} \vee \bar{\xi},\end{aligned}$$

while the reverse inequality follows from the observation that for any $\nu, \nu' \in \mathfrak{U}$, $\nu\langle\mu\rangle \vee \nu'\langle\xi\rangle \geqslant \nu \wedge \nu'\langle\mu \vee \xi\rangle$.

To prove (FC4) let $\mu \in I^X$, then it suffices to show that for any $\varepsilon \in I_0$ and any $\nu \in \mathfrak{U}$ there exists a $\nu' \in \mathfrak{U}$ such that $\nu\langle\mu\rangle \geqslant \nu'\langle\bar{\mu}\rangle - \varepsilon$. Choose $\nu' \in \mathfrak{U}$ such that $\nu' \circ \nu' - \varepsilon \leqslant \nu$; then for any $x \in X$, we have

$$\begin{aligned}\nu\langle\mu\rangle(x) &= \sup_{y \in X} \mu(y) \wedge \nu(y, x) \\ &\geqslant \sup_{y \in X} \mu(y) \wedge (\nu' \circ \nu')(y, x) - \varepsilon \\ &= \sup_{y \in X} \sup_{z \in X} \mu(y) \wedge \nu'(y, z) \wedge \nu'(z, x) - \varepsilon \\ &\geqslant \sup_{z \in X} \nu'(z, x) \wedge \big(\inf_{\nu'' \in \mathfrak{U}} \sup_{y \in X} \mu(y) \wedge \nu''(y, z)\big) - \varepsilon \\ &= \sup_{z \in X} \nu'(z, x) \wedge \bar{\mu}(z) - \varepsilon \\ &= \nu'\langle\bar{\mu}\rangle(x) - \varepsilon.\end{aligned}$$

PROPOSITION 2.4. *If* $\mathfrak{B}$ *is a basis for the fuzzy uniformity* $\mathfrak{U}$, *then for all* $\mu \in I^X$ *we have*

$$\bar{\mu} = \inf_{\theta \in \hat{\mathfrak{B}}} \theta\langle\mu\rangle = \inf_{\kappa \in [\mathfrak{B}]} \kappa\langle\mu\rangle = \inf_{\beta \in \mathfrak{B}} \beta\langle\mu\rangle.$$

Proof. Since $\mathfrak{B} \subset \hat{\mathfrak{B}} \subset \mathfrak{U}$ and $\mathfrak{B} \subset [\mathfrak{B}] \subset \mathfrak{U}$, it suffices to show the following. Let $\mu \in I^X$ and let $\varepsilon \in I_0$; then since for all $\nu \in \mathfrak{U}$ there exists $\beta \in \mathfrak{B}$ such that $\beta - \varepsilon \leqslant \nu$, it follows that

$$\bar{\mu} \geqslant \inf_{\beta \in \mathfrak{B}} [(\beta - \varepsilon) \vee 0]\langle\mu\rangle \geqslant \inf_{\beta \in \mathfrak{B}} \beta\langle\mu\rangle - \varepsilon$$

which proves that $\bar{\mu} \geqslant \inf_{\beta \in \mathfrak{B}} \beta\langle\mu\rangle$.

DEFINITION 2.4. Let $(X, \mathfrak{U})$ and $(X', \mathfrak{U}')$ be fuzzy uniform spaces and $f: X \to X'$. We say f is uniformly cntinuous if and only if for all $\nu' \in \mathfrak{U}'$ there exists $\nu \in \mathfrak{U}$ such that $\nu \leqslant (f \times f)^{-1}(\nu')$.

This is obviously equivalent to saying that for all $\nu' \in \mathfrak{U}'$ $(f \times f)^{-1}(\nu') \in \mathfrak{U}$ or that there exists $\nu \in \mathfrak{U}$ suchs that $(f \times f)(\nu) \leqslant \nu'$.

PROPOSITION 2.5. *If $(X, \mathfrak{U})$ and $(X', \mathfrak{U}')$ are fuzzy uniform spaces, $\mathfrak{B}$ and $\mathfrak{B}'$ are bases for $\mathfrak{U}$ and $\mathfrak{U}'$, respectively, and $f: X \to X'$, then f is uniformly continuous if and only if for all $\beta' \in \mathfrak{B}'$ and for all $\varepsilon \in I_0$ there exists $\beta \in \mathfrak{B}$ such that $\beta - \varepsilon \leqslant (f \times f)^{-1}(\beta')$.*

Proof. Follows at once from Definition 2.4 and (FU2).

COROLLARY 2.6. *If $(X, \mathfrak{U})$ and $(X', \mathfrak{U}')$ are fuzzy uniform spaces and $f: X \to X'$, then f is uniformly continuous if and only if for all $\nu' \in \mathfrak{U}'$ and for all $\varepsilon \in I_0$ there exists $\nu \in \mathfrak{U}$ such that $\nu - \varepsilon \leqslant (f \times f)^{-1}(\nu')$.*

Proof. Since each fuzzy uniformity is obviously a basis for itself, this follows at once from Proposition 2.5.

THEOREM 2.7. *If $(X, \mathfrak{U})$ and $(X', \mathfrak{U}')$ are fuzzy uniform spaces and $f: X \to X'$ is uniformly continuous, then f is continuous.*

Proof. Let $\mu \in I^X$ and $x' \in f(X)$; then

$$\overline{f(\mu)}(x') = \inf_{\nu' \in \mathfrak{U}'} \nu'\langle f(\mu)\rangle(x')$$

$$= \inf_{\nu' \in \mathfrak{U}'} \sup_{t \in X} \mu(t) \wedge \nu'(f(t), x).$$

On the other hand, since f is uniformly continuous, for any $\nu' \in \mathfrak{U}'$ there exists a $\nu \in \mathfrak{U}$ such that for all $s, t \in X$, $\nu'(f(t), f(s)) \geqslant \nu(t, s)$; consequently, for all $x \in f^{-1}(x')$

$$\overline{f(\mu)}(x') = \inf_{\nu' \in \mathfrak{U}'} \sup_{t \in X} \mu(t) \wedge \nu'(f(t), f(x))$$

$$\geqslant \inf_{\nu \in \mathfrak{U}} \sup_{t \in X} \mu(t) \wedge \nu(t, x) = \bar{\mu}(x).$$

Consequently,

$$\overline{f(\mu)}(x') \geqslant \sup_{x \in f^{-1}(x')} \bar{\mu}(x) = f(\bar{\mu})(x').$$

3. The Operators ω_u and ι_u

The following operators are the uniform analogues of the operators ω and ι introduced in [3].

If $\mathscr{U}$ is a uniformity on X, then we define

$$\omega_u(\mathscr{U}) = \{\mu \in I^{X \times X} : \forall \varepsilon \in I_1 \ \overset{-1}{\mu}]\varepsilon, 1] \in \mathscr{U}\};$$

and if $\mathfrak{U}$ is a fuzzy uniformity on X, we define

$$\iota_u(\mathfrak{U}) = \{ \overset{-1}{\mu}]\varepsilon, 1] : \mu \in \mathfrak{U}, \varepsilon \in I_1\}.$$

In Theorem 3.1 we prove that $\omega_u(\mathscr{U})$ is a fuzzy uniformity on X. A fuzzy uniformity of this type is called generated (by $\mathscr{U}$).

We denote the topology associated with a uniformity $\mathscr{U}$ by $T(\mathscr{U})$ and the fuzzy topology associated with a fuzzy uniformity $\mathfrak{U}$ by $t(\mathfrak{U})$, i.e., $t(\mathfrak{U})$ is the fuzzy topology whose fuzzy closure operator was introduced in Proposition 2.3.

Theorem 3.1. *If $\mathscr{U}$ is a uniformity, and $\mathfrak{U}$ a fuzzy uniformity on X, then*

(i) *$\omega_u(\mathscr{U})$ is a fuzzy uniformity on X.*

(ii) *$\iota_u(\mathfrak{U})$ is a uniformity on X.*

(iii) *$\iota_u(\omega_u(\mathscr{U})) = \mathscr{U}$.*

(iv) *$\omega_u(\iota_u(\mathfrak{U}))$ is the coarsest fuzzy uniformity generated by a uniformity and which is finer than $\mathfrak{U}$. We denote $\omega_u(\iota_u(\mathfrak{U}))$ by $\overline{\mathfrak{U}}$.*

(v) *$t(\omega_u(\mathscr{U})) = \omega(T(\mathscr{U}))$.*

(vi) *$T(\iota_u(\mathfrak{U})) = \iota(t(\mathfrak{U}))$.*

(vii) *$t(\overline{\mathfrak{U}}) = \overline{t(\mathfrak{U})}$.*

Proof. (i) (FU1) follows at once from the fact that $\omega_u(\mathscr{U}) = \omega_{I_1}(\mathscr{U})$. To show (FU2) let $(\nu_\epsilon)_{\epsilon \in I_0} \in \omega_u(\mathscr{U})^{I_0}$ and $\delta \in I_1$. Put $\varepsilon_0 = (1-\delta)/2$; then $\varepsilon_0 + \delta \in I_1$ and

$$(\sup_{\epsilon \in I_0} (\nu_\epsilon - \varepsilon))^{-1}]\delta, 1] = \bigcup_{\epsilon \in I_0} \nu_\epsilon^{-1}]\varepsilon + \delta, 1] \supset \nu_{\varepsilon_0}^{-1}]\varepsilon_0 + \delta, 1]$$

which implies $\sup_{\epsilon \in I_0}(\nu_\epsilon - \varepsilon) \in \omega_u(\mathscr{U})$.

(FU3) and (FU4) are immediate. To prove (FU5) let $\nu \in \omega_u(\mathscr{U})$ and let $\varepsilon \in I_0$. Put $\delta = 1 - \varepsilon \in I_1$; then since $\overset{-1}{\nu}]\delta, 1] \in \mathscr{U}$, there exists $U \in \mathscr{U}$ such that $U \circ U \subset \overset{-1}{\nu}]\delta, 1]$. Now if we put $\xi = 1_U$, then $\xi \in \omega_u(\mathscr{U})$ and $\xi \circ \xi \leqslant \nu + \varepsilon$ which proves (FU5).

(ii) (U1) follows at once from the observation that $\iota_u(\mathfrak{U}) = \iota_{I_1}(\mathfrak{U})$. (U2) and (U3) are immediate.

To prove (U4) let $\mu \in \mathfrak{U}$ and $\varepsilon \in I_1$. Since $(1-\varepsilon)/2 \in I_0$, it follows from (FU5) that there exists $\xi \in \mathfrak{U}$ such that $\xi \circ \xi \leqslant \mu + (1-\varepsilon)/2$. Since $(1+\varepsilon)/2 \in I_1$, we have that $U = \xi^{-1}](1+\varepsilon)/2, 1] \in \iota_u(\mathfrak{U})$. It is easily checked that $U \circ U \subset \overset{-1}{\mu}\,]\varepsilon, 1]$.

(iii) Immediate by definition.

(iv) Follows from (iii) and the fact that ω_u and ι_u are isotone maps.

(v) Suppose μ is $t(\omega_u(\mathscr{U}))$-closed. To show it is $\omega(T(\mathscr{U}))$-closed it suffices to show that for all $\varepsilon \in I$, $\overset{-1}{\mu}[\varepsilon, 1]$ is $T(\mathscr{U})$-closed. Let $x \in \bigcap_{V \in \mathscr{U}} V(\overset{-1}{\mu}[\varepsilon, 1])$. This implies that for all $V \in \mathscr{U}$, $1_V\langle\mu\rangle(x) \geqslant \varepsilon$. Remark that the family $\{1_V : V \in \mathscr{U}\}$ is a basis for the fuzzy uniformity $\omega_u(\mathscr{U})$. This follows easily from the definitions of basis and of $\omega_u(\mathscr{U})$. Consequently, from Proposition 2.4, $\mu(x) = \bar{\mu}(x) \geqslant \varepsilon$ which proves $\overset{-1}{\mu}[\varepsilon, 1]$ is $T(\mathscr{U})$-closed.

Conversely let μ be $\omega(T(\mathscr{U}))$-closed. Suppose $x \in X$ and $\varepsilon \in I_0$ are such that $\mu(x) < \varepsilon$. From

$$\overset{-1}{\mu}[\varepsilon, 1] \supset \bigcap_{V \in \mathscr{U}} V(\overset{-1}{\mu}[\varepsilon, 1])$$

it follows that there exists $V_0 \in \mathscr{U}$ such that $x \notin V_0(\overset{-1}{\mu}[\varepsilon, 1])$, i.e., for all $z \in X$, $\mu(z) \wedge 1_{V_0}(z, x) < \varepsilon$. Consequently $\bar{\mu}(x) \leqslant \varepsilon$. Since this holds for all $\varepsilon > \mu(x)$, it follows that $\bar{\mu}(x) \leqslant \mu(x)$.

Further if $\mu(x) = 1$, then $\bar{\mu}(x) = 1$. Thus, we have shown that $\bar{\mu} = \mu$.

(vi) Suppose A is $T(\iota_u(\mathfrak{U}))$-closed, i.e.,

$$\begin{aligned}
A &= \bigcap \{\overset{-1}{\mu}\,]\varepsilon, 1](A) : \mu \in \mathfrak{U}, \varepsilon \in I_1\} \\
&= \{x : \forall \mu \in \mathfrak{U}, \forall \varepsilon \in I_1 \sup_{z \in X} 1_A(z) \wedge \mu(z, x) > \varepsilon\} \\
&= \{x : \inf_{\mu \in \mathfrak{U}} \sup_{z \in X} 1_A(z) \wedge \mu(z, x) = 1\} \\
&= (\overline{1_A}^{t(\mathfrak{U})})^{-1}(1).
\end{aligned}$$

Now since $t(\mathfrak{U}) \subset \overline{t(\mathfrak{U})}(\overline{t(\mathfrak{U})} = \omega(\iota(t(\mathfrak{U})))$ [3]), it follows that

$$\overline{1_A}^{t(\mathfrak{U})} \geqslant \overline{1_A}^{\overline{t(\mathfrak{U})}} = 1_{\bar{A}^{\iota(t(\mathfrak{U}))}}$$

which implies

$$\bar{A}^{\iota(t(\mathfrak{U}))} = (1_{\bar{A}^{\iota(t(\mathfrak{U}))}})^{-1}(1) \subset (\overline{1_A}^{t(\mathfrak{U})})^{-1}(1) = A.$$

Consequently, A is $\iota(t(\mathfrak{U}))$-closed.

Conversely, suppose $A = \overset{-1}{\mu}[\delta, 1]$ where μ is $t(\mathfrak{U})$-closed and $\delta \in I$. Let $x \in \bar{A}^{T(\iota_u(\mathfrak{U}))}$, then for all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_1$, $x \in \overset{-1}{\nu}\,]\varepsilon, 1](A)$, i.e., for all $\nu \in \mathfrak{U}$ and for all $\varepsilon \in I_1$ there exists $z_\epsilon \in X$ such that $\mu(z_\epsilon) \geqslant \delta$ and $\nu(z_\epsilon, x) > \varepsilon$; consequently,

(i) if $\delta < 1$, then choose $\varepsilon > \delta \Rightarrow \mu(z_\epsilon) \wedge \nu(z_\epsilon, x) \geqslant \delta$ such that $\sup_{z \in X} \mu(z) \wedge \nu(z, x) \geqslant \delta$,

(ii) if $\delta = 1$, then for all $\varepsilon < 1$, $\mu(z_\epsilon) = 1$ so that $\mu(z_\epsilon) \wedge \nu(z_\epsilon, x) = \nu(z_\epsilon, x) > \varepsilon$ which again implies $\sup_{z \in X} \mu(z) \wedge \nu(z, x) \geqslant 1 = \delta$.

Since this holds for all $\nu \in \mathfrak{U}$, we have

$$\inf_{\nu \in \mathfrak{U}} \sup_{z \in X} \mu(z) \wedge \nu(z, x) \geqslant \delta,$$

i.e.,

$$\mu(x) = \bar{\mu}(x) \geqslant \delta$$

which implies $x \in A$. Consequently, A is $T(\iota_u(\mathfrak{U}))$-closed.

(vii) Follows at once from (v) and (vi).

DEFINITION 3.1. We say that a fuzzy topological space (X, Δ) is fuzzy uniformizable if and only if there exists a fuzzy uniformity $\mathfrak{U}$ on X such that $\Delta = t(\mathfrak{U})$.

COROLLARY 3.2. *A topological space $(X, \mathscr{T})$ is uniformizable if and only if $(X, \omega(\mathscr{T}))$ is fuzzy uniformizable.*

Proof. If $\mathscr{T} = T(\mathscr{U})$, then $\omega(\mathscr{T}) = \omega(T(\mathscr{U})) = t(\omega_u(\mathscr{U}))$ and conversely if $\omega(\mathscr{T}) = t(\mathfrak{U})$, then $\mathscr{T} = \iota(t(\mathfrak{U})) = T(\iota_u(\mathfrak{U}))$.

THEOREM 3.3. (i) *If $(X, \mathscr{U})$ and $(X', \mathscr{U}')$ are uniform spaces, then $f: X \to X'$ is uniformly continuous in the usual sense if and only if it is uniformly continuous when considered as a map between the fuzzy uniform spaces $(X, \omega_u(\mathscr{U}))$ and $(X', \omega_u(\mathscr{U}'))$.*

(ii) *If $(X, \mathfrak{U})$ and $(X', \mathfrak{U}')$ are fuzzy uniform spaces and $f: X \to X'$ is uniformly continuous, then it is also uniformly continuous when considered as a function between the uniform spaces $(X, \iota_u(\mathfrak{U}))$ and $(X', \iota_u(\mathfrak{U}'))$.*

Proof. This is an easy consequence of the definitions and the obvious fact that for any $\mu' \in I^{X' \times X'}$ and any $\varepsilon \in I$, $(f \times f)^{-1}(\mu'^{-1}]\varepsilon, 1]) = ((f \times f)^{-1}(\mu'))^{-1}]\varepsilon, 1]$.

4. Initial Fuzzy Uniformities

An important construction is that of initial—or weak—fuzzy uniformities. More precisely, given a family $(X_j, \mathfrak{U}_j, f_j)_{j\in J}$ where for each $j \in J$, $(X_j, \mathfrak{U}_j)$ is a fuzzy uniform space and f_j a function from some set X to X_j, we want to construct on X a coarsest fuzzy uniformity making each function f_j uniformly continuous.

By methods which, apart from some technicalities, are perfect duplicates of the classical ones the reader can easily establish the proof of the following theorems.

Theorem 4.1. *Given a family $(X_j, \mathfrak{U}_j, f_j)_{j\in J}$ where for each $j \in J$, $(X_j, \mathfrak{U}_j)$ is a fuzzy uniform space and f_j a function from some set X to X_j there exists a coarsest fuzzy uniformity on X making each f_j uniformly continuous. This fuzzy uniformity has as basis the family*

$$\{ \inf_{j\in J_0} (f_j \times f_j)^{-1}(\nu_j) \colon \nu_j \in \mathfrak{U}_j, J_0 \in 2^{(J)} \}$$

and is called the initial—or weak—fuzzy uniformity on X for the family $(X_j, \mathfrak{U}_j, f_j)_{j\in J}$. It is denoted $\sup_{j\in J}(f_j \times f_j)^{-1}(\mathfrak{U}_j)$.

If $(X', \mathfrak{U}')$ is a fuzzy uniform space and $f\colon X' \to X$, then if X carries the initial fuzzy uniformity, f is uniformly continuous ifand only if each $f_j \circ f$ is uniformly continuous.

Theorem 4.2. *Given a set X, a family of fuzzy uniform spaces $(X_j, \mathfrak{U}_j)_{j\in J}$, a partition $(J_\lambda)_{\lambda\in L}$ of J, a family of sets $(X'_\lambda)_{\lambda\in L}$, for each $\lambda \in L$ a function h_λ from X to X'_λ and for each $\lambda \in L$ and $j \in J_\lambda$ a function $g_{j\lambda}$ from X'_λ to X_j. We equip each X'_λ with the initial fuzzy uniformity for the family $(X_j, \mathfrak{U}_j, g_{j\lambda})_{j\in J_\lambda}$. Then the initial fuzzy uniformity on X for the family $(X_j, \mathfrak{U}_j, g_{j\lambda} \circ h_\lambda)_{j\in J}$ is equal to the initial fuzzy uniformity on X for the family $(X'_\lambda, \sup_{j\in J_\lambda}(g_{j\lambda} \times g_{j\lambda})^{-1}(\mathfrak{U}_j), h_\lambda)_{\lambda\in L}$.*

Three particular cases of the result of Theorem 4.1 are worth considering.

Given a set X a fuzzy uniform space $(X', \mathfrak{U}')$ and a function $f\colon X \to X'$, the initial fuzzy uniformity on X for $(X', \mathfrak{U}', f)$ is also called the reciprocal fuzzy uniformity and it follows that a basis for it is given by the family

$$\{(f \times f)^{-1}(\nu') \colon \nu' \in \mathfrak{U}'\}.$$

Given a fuzzy uniform space $(X, \mathfrak{U})$ and a subset $Y \subset X$, the fuzzy uniformity induced on Y is the reciprocal fuzzy uniformity for $(X, \mathfrak{U}, i_Y)$ where $i_Y\colon Y \hookrightarrow X$ is the canonical injection. The fuzzy entourages for this structure are given by

$$\{\nu|_{Y\times Y} \colon \nu \in \mathfrak{U}\};$$

we denote it by $\mathfrak{U}|_{Y\times Y}$. $(Y, \mathfrak{U}|_{Y\times Y})$ is called a fuzzy uniform subspace of $(X, \mathfrak{U})$.

Given a family $(X_j, \mathfrak{U}_j)_{j\in J}$ of fuzzy uniform spaces, the product fuzzy uniformity on $\prod_{j\in J} X_j$ is the initial fuzzy uniformity on $\prod_{j\in J} X_j$ for the family $(X_j, \mathfrak{U}_j, \mathrm{pr}_j)_{j\in J}$, where pr_j denotes the jth projection.

We now return to the general case and show the following theorem which proves that the operation of taking an initial fuzzy uniformity is compatible with that of taking an initial fuzzy topology [4].

THEOREM 4.3. *If X is a set, $(X_j, \mathfrak{U}_j)_{j\in J}$ a family of fuzzy uniform spaces and for each $j\in J$, $f_j: X \to X_j$, then*

$$t(\sup_{j\in J} (f_j \times f_j)^{-1}(\mathfrak{U}_j)) = \sup_{j\in J} f_j^{-1}(t(\mathfrak{U}_j)).$$

LEMMA 4.4. *If X is a set, (X', Δ') a fuzzy topological space and $f: X \to X'$, then the fuzzy closure operator associated with the reciprocal fuzzy topology on X is given by*

$$\bar{\mu} = f^{-1}(\overline{f(\mu)}) \qquad \text{for all} \quad \mu \in I^X.$$

Proof. Let $\mu \in I^X$, then

$$\begin{aligned}\bar{\mu} &= \inf\{f^{-1}(\mu') : \mu' \text{ closed, } f^{-1}(\mu') \geqslant \mu\}\\ &= f^{-1}(\inf\{\mu' : \mu' \text{ closed } \mu' \geqslant f(\mu)\})\\ &= f^{-1}(\overline{f(\mu)}).\end{aligned}$$

LEMMA 4.5. *If X is a set, $(X', \mathfrak{U}')$ a fuzzy uniform space and $f: X \to X'$, then $t((f\times f)^{-1}(\mathfrak{U}')) = f^{-1}(t(\mathfrak{U}'))$.*

Proof. Denote the fuzzy closure operators associated with $t((f\times f)^{-1}(\mathfrak{U}'))$ and $f^{-1}(t(\mathfrak{U}'))$ respectively $^{-1}$ and $^{-2}$. Let $\mu \in I^X$ and $x \in X$, then from Lemma 4.4

$$\begin{aligned}\bar{\mu}^2(x) &= f^{-1}(\overline{f(\mu)})(x)\\ &= \inf_{\nu' \in \mathfrak{U}'} \nu'\langle f(\mu)\rangle(f(x))\\ &= \inf_{\nu' \in \mathfrak{U}'} \sup_{x' \in X'} (\sup_{t \in f^{-1}(x')} \mu(t)) \wedge \nu'(x', f(x))\\ &= \inf_{\nu' \in \mathfrak{U}'} \sup_{y \in X} \mu(y) \wedge \nu'(f(y), f(x))\\ &= \inf_{\nu' \in \mathfrak{U}'} (f\times f)^{-1}(\nu')\langle \mu\rangle(x)\\ &= \bar{\mu}^1(x).\end{aligned}$$

LEMMA 4.6. *If X is a set and $(\mathfrak{U}_j)_{j\in J}$ a family of fuzzy uniformities on X, then*

$$t(\sup_{j\in J} \mathfrak{U}_j) = \sup_{j\in J} t(\mathfrak{U}_j).$$

Proof. $t(\sup_{j\in J} \mathfrak{U}_j) \supset \sup_{j\in J} t(\mathfrak{U}_j)$ is clear. To prove the other inclusion we shall denote closures in $t(\sup_{j\in J} \mathfrak{U}_j)$ and in $\sup_{j\in J} t(\mathfrak{U}_j)$ respectively by $^{-1}$ and $^{-2}$.

Closure in $t(\mathfrak{U}_j)$ shall be denoted $^{-j}$. Let $\mu \in I^X$ and $x \in X$. Now put

$$\mathscr{H} = 2^{(\bigcup_{j\in J} \mathfrak{U}_j)},$$

then

$$\begin{aligned}\bar{\mu}^1(x) &= \inf_{H\in\mathscr{H}} \sup_{y\in X} \inf_{\nu\in H} \mu(y) \wedge \nu(y, x)\\ &= \inf_{H\in\mathscr{H}} \inf_{h\in H^X} \sup_{y\in X} \mu(y) \wedge h(y)(y, x) \qquad (1)\end{aligned}$$

Next put $\mathscr{K} = \{K \in 2^{(J\times I^X)}: \sup_{(j,\xi)\in K} \bar{\xi}^j \geqslant \mu\}$

$$\begin{aligned}\bar{\mu}^2(x) &= \inf_{K\in\mathscr{K}} \sup_{(j,\xi)\in K} \bar{\xi}^j(x)\\ &= \inf_{K\in\mathscr{K}} \sup_{(j,\xi)\in K} \inf_{\nu_j\in\mathfrak{U}_j} \sup_{y\in X} \xi(y) \wedge \nu_j(y, x)\\ &= \inf_{K\in\mathscr{K}} \inf_{g\in G_K} \sup_{(j,\xi)\in K} \sup_{y\in X} \xi(y) \wedge g(j, \xi)(y, x), \qquad (2)\end{aligned}$$

where we have put

$$G_K = \left\{ g: K \to \bigcup_{(j,\xi)\in K} \mathfrak{U}_j, \forall (j, \xi) \in K, g(j, \xi) \in \mathfrak{U}_j \right\}.$$

Since we want to show that $\bar{\mu}^1(x) \geqslant \bar{\mu}^2(x)$, it suffices from (1) and (2) to show that for all $H \in \mathscr{H}$ and for all $h \in H^X$ there exists $K \in \mathscr{K}$ and $g \in G_K$ such that for all $y \in X$

$$\mu(y) \wedge h(y)(y, x) \geqslant \sup_{(j,\xi)\in K} \xi(y) \wedge g(j, \xi)(y, x).$$

Let $H = \{\nu_1 \in \mathfrak{U}_{j_1}, \ldots, \nu_n \in \mathfrak{U}_{j_n}\}$ and $h \in H^X$.

Then for all $i = 1, \ldots, n$, put $X_i = h^{-1}(\nu_i)$. Clearly, the family $(X_i)_{i=1}^n$ covers X and is mutually disjoint. Then we define

$$K = \{(j_1, \mu|_{X_1}), \ldots, (j_n, \mu|_{X_n})\}.$$

Since $\sup_{i=1}^{n} \overline{(\mu|_{X_i})}^{j_i} \geqslant \sup_{i=1}^{n} \mu|_{X_i} = \mu$, we thus have that $K \in \mathscr{K}$. Next define $g \in G_K$ by

$$g(j_i, \mu|_{X_i}) = \nu_i.$$

Now if $y \in X$, then if k is such that $y \in X_k$, it follows that

$$\begin{aligned} \sup_{(j,\xi) \in K} \xi(y) \wedge g(j, \xi)(y, x) &= \sup_{i=1}^{n} \mu|_{X_i}(y) \wedge \nu_i(y, x) \\ &= \mu(y) \wedge \nu_k(y, x) \\ &= \mu(y) \wedge h(y)(y, x) \end{aligned}$$

which completes the proof of Lemma 4.6.

Proof of Theorem 4.3. This now follows at once from Lemmas 4.5 and 4.6 since

$$\begin{aligned} t(\sup_{j \in J} (f_j \times f_j)^{-1}(\mathfrak{U}_j)) &= \sup_{j \in J} t((f_j \times f_j)^{-1}(\mathfrak{U}_j)) \\ &= \sup_{j \in J} f_j^{-1}(t(\mathfrak{U}_j)). \end{aligned}$$

That the operation of taking an initial fuzzy uniformity or an initial uniformity also behaves well when connected to the operators ω_u and ι_u is shown in the next two theorems.

THEOREM 4.7. *If X is a set, $(X_j, \mathscr{U}_j)_{j \in J}$ a family of uniform spaces and for each $j \in J$, $f_j: X \to X_j$, then*

$$\omega_u(\sup_{j \in J} (f_j \times f_j)^{-1}(\mathscr{U}_j)) = \sup_{j \in J} (f_j \times f_j)^{-1}(\omega_u(\mathscr{U}_j)).$$

Proof. Since for each choice of a finite number of entourages $U_{j_1} \in \mathscr{U}_{j_1}, \ldots, U_{j_n} \in \mathscr{U}_{j_n}$

$$\begin{aligned} &1_{(f_{j_1} \times f_{j_1})^{-1}(U_{j_1}) \cap \cdots \cap (f_{j_n} \times f_{j_n})^{-1}(U_{j_n})} \\ &\quad = (f_{j_1} \times f_{j_1})^{-1}(1_{U_{j_1}}) \wedge \cdots \wedge (f_{j_n} \times f_{j_n})^{-1}(1_{U_{j_n}}) \end{aligned}$$

and since the fuzzy sets on the left-hand side form a basis for $\omega_u(\sup_{j \in J}(f_j \times f_j)^{-1}(\mathscr{U}_j))$ and those on the right-hand side form a basis for $\sup_{j \in J}(f_j \times f_j)^{-1}(\omega_u(\mathscr{U}_j))$ the result follows.

THEOREM 4.8. *If X is a set, $(X_j, \mathfrak{U}_j)_{j \in J}$ a family of fuzzy uniform spaces and for each $j \in J$, $f_j: X \to X_j$, then*

$$\iota_u(\sup_{j \in J} (f_j \times f_j)^{-1}(\mathfrak{U}_j)) = \sup_{j \in J} (f_j \times f_j)^{-1}(\iota_u(\mathfrak{U}_j)).$$

Proof. Since for each choice of a finite number of fuzzy entourages $\nu_{j_1} \in \mathfrak{U}_{j_1}, \ldots, \nu_{j_n} \in \mathfrak{U}_{j_n}$,

$$((f_{j_1} \times f_{j_1})^{-1}(\nu_{j_1}) \wedge \cdots \wedge (f_{j_n} \times f_{j_n})^{-1}(\nu_{j_n}))^{-1}]\varepsilon, 1]$$
$$= (f_{j_1} \times f_{j_1})^{-1}(\nu_{j_1}^{-1}]\varepsilon, 1]) \cap \cdots \cap (f_{j_n} \times f_{j_n})^{-1}(\nu_{j_n}^{-1}]\varepsilon, 1])$$

and since the fuzzy sets on the left-hand side form a basis for $\iota_u(\sup_{j \in J}(f_j \times f_j)^{-1}(\mathfrak{U}_j))$ and those on the right-hand side form a basis for $\sup_{j \in J}(f_j \times f_j)^{-1}(\iota_u(\mathfrak{U}_j))$ the result follows.

As an immediate consequence of Theorems 4.3, 4.7, and 4.8 we have the following corollary:

COROLLARY 4.9. *If X is a set, $(X_j, \mathfrak{U}_j)_{j \in J}$ a family of fuzzy uniform spaces, $(X_j, \mathscr{U}_j)_{j \in J}$ a family of uniform spaces and for each $j \in J$, $f_j: X \to X_j$, then*

(i) $t(\omega_u(\sup_{j \in J}(f_j \times f_j)^{-1}(\mathscr{U}_j))) = \sup_{j \in J}(f_j \times f_j)^{-1}(t(\omega_u(\mathscr{U}_j)))$;

(ii) $T(\iota_u(\sup_{j \in J}(f_j \times f_j)^{-1}(\mathfrak{U}_j))) = \sup_{j \in J}(f_j \times f_j)^{-1}(T(\iota_u(\mathfrak{U}_j)))$;

(iii) $\overline{\sup_{j \in J}(f_j \times f_j)^{-1}(\mathfrak{U}_j)} = \sup_{j \in J}(f_j \times f_j)^{-1}(\bar{\mathfrak{U}}_j)$.

5. SOME RESULTS ON THE FUZZY TOPOLOGY ASSOCIATED WITH A FUZZY UNIFORMITY

PROPOSITION 5.1. *If $(X, \mathfrak{U})$ is a fuzzy uniform space and $\mu \in I^{X \times X}$, then for $t(\mathfrak{U}) \times t(\mathfrak{U})$ we have*

$$\bar{\mu} = \inf_{\nu \in {}_s\mathfrak{U}} \nu \circ \mu \circ \nu.$$

Proof. If pr_1 and pr_2 denote first and second projection, respectively, then for any $\xi \in I^{X \times X}$ and any (x, y) and (x', y') in $X \times X$

$$(\mathrm{pr}_1 \times \mathrm{pr}_1)^{-1}(\xi)((x, y), (x', y')) = \xi(x, x'),$$
$$(\mathrm{pr}_2 \times \mathrm{pr}_2)^{-1}(\xi)((x, y), (x', y')) = \xi(y, y').$$

If for all $\nu, \nu' \in {}_s\mathfrak{U}$ we put

$$\nu * \nu'((x, y), (x', y')) = \nu(x, x') \wedge \nu'(y, y');$$

then it follows at once from Theorem 4.1 that $\mathfrak{B} = \{\nu * \nu' : \nu, \nu' \in {}_s\mathfrak{U}\}$ is a basis for $\mathfrak{U} \times \mathfrak{U}$. Since for any $\nu, \nu' \in {}_s\mathfrak{U} (\nu \wedge \nu') * (\nu \wedge \nu') \leqslant \nu * \nu'$, it is also clear that $\mathfrak{B}' = \{\nu * \nu : \nu \in {}_s\mathfrak{U}\}$ is a basis for $\mathfrak{U} \times \mathfrak{U}$. Consequently, from

Proposition 2.4 and Theorem 4.3 it follows that for any $\mu \in I^{X \times X}$ and any $(x, y) \in X \times X$

$$\begin{aligned}\bar{\mu}(x, y) &= \inf_{\nu \in {}_s\mathfrak{U}} \nu * \nu\langle\mu\rangle(x, y)\\ &= \inf_{\nu \in {}_s\mathfrak{U}} \sup_{(t,s) \in X \times X} \nu(x, t) \wedge \mu(t, s) \wedge \nu(s, y)\\ &= \inf_{\nu \in {}_s\mathfrak{U}} \nu \circ \mu \circ \nu(x, y).\end{aligned}$$

COROLLARY 5.2. *If* $(X, \mathfrak{U})$ *is a fuzzy uniform space, then the closed fuzzy entourages form a basis for* $\mathfrak{U}$.

Proof. Let $\nu \in \mathfrak{U}$ and let $\varepsilon \in I_0$. Choose $\nu_0 \in {}_s\mathfrak{U}$ such that $\overset{3}{\nu}_0 - \varepsilon \leqslant \nu$. Then

$$\bar{\nu}_0 - \varepsilon = \inf_{\nu' \in {}_s\mathfrak{U}} \nu' \circ \nu_0 \circ \nu' - \varepsilon \leqslant \overset{3}{\nu}_0 - \varepsilon \leqslant \nu.$$

We recall that a fuzzy topological space is called Hausdorff [6] if and only if each (prime-) prefilter has a limit which is non-zero in at most one point.

THEOREM 5.3. *$(X, t(\mathfrak{U}))$ is Hausdorff if and only if* $\inf_{\nu \in \mathfrak{U}} \nu = 1_{D(X)}$.

Proof. Suppose $(X, t(\mathfrak{U}))$ is Hausdorff and let $x \neq y \in X$ and $\varepsilon \in I_0$ be such that for all $\nu \in \mathfrak{U}$ $\nu(x, y) > \varepsilon$. It is very easy to see that

$$\mathfrak{F}_x = \{\nu_x \colon \nu \in \mathfrak{U}\}$$

and

$$\mathfrak{F}_y = \{\nu_y \colon \nu \in \mathfrak{U}\}$$

are prefilters in X. Since for all $\nu, \nu' \in \mathfrak{U}$ $\nu_x \wedge \nu'_y(y) = \nu(x, y) > \varepsilon$, it follows that $\mathfrak{F}_x \vee \mathfrak{F}_y$ exists and that $c(\mathfrak{F}_x \vee \mathfrak{F}_y) \geqslant \varepsilon$. Consequently, it follows from Proposition 2.1 in [5] that there exists $\mathfrak{G} \in P_m(\mathfrak{F}_x \vee \mathfrak{F}_y)$ such that $c(\mathfrak{G}) \geqslant \varepsilon/2$. Then it follows that

$$\begin{aligned}\lim \mathfrak{G}(x) &= \inf_{\xi \in \mathfrak{G}} \bar{\xi}(x) = \inf_{\xi \in \mathfrak{G}} \inf_{\nu \in \mathfrak{U}} \sup_{z \in X} \xi(z) \wedge \nu(x, z)\\ &= \inf_{\xi \in \mathfrak{G}} \inf_{\mu \in \mathfrak{F}_x} \sup_{z \in X} \xi \wedge \mu(z)\\ &= \inf_{\xi \in \mathfrak{G}} \sup_{z \in X} \xi(z) \geqslant \varepsilon/2.\end{aligned}$$

Analogously, however,

$$\lim \mathfrak{G}(y) \geqslant \varepsilon/2$$

which is in contradiction with the Hausdorffness of $(X, t(\mathfrak{U}))$. Conversely, suppose $\mathfrak{G}$ is a prime prefilter, $x \neq y \in X$, and $\varepsilon \in I_0$ such that $\lim \mathfrak{G}(x) > \varepsilon$ and $\lim \mathfrak{G}(y) > \varepsilon$. From $\inf_{\nu \in \mathfrak{U}} \nu = 1_{D(X)}$ it follows that there exist $\nu \in \mathfrak{U}$ such that $\nu(x, y) \leqslant \varepsilon/2$. Choose $\nu' \in {}_s\mathfrak{U}$ such that $\nu' \circ \nu' - \varepsilon/2 \leqslant \nu$; clearly then $\nu'_x \wedge \nu'_y \leqslant \varepsilon$. If we put

$$N_x = \nu_x'^{-1}[0, \varepsilon],$$
$$N_y = \nu_y'^{-1}[0, \varepsilon],$$

then from the fact that $\mathfrak{G}$ is prime it follows that, for instance, $1_{N_x} \in \mathfrak{G}$. Consequently,

$$\begin{aligned} \lim \mathfrak{G}(x) &= \inf_{\mu \in \mathfrak{G}} \bar{\mu}(x) \\ &\leqslant \overline{1_{N_x}}(x) \\ &\leqslant \sup_{z \in X} 1_{N_x} \wedge \nu'_x(z) \leqslant \varepsilon, \end{aligned}$$

which is a contradiction.

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