Weak Copositive and Intertwining Approximation

Y. K. Hu
Georgia Southern University

K. A. Kopotun
Vanderbilt University

and

X. M. Yu
Southwest Missouri State University

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It is known that shape preserving approximation has lower rates than unconstrained approximation. This is especially true for copositive and intertwining approximations. For \( f \in L^p \), \( 1 \leq p < \infty \), the former only has rate \( \| f \|_{L^p} \), and the latter cannot even be bounded by \( C \| f \|_p \). In this paper, we discuss various ways to relax the restrictions in these approximations and conclude that the most sensible way is the so-called almost copositive/intertwining approximation in which one relaxes the restriction on the approximants in a neighborhood of radius \( 2n \) of each sign change \( y_j \).

1. INTRODUCTION

Let \( C[a, b] \) and \( C^k[a, b] \) be the sets of all continuous and all \( k \)-times continuously differentiable functions on \( [a, b] \), respectively, and let \( L^p[a, b] \), \( 0 < p < \infty \), be the set of measurable functions on \( [a, b] \) such that \( \| f \|_{L^p[a, b]} < \infty \), where

\[
\| f \|_{L^p[a, b]} := \left( \int_a^b |f(x)|^p \, dx \right)^{1/p}.
\]
Throughout this paper $L_{\alpha, \beta}$ is understood as $C[a, b]$ with the usual uniform norm, to simplify the notation, and $1 \leq p \leq \infty$ is always assumed unless otherwise indicated. We also denote by $W^k[a, b]$ the Sobolev space, the set of all functions $f$ on $[a, b]$ such that $f^{k-1}$ are absolutely continuous and $f^{(k)} \in L_p$, and by $P_n$ the set of all polynomials of degree $\leq n$.

Let us recall some definitions of moduli of smoothness used throughout this paper. The $m$th symmetric difference of $f$ is given by

$$A_m^n(f, x, [a, b]) := \left\{ \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f \left( x - \frac{m}{2} + ih \right) \right\}.$$  

Then the $m$th (usual) modulus of smoothness of $f \in L_p[a, b]$ is defined by

$$\omega_m(f, t, [a, b]) := \sup_{0 < h < t} \| A_m^n(f, \cdot, [a, b]) \|_{L_p[a, b]}.$$  

We shall also use the so-called $\tau$-modulus, or Sendov–Popov modulus, an averaged modulus of smoothness, defined for all bounded measurable functions on $[a, b]$ by

$$\tau_m(f, t, [a, b]) := \| \omega_m(f, \cdot, t) \|_{L_p[a, b]},$$

where

$$\omega_m(f, x, t) := \sup \{ |A_m^n(f, y)| : y + mh/2 \in [x - mt/2, x + mt/2] \cap [a, b] \}$$

is the $m$th local modulus of smoothness of $f$. (We set $\tau_m(f, t, [a, b]) := \infty$ if the function $f$ is unbounded.) If the interval $[-1, 1]$ is used in any of the above notations, it will be omitted for the sake of simplicity, for example,

$$\| f \|_p := \| f \|_{L_p[-1, 1]}, \quad \omega_m(f, t) := \omega_m(f, t, [-1, 1]).$$

The $\omega$- and $\tau$-moduli measure the smoothness of $f$ over the interval uniformly. The “non-uniform” modulus that we use is the $m$th Ditzian–Totik modulus of smoothness, defined for $f \in L_p[-1, 1]$ by

$$\omega_m^{dt}(f, t) := \sup_{0 < h < t} \| A_m^n(f, \cdot, [-1, 1]) \|_p,$$

with $\varphi(x) := \sqrt{1 - x^2}$. Let $A_m(x) := n^{-1} \sqrt{1 - x^2} + n^{-2}$. The term of $\omega_m^{dt}(f, A_m(x))$ is also used in this paper.

Let $Y_s := \{ y_1, \ldots, y_s : y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 = y_{s+1} \}, \ s \geq 0$. We denote by $A^0(Y_s)$ the set of all functions $f$ such that $(1)^{r-1} f(x) \geq 0$
for \(x \in \{y_j, y_{j+1}\}\), \(k = 0, \ldots, s\), i.e., those that have \(0 \leq s < \infty\) sign changes at the points in \(Y_s\) and are nonnegative near 1. In particular, \(A^0 := A^0(Y_0)\) denotes the set of all nonnegative functions on \([-1, 1]\). Functions \(f\) and \(g\) which belong to the same class \(A^0(Y_s)\) are said to be copositive on \([-1, 1]\). Copositive approximation is the approximation of functions \(f\) from \(A^0(Y_s)\) class by polynomials or splines that are copositive with \(f\). For \(f \in L_p[-1, 1]\) let
\[
E_n(f)_p := \inf_{P_n \in P_n} \|f - P_n\|_p
\]
denote the degree of unconstrained approximation, and let
\[
E_n^0(f, Y_s)_p := \inf_{P_n \in P_n, A^0(Y_s)} \|f - P_n\|_p
\]
be the degree of copositive polynomial approximation of \(f\). In particular,
\[
E_n^0(f)_p := E_n^0(f, Y_0)_p := \inf_{P_n \in P_n, A^0} \|f - P_n\|_p
\]
is the degree of positive approximation. The degree of intertwining polynomial approximation of functions \(f \in L_p[-1, 1]\) with respect to \(Y_s\) is given by
\[
\tilde{E}_n(f, Y_s)_p := \inf\{\|P - Q\|_p : P, Q \in P_n, P - f \in A^0(Y_s) \text{ and } f - Q \in A^0(Y_s)\}.
\]
We call \(\{P, Q\}\) an intertwining pair of polynomials for \(f\) with respect to \(Y_s\) if \(P - f, f - Q \in A^0(Y_s)\). In particular, when \(s = 0\), \(\tilde{E}_n(f, Y_0)_p = \tilde{E}_n(f)_p\) is the degree of one-sided polynomial approximation of \(f\).

While intertwining approximation was introduced by the authors [10] not long ago, positive, copositive, and one-sided approximations have been studied extensively in recent years.

Some main results are summarized in Tables I–III. (See [9–10] and the references therein.) From these tables we see the degrees are astonishingly low in \(L_p\), \(p < \infty\). As an extreme, the degree of intertwining approximation is not even bounded by \(\|f\|_p\) or \(\tau(f, 1)_p\). Recently, Leviatan and Shevchuk [20] obtained higher degree of comonotone approximation in \(C[-1, 1]\) by relaxing the restriction in a neighborhood of radius \(A_d(y_j)\) of each sign change \(y_j\). Inspired by their idea, we discuss in this paper various ways to relax the restrictions in copositive and intertwining approximations and conclude that the most sensible way is the so-called almost copositive/ intertwining approximation, in which one gives up the “right” amount of restriction in change for higher degrees than those in Tables I–III. All these are defined in Section 2 and summarized in Section 3.
### TABLE I

Positive Approximation

| $p = \infty$ | $f \in \mathbb{C}$ | $3P_{\ast}P_{a}(x) \geq 0$, such that \( |f(x) - P_{a}(x)| \leq C_{a}^{\ast}(f, \Delta_{a}(x))_{\infty} \) |
|-------------|-----------------|----------------------------------|
| $1 \leq p < \infty$ | $f \in \mathbb{C}_{p}$ | $E_{n}(f)_{p} \leq C_{n}^{\ast}(f, n^{-1})_{p}$ |
|             | $f \in \mathbb{W}_{p}$ | $E_{n}(f)_{p} \leq C_{n}(f, \Delta_{a}(x))_{p}$ |
|             | $f \in \mathbb{W}_{1}$ | $E_{n}(f)_{1} \leq C_{n}^{\ast}(f, 1)_{p}$ |

### TABLE II

Copositive Approximation

| $p = \infty$ | $f \in \mathbb{C}$ | $3P_{\ast}$, copositive with $f$, such that \( |f(x) - P_{a}(x)| \leq C_{a}^{\ast}(f, \Delta_{a}(x))_{\infty} \) |
|-------------|-----------------|----------------------------------|
| $1 \leq p < \infty$ | $f \in \mathbb{C}_{p}$ | $E_{n}(f)_{p} \leq C_{n}^{\ast}(f, n^{-1})_{p}$ |
|             | $f \in \mathbb{W}_{p}$ | $E_{n}(f)_{p} \leq C_{n}(f, \Delta_{a}(x))_{p}$ |
|             | $f \in \mathbb{W}_{1}$ | $E_{n}(f)_{1} \leq C_{n}^{\ast}(f, 1)_{p}$ |

### TABLE III

Intertwining Approximation

<table>
<thead>
<tr>
<th>$p = \infty$</th>
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<th>$E_{n}(f, Y_{a})<em>{\infty} \leq C(f)</em>{\infty}$</th>
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2. NOTATIONS AND DEFINITIONS

We denote $J_j(n, \varepsilon) := \{ y_j - A_j(y_j) n^j, y_j + A_j(y_j) n^j \} \cap [-1, 1], j = 0, 1, \ldots, s + 1$, and denote $O_j(Y, \varepsilon) := \bigcup_{j=0}^s J_j(n, \varepsilon)$ and $O^*_j(Y, \varepsilon) := \bigcup_{j=0}^s J_j(n, \varepsilon)$. If $\varepsilon = 0$, we shall also use the simpler notation $J_j := J_j(n, 0)$, $O_j(Y) := O_j(Y, 0)$, and $O^*_j(Y) := O^*_j(Y, 0)$. Functions $f$ and $g$ are said to be copositive on $J \subset I := [-1, 1]$ if $f(x) g(x) \geq 0, \forall x \in J$. Functions $f$ and $g$ are called almost copositive on $I$ with respect to $Y$, if they are copositive on $I \setminus O^*_n(Y)$. We say that $f$ and $g$ are strongly (weakly) almost copositive on $I$ with respect to $Y$, if they are copositive on $I \setminus O^*_n(Y, \varepsilon)$, where $\varepsilon < 0 (\varepsilon > 0)$. In particular, if $\varepsilon = -\infty$, then strongly almost copositive functions are just copositive. We define a function class

$$(\varepsilon\text{-alm } A^0_n(Y)) := \{ f : (-1)^{n-k} f(x) \geq 0 \text{ for } x \in I \setminus O^*_n(Y, \varepsilon) \}.$$ 

If $x = 0$, it becomes

$$(\varepsilon\text{-alm } A^0_n(Y)) := (\varepsilon\text{-alm } A^0_n(Y_0)) := \{ f : f(x) \geq 0 \text{ for } x \in [-1 + n^{-2+\varepsilon}, 1 - n^{-2+\varepsilon}] \},$$

the set of all strongly (weakly) almost nonnegative functions on $I$ if $\varepsilon < 0 (\varepsilon > 0)$. Again, if $\varepsilon = 0$, we omit the letter $\varepsilon$ in the notation and use $(\text{alm } A^0_n(Y))$ and $(\text{alm } A^0_n(Y))$. The latter is the set of almost nonnegative function on $I$. If $\varepsilon = -\infty$, strongly almost nonnegative functions are just nonnegative.

**Definition.** The degree of almost positive polynomial approximation of $f \in L_p[-1, 1]$ is

$$E_\varepsilon^0(f, \text{alm } Y_0)_p := \inf \{ \| f - P \|_p : P \in P_n \cap (\text{alm } A^0_n) \}.$$ 

Similarly, we define $E_\varepsilon^0(f, \text{alm } Y_0)_p$, the degree of strongly (weakly) almost positive polynomial approximation of $f \in L_p[-1, 1]$, by means of $P \in P_n \cap (\text{alm } A^0_n)$.

**Definition.** The degree of almost copositive polynomial approximation of $f \in L_p[-1, 1] \cap A^0(Y)$ is

$$E_\varepsilon^0(f, \text{alm } Y_0)_p := \inf \{ \| f - P \|_p : P \in P_n \cap (\text{alm } A^0_n(Y)) \}.$$ 

Similarly, we define $E_\varepsilon^0(f, \text{alm } Y_0)_p$, the degree of strongly (weakly) almost copositive polynomial approximation of $f \in L_p[-1, 1] \cap A^0(Y)$, by means of $P \in P_n \cap (\text{alm } A^0_n(Y))$. 


Definition. The degree of almost intertwining polynomial approximation of \( f \in L_p[-1, 1] \) with respect to \( Y_s \) is

\[
\bar{E}_n(f, \text{alm } Y_s)_p := \inf \left\{ \|P - f\|_p + \|Q - f\|_p : P, Q \in P_n, (-1)^{s-j}(P(x) - f(x)) \geq 0 \right. \\
\left. \quad \text{and } (-1)^{s-j}(f(x) - Q(x)) \geq 0 \right\}
\]

if \( x \in [y_j, y_{j+1}] \setminus O_n(Y_s), j = 0, ..., s \).

We call \( \{P, Q\} \) an almost intertwining pair of polynomials for \( f \) with respect to \( Y_s \) if \( P \) and \( Q \) satisfy the restrictions in the above infimum.

Note. We do not use \( \|P - Q\|_p \) in the definition since \( f(x) \) does not have to be between \( P(x) \) and \( Q(x) \) when \( x \) is close to \( y_j \).

Definition. The degree of nearly intertwining polynomial approximation of \( f \in L_p[-1, 1] \) with respect to \( Y_s \) is

\[
\bar{E}_n(f, \text{nearly } Y_s)_p := \inf \left\{ \|P - Q\|_p : P, Q \in P_n, P - f \in A^0(\tilde{Y}_s), f - Q \in A^0(\tilde{Y}_s), \\
\text{where } \tilde{Y}_s = \{\tilde{y}_1, ..., \tilde{y}_s\}, \tilde{y}_0 := -1 < \tilde{y}_1 < ... < \tilde{y}_s < 1 := \tilde{y}_{s+1} \}, \\
\text{and } |\tilde{y}_j - y_j| \leq A_n(y_j) \text{ for } j = 1, 2, ..., s \}.
\]

We call \( \{P, Q\} \) a nearly intertwining pair of polynomials for \( f \) with respect to \( Y_s \) if \( P - f, f - Q \in A^0(\tilde{Y}_s) \).

Remark. We have the following relationships among the above quantities:

\[
\bar{E}_n(f, \text{alm } Y_s)_p \leq \bar{E}_n(f, \text{nearly } Y_s)_p \leq \bar{E}_n(f, Y_s)_p;
\]

and for \( f \in A^0(Y_s) \),

\[
E^{(0)}_n(f, \text{alm } Y_s)_p \leq \bar{E}_n(f, \text{alm } Y_s)_p
\]

and

\[
E^{(0)}_n(f, \text{alm } Y_s)_p \leq E^{(0)}_n(f, Y_s)_p.
\]
3. MAIN RESULTS

We summarize all the results in this paper in Tables IV–VII. Compared with Tables I–III we see significant improvements made by switching from positive/copositive/intertwining approximations to almost positive/copositive/intertwining approximations, due to relaxing restrictions in a neighborhood of radius \( A_d(y) \) of each \( y_j \). For example, almost positive approximation improves to the order of \( \omega^2(f, n^{-1})_p \) from \( \omega(f, n^{-1})_p \) for \( 1 \leq p < \infty \). Almost intertwining approximation also achieves a better order of \( \omega^2(f, n^{-1})_p \) or \( \omega^2(f, A_d(x))_p \) in \( C \), and the order of \( \tau^p(f, n^{-1})_p \) in \( L_p \). While it still fails to reach the order of \( \| f \|_p \) for \( 1 \leq p < \infty \), (for example, \( f(x) = 0 \) if \( x \neq 0 \), and \( f(0) = 1 \), and 0 is “far from” the set \( Y_s \)).

Nearly intertwining approximation, in which the intertwining points are allowed to shift by an amount no larger than \( A_d(y) \) (using \( Y_s \) instead of \( Y_s \)), improves to the order of \( n^{-1} \| f \|_p \) from \( n^{-1} \| f \|_p \) for \( f \in W^1_p \). Unfortunately, it shows no improvements in \( L_p \). We emphasize that all rates we obtain in this paper are exact in the sense that one can not raise the order of the modulus used in the upper bound.

At the same time, we find that strongly almost positive/copositive approximations, in which restrictions are relaxed in intervals smaller than \([ y_j - A_d(y), y_j + A_d(y) ]\), do not do better than the ordinary positive/copositive approximations; while weakly almost positive/copositive approximations, in which restrictions are relaxed in intervals larger than \([ y_j - A_d(y), y_j + A_d(y) ]\), fail to bring a further improvement to the approximation order. In this sense, the “almost” version \( (\epsilon = 0) \) is the most sensible weak version.

### TABLE IV

<table>
<thead>
<tr>
<th>( \epsilon = 0 ): Almost Positive Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(f, \text{alm} Y_0) \leq C \omega^2(f, n^{-1})_p )</td>
</tr>
<tr>
<td>( E(f, \text{alm} Y_0) \not\leq C \omega^2(f, 1)_p )</td>
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<table>
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<tr>
<th>( \epsilon &lt; 0 ): Strongly Almost Positive Approximation</th>
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<tr>
<td>( E(f, \text{alm} Y_0) \leq C \omega^2(f, n^{-1})_p )</td>
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<tbody>
<tr>
<td>becomes unconstrained approximation</td>
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Note. If \( f \in C \) or \( W^1_p \), then (almost) positive approximation has the same order as the unconstrained case (see Table I or [10]).
<table>
<thead>
<tr>
<th>$\varepsilon = 0$: Almost Copositive Approximation</th>
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</table>
| \[ f \in C \]
| \[ E_{n}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| \[ \exists \: P \in P_{\varepsilon} \cap (\text{alm} \Delta_{n}(Y_{\varepsilon})) \text{ such that} \]
| \[ |f(x) - P(x)| \leq C_{\varepsilon}^{0}(f, \Delta_{n}(x))_{p} \] |
| Theorem 10 |
| $1 \leq p < \infty$ |

| $f \in L_{p}$ |
| \[ E_{n}^{0}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| Theorem 11 |
| $1 \leq p < \infty$ |

| $f \in W_{1}^{1}$ |
| \[ E_{n}^{0}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| Theorem 11 |
| $\varepsilon < 0$: Strongly Almost Copositive Approximation |
| $1 \leq p < \infty$ and $s = 1$ |

| $f \in L_{p}$ |
| \[ E_{n}^{s}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{s}(f, 1)_{p} \] |
| Theorem 8 |
| $1 \leq p < \infty$ and $s > 1$ |

| $f \in L_{p}$ |
| \[ E_{n}^{s}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{s}(f, n^{-1})_{p} \] |
| Theorem 9 |
| $f \in W_{1}^{1}$ |
| \[ E_{n}^{s}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{s}(f, n^{-1})_{p} \] |
| Corollary 6 |
| $1 < p < \infty$ and $s = 1$ |

| $f \in L_{p}$ |
| \[ E_{n}^{s}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{s}(f, 1)_{p} \] |
| Theorem 10 |
| $1 \leq p < \infty$ and $s > 1$ |

| $f \in L_{p}$ |
| \[ E_{n}^{s}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{s}(f, n^{-1})_{p} \] |
| Theorem 11 |
| $0 \leq \varepsilon < 1$: Weakly Almost Copositive Approximation |

| $\varepsilon > 1$: Weakly Almost Copositive Approximation |
| Becomes unconstrained approximation for large $n$ |

### TABLE VI

#### Almost Intertwining Approximation

<table>
<thead>
<tr>
<th>$\varepsilon = 0$: Almost Intertwining Approximation</th>
</tr>
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</table>
| \[ f \in C \]
| \[ E_{n}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| \[ \exists \: P \in P \cap (\text{alm} \Delta_{n}(Y_{\varepsilon})) \text{ such that} \]
| \[ |P(x) - f(x)| + |f(x) - Q_{\varepsilon}(x)| \leq C_{\varepsilon}^{0}(f, \Delta_{n}(x))_{p} \] |
| Theorem 13 |
| $1 \leq p < \infty$ |

| $f \in L_{p}$ |
| \[ E_{n}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| Theorem 14 |
| $1 \leq p < \infty$ |

| $f \in W_{1}^{1}$ |
| \[ E_{n}(f, \text{alm} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| Example on page 7 |

### TABLE VII

#### Nearly Intertwining Approximation

<table>
<thead>
<tr>
<th>$\varepsilon &gt; 0$: Nearly Intertwining Approximation</th>
</tr>
</thead>
</table>
| \[ f \in L_{p} \]
| \[ E_{n}(f, \text{near} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, 1)_{p} \] |
| Theorem 15 and |
| Example on page 7 |

| $f \in W_{1}^{1}$ |
| \[ E_{n}(f, \text{near} Y_{\varepsilon})_{p} \leq C_{\varepsilon}^{0}(f, n^{-1})_{p} \] |
| Theorem 17 |
4. WEAK POSITIVE APPROXIMATION

Although positive approximation is a special case of copositive approximation, it very often has a better rate; at least this is the case in the ordinary positive approximation. We begin with almost and weakly/strongly almost positive approximation for \( 1 < p < \infty \). Theorems 1 and 2 show that almost positive approximation has an order of \( \omega^2(f, n^{-1})_p \), compared with \( \omega(f, n^{-1})_p \) for the ordinary positive approximation. The rate is exact in the sense that one cannot replace it even by \( \omega^3(f, 1)_p \). This is obtained by relaxing the restriction on intervals of length \( n^{-2} \) at \( x = \pm 1 \). Using larger intervals (of length \( n^{-2+\varepsilon} \), \( 0 < \varepsilon < 2 \)) gains no more than this, unless giving up the restriction on the whole interval \([-1, 1]\) (\( \varepsilon \geq 2 \)), that is, back to unconstrained approximation; while using smaller intervals (\( \varepsilon < 0 \)) yields no improvement over the ordinary positive approximation. For the case of \( p = \infty \), see the note below Table IV.

**Theorem 1.** Let \( f \in L_p([-1, 1] \cap A^d) \), \( 0 \leq \varepsilon < 2 \), and \( 1 < p < \infty \). Then

\[
E_n^{(0)}(f, \varepsilon \text{-alm } Y_0)_p \leq C \omega^2(f, n^{-1})_p,
\]

where \( C \) is an absolute constant. On the other hand, given any \( A > 0 \), \( n \in \mathbb{N}^* \), \( 1 < p < \infty \), and \( 0 \leq \varepsilon < 2 \), \( f \in L_p([-1, 1] \cap A^d) \) such that

\[
E_n^{(0)}(f, \varepsilon \text{-alm } Y_0)_p > A \omega^3(f, 1)_p.
\]

**Proof.** Inequality (4.1) follows from Theorem 4. To prove (4.2), we let \( Q(x) := x^2 - b^{-1} \), where \( b > 1 \) is a constant to be chosen later, and define

\[
f(x) := Q(x)_+ := \begin{cases} Q(x), & |x| > \sqrt{b^{-1}}, \\ 0, & \text{otherwise}. \end{cases}
\]

Then

\[
\omega^3(f, 1)_p = \omega^3(f - Q, 1)_p \leq C \| f - Q \|_p
\]
\[
\leq C \| Q \|_{L_p([-\sqrt{b^{-1}}, \sqrt{b^{-1}}], \mathbb{R})} \leq C b^{1-1/(2p)}.
\]
Suppose, towards a contradiction, \( P_n \) is a polynomial from \( \mathbb{P}_n \) such that 
\[
\| P_n(0) \|_p > 0 \quad \text{and} \quad \| P_n - f \|_p \leq \alpha \omega^3(f, 1)_p.
\]
Then
\[
\| P_n(0) - Q(0) \|_p \leq \| P_n - Q \|_p \leq C \| P_n - f \|_p + C \| f - Q \|_p \\
\leq C \alpha \omega^3(f, 1)_p + C \| f - Q \|_p < C_1 b^{-1 - \frac{1}{3} (2p)},
\]
where \( C_1 \) depends on \( n \) but not on \( b \). Therefore,
\[
P_n(0) + C_1 b^{-1 - \frac{1}{3} (2p)} = b^{-1} + C_1 b^{-1 - \frac{1}{3} (2p)} < 0
\]
for sufficiently large \( b \), which is the desired contradiction.

Remark. The same proof can be used to show that for any \( \beta > 0 \) and \( 0 \leq \varepsilon < 2 \),
\[
E_n(0; f, e-\text{alm } Y_0) \supseteq C n^\beta \omega^3(f, 1)_p, \quad 1 \leq p < \infty.
\]

Theorem 5. For any given \( A > 0 \) and \( 0 \leq \varepsilon < 2 \), \( n \), \( n \) sufficiently large, \( f \in L_p([-1, 1] \cap A^1) \) such that
\[
E_n(0; f, e-\text{alm } Y_0) > A \omega^3(f, 1)_p.
\]

Proof. We only give a sketch since the proof is similar to that of Theorem 1. Let
\[
\{ L(x) := x + 1 - n^{-2} a, \ f(x) := L(x) := \max\{ L(x), 0 \} \}
\]
By choosing the value of the parameter \( a > 4n^2 \) carefully one can readily prove that
\[
P_n(\cdot + 1 + 2n^{a-2}) < 0
\]
for any polynomial \( P_n \in \mathbb{P}_n \) with \( \| P_n - f \|_p \leq \omega^3(f, 1)_p \) provided \( n \) is sufficiently large. Therefore \( P_n \) is not strongly almost positive.

5. WEAK COPOSITIVE APPROXIMATION

In this section, we first show in Theorem 4 that almost copositive approximation in \( L_p \), \( 1 \leq p < \infty \), improves the rate to \( \omega^3(f, n^{-1})_p \) from \( \omega^2(f, n^{-1})_p \), the rate for the ordinary copositive approximation. We first need an analog of this for splines. The result for polynomials will then come from the following theorem by the authors [10].

Theorem A. Let \( Y_s (s \geq 0) \) be given, \( m \in \mathbb{N}, \ \mu \geq 2m + 30, \ 0 < p \leq \infty, \) and let \( S(x) \) be a spline of an odd order \( r \) \((r = 2m + 1)\) on the knot sequence \( \{ x_i = \cos(\pi n) \}_{i=0}^{n+1} \), where \( n > C(Y_s) \) is such that there are at least 4
knots \( x_i \) in each interval \( (y_j, y_{j+1}) \), \( j = 0, \ldots, s \), and \( I_s(Y) := \{ 1, \ldots, n \} \) \( i, i-1 \): \( x_i < x_{i-1} \) for some \( 1 \leq j \leq s \). Then there exists an intertwining pair of polynomials \( \{ P_1, P_2 \} \subset \mathbb{P}_{C(r)}^n \) for \( S \) with respect to \( Y \), such that

\[
\| P_1 - P_2 \|_p \leq C(r, s, \min\{ p, 1 \}) \sum_{i=1}^{n-1} E_{r-1}(S, I_i \cup I_{i+1})^p,
\]

if \( 0 < p < \infty \),

and

\[
| P_1(x) - P_2(x) | \leq C(r, \mu, s) \sum_{i=1}^{n-1} E_{r-1}(S, I_i \cup I_{i+1}) \left( \frac{|I_i|}{|x - x_i| + |I_i|} \right)^\mu,
\]

if \( p = \infty \),

where \( I_i := [x_i, x_{i-1}] \) and \( E_n(f, [a, b])_p := \inf_{P_n \in \mathbb{P}_n} \| f - P_n \|_{L_p(a, b)} \).

Let \( k > 0 \) be an integer, and \(-1 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1\) be a partition of \( I = [-1, 1] \). Define the so-called auxiliary knots (DeVore and Lorentz [5, p. 140]) by \( t_i := 1 + i \Delta t_0, i = - r + 1, \ldots, - 1, \) and \( t_{k-i} := 1 + (i - k) \Delta t_{k-1}, i = k + 1, \ldots, k + r - 1 \), where \( \Delta t_i := t_{i+1} - t_i \). Let \( I_i := [t_i, t_{i+1}] \), and \( J_i := [t_{i-r+1}, t_{i+1}] \). Denote \( T_k := \{ t_i \}_{i=-r+1}^{k+r-1} \), \( \Delta T_k := \max\{ |T_i| \} \), and for \( i = - r + 1, \ldots, k - 1 \), \( d_i := (t_{i+1} - t_i)/r \), \( t^*_i := (t_{i+1} + \cdots + t_{i+r-1})/(r - 1) \), \( d^*_i := 2 \min(t^*_{i-r}, t^*_{i-r+1}, t^*_i) \). For any \( f \in L_1([-r+1, k+r-1]) \), we define a linear operator \( T \) by

\[
Tf := \sum_{i=-r+1}^{k-r} c_i N_{t_i}, \quad c_i := c_i(f) := d_i^{*,-1} \int_{t_i}^{t_i^*} f,
\]

where \( N_{t_i}(x) := N_{t, t_i}(x) := N(x; t_i, \ldots, t_{i+r}) \) is the B-spline on \( t_i, \ldots, t_{i+r} \), normalized so that \( \sum N_{t_i}(x) = 1 \). Note that \( T \) preserves linearity, that is, \( TL = L \) for any \( L \in \mathbb{P}_1 \), because \( T \) becomes the Schoenberg variation diminishing operator in this case [2, Chap. XI–XII].

For any function \( f \in L_1[-1, 1] \), we use its Whitney's Extension to \([t_{r+1}, t_{k+r-1}]\) so that \( T \) can be applied. This will only enlarge the constant \( C \) in (5.3) and (5.4) by a factor depending only on \( r \) (see Theorem 6.4.1 and its proof in [5]). And we shall still use the letter \( f \) for the extension for the sake of simple notation. With the notation above, we prove

**Lemma 3.** Let \( f \in L_p([-1, 1]) \). Then the spline \( S := Tf \) of order \( r \) on the knot sequence \( T_k \) satisfies...
\[ \| f - S \|_p \leq C \omega^2(f, A \mathbf{T}_k)_p, \quad (5.3) \]

\[ \| f - S \|_p^p \leq C \sum_{i = 1}^{k+1} \omega^2(f, A \mathbf{T}_i, J_i)_p^p, \quad (5.4) \]

where the constant C depends on r and the ratios $A_{i+r}/A_i$ of lengths of neighboring subintervals.

**Proof.** By Hölder’s Inequality we have

\[ |c_j| \leq d_j^{p-1} \int_{I_{i+1}} |f| d_{i+1}^{p-1} \| f \|_{L_p(I_{i+1})} d_i^{1/p} = d_j^{p-1} \| f \|_{L_p(I_i)}, \]

By the well-known relationship between a spline and its B-spline series coefficients (de Boor, see [2, Chap. XI] and [21, Section 4.6] for $p = \infty$, and [5, Section 5.4] for $0 < p < \infty$)

\[ \| T f \|_{L_p(I_i)} = \| S \|_{L_p(I_i)} = \sum_{j = i-r+1}^{i} c_j N_j \| f \|_{L_p(I_{i+1})} d_i^{1/p} \leq C \sum_{j = i-r+1}^{i} |c_j| d_j^{1/p} \leq C \| f \|_{L_p(I_i)}. \quad (5.5) \]

Let $l_i$ be a best linear approximation to $f$ on $I_i$; then

\[ \| f - l_i \|_{L_p(I_i)} \leq C \omega^2(f, A \mathbf{T}_i, I_i)_p. \]

This $l_i$ is also a near best linear approximation on $J_i = [t_i, t_i + r]$ (DeVore and Popov [6]) and therefore satisfies

\[ \| f - l_i \|_{L_p(I_i)} \leq C \omega^2(f, A \mathbf{T}_i, J_i)_p. \]

Applying (5.5) to $T(f - l_i)$ gives (5.4);

\[ \| f - S \|_{L_p(I_i)} = \| f - T f \|_{L_p(I_i)} \leq \| f - l_i \|_{L_p(I_i)} + \| T (f - l_i) \|_{L_p(I_i)} \leq C \omega^2(f, A \mathbf{T}_i, I_i)_p + \| f - l_i \|_{L_p(I_i)} \leq C \omega^2(f, A \mathbf{T}_i, J_i)_p, \]

which in turn gives (5.3) (Leviatan and Mhaskar [19], also see Hu [8]):

\[ \| f - S \|_p \leq C \omega^2(f, A \mathbf{T}_k, [t_{i+r+1}, t_{i+r}], [t_{i+r+1}, t_{i+r+1}]_p \leq C \omega^2(f, A \mathbf{T}_k, [-1, 1])_p. \]

The following theorem gives affirmative results on almost copositive approximation. Here and throughout the rest of the paper, we denote $J^+_y := [y_j - A_d(y_j), y_j]$ and $J^-_y := [y_j, y_j + A_d(y_j)]$.
Theorem 4. Suppose \( f \in L^p[-1, 1] \cap A^0(Y_s) \).

(i) Let \( T_k \) be a given single-knot sequence such that there are at least \( r \) knots in each of \( J_j^+ \), \( j = 1, ..., s+1 \), and in each of \( J_j^- \), \( j = 0, ..., s \), if they do not intersect \( J_{j-1}^+ \) or \( J_{j+1}^- \), respectively. Then \( S := Tf = \sum_{i=-r+1}^{k-1} c_i N_i \) is almost copositive with \( f \).

(ii) For any \( n > C(Y_s) \), there exists a polynomial \( P \in P_n \) that is almost copositive with \( f \) and satisfies

\[
\|f - P\|_p \leq C(s) \omega^2_p(f, n^{-1})_p. \tag{5.6}
\]

Remark. Although we require \( n > C(Y_s) \) in (ii) for simplicity, it seems unnecessary. In many cases of constrained approximation how large \( n \) is depends on \( Y_s \) because two \( y_j \)'s may be very close to each other, and the degree of a polynomial will then have to be very large to follow the trend of the graph. In this paper, however, we relax the shape-preserving requirement in a neighborhood of each \( y_j \) of radius \( 2n(y_j) \). When some points in \( Y_s \) get too close to one another, these neighborhoods will be connected and we will not have to worry about the sign changes at these points. In other words, the set \( Y_s \) can be “thinned out” if its points are dense (or, equivalently, if \( n \) is small).

Proof. For (i), we only prove that \( S \) is copositive with \( f \) between \( J_0^+ \) and \( J_0^- \), which is \( A_0 := [-1 + n^{-2}, y_1 - A_d(y_1)] \), if the two intervals do not intersect. For the sake of certainty, we assume \( f \) is nonnegative on \( A_0 \).

Since there are at least \( r \) knots in \( J_0^+ \), if we denote the last knot less than \( y_1 - 2n(y_1) \) by \( t_{j_1} \), then the restriction of \( S \) on \( A_0 \) can be written as \( S|_{A_0} = \sum_{i=0}^{r} c_i N_i \). From the facts that \( c_i \) is the integral average of \( f \) on \( I_i \subset [t_i, t_{i+1}] \), and that there are at least \( r \) knots in each of \( J_0^+ \) and \( J_0^- \), we know \( c_i \geq 0, i = 0, ..., j_1 \). Thus \( S|_{A_0} \geq 0 \).

To prove (ii), we take \( r = 3 \) in Lemma 3 and choose \( k \geq C_1n \) large enough so that the knot sequence \( T_{k-2r} := \{ x_i = \cos(\pi/k) \}_{i \in \mathbb{N}} \) has at least 4 knots in each of \( J^- \), \( j = 1, ..., s+1 \), and \( J^+ \), \( j = 0, ..., s \), if they do not intersect \( J_{j-1}^+ \) or \( J_{j+1}^- \), respectively. We add auxiliary knots to \( T_{k-2r} \) as before Lemma 3, and define a quadratic spline on \( T_{k-2r} \) by \( S := Tf \). By Lemma 3 and Theorem A,

\[
\|S - P\|_p \leq C_p \sum_{i=1}^{k-1} E_2(S, \hat{I}_i \cup \hat{I}_{i+1})_p \leq C_p \sum_{i=1}^{k-1} \omega^2(f |\hat{I}_i \cup \hat{I}_{i+1}|, \hat{I}_i \cup \hat{I}_{i+1})_p \leq C_p \omega^2_p(f, n^{-1})_p,
\]
where in the last step we have used an inequality established and/or used in [3, 4, 9, 16, 10]. The fact that $P$ is almost copositive with $f$ follows from (i) and Theorem A.

In the following four theorems and corollary, we show that (5.3), (5.4), and (5.6) are exact for $1 < p < \infty$ in the sense that one cannot replace $\alpha^2$ or $\omega_p^2$ by $\omega^3$ in them (Corollary 6); that weakly almost copositive approximation does not do better than these (Corollary 6), and strongly almost copositive approximation does not do better than the ordinary copositive approximation (Theorems 8 and 9), in spite of larger intervals in which the restriction is relaxed.

**Theorem 5.** Let $Y_s$ be fixed. For any given $A > 0$, $1 \leq p < \infty$, and sufficiently large $n \in \mathbb{N}$, there exists a function $f \in C[-1, 1] \cap A^p(Y_s)$ such that for every polynomial $P_n \in \mathbb{P}_n$, which is copositive with $f$ on $[y_s + (1 - y_s)/3, 1 - (1 - y_s)/3]$, the following inequality holds,

$$
\|f - P_n\|_p > A n^\beta \omega_m(f, n^{-1})_p,
$$

where $m = 3$ and $\beta < (p - 1)/(2p + 1)$ if $1 < p < \infty$, and $m = 4$ and $\beta < 1/3$ if $p = 1$.

**Corollary 6.** Let $Y_s$ be fixed. For any given $0 \leq \varepsilon < 1$, $A > 0$, $1 \leq p < \infty$, and sufficiently large $n \in \mathbb{N}$, there exists $f \in C[-1, 1] \cap A^p(Y_s)$ such that

$$
E_n^{(\varepsilon)}(f, \varepsilon \text{-alm } Y_s)_p > A \begin{cases} \omega^3(f, n^{-1})_p, & \text{if } 1 < p < \infty \\ \omega^4(f, n^{-1})_p, & \text{if } p = 1. \end{cases}
$$

To prove Theorem 5, we need the following inequality for polynomials. We were aware of its usage in Zhou [23] through communication with him, but could not find a handy reference. The following is a modification of the proof Professor Zhou outlined to the authors.

**Lemma 7.** Let $P_n \in \mathbb{P}_n$ and $1 \leq p < \infty$. Then

$$
|P_n(x) \sqrt{1 - x^2}| \leq C_p n^{1/p} \|P_n\|_p, \quad x \in [-1, 1].
$$

**Proof.** Let $x := \cos \theta$ and $t_n(\theta) := P_n(\cos \theta) \sin \theta$. Since $t_n$ is a trigonometric function of degree $n + 1$, applying the Nikolskii’s inequality, we have

$$
|t_n(\theta)| \leq C_p n^{1/p} \|t_n\|_{L_p(0, 2\pi)}.
$$
Notice that
\[
\|t_n\|_{L^1[0,2\pi]} = \int_0^{2\pi} |t_n(\theta)|^p \, d\theta = 2 \int_0^{\pi} |P_n(\cos \theta)|^p (\sin \theta)^p \, d\theta
\]
\[
\leq 2 \int_0^{\pi} |P_n(\cos \theta)|^p \sin \theta \, d\theta = 2 \int_{-1}^{1} |P_n(x)|^p \, dx = 2 \|P_n\|_p.
\]

Therefore, we obtain
\[
|P_n(x)\sqrt{1-x^2}| = |t_n(\theta)| \leq C_p n^{1/p} \|P_n\|_p.
\]

**Proof of Theorem 5.** The following is a modification of the proof used by Gilewicz and Shevchuk [7]. Let \( n \geq s + 2, \) \( x_0 := (1 + y_s)/2, \)
\[
L(x) := (x - x_0 + b)(x - x_0 - b) \prod_{j=1}^{s} (x - y_j),
\]
where \( b < (1 - y_s)/6 \) is a constant to be chosen later, and let
\[
f(x) := \begin{cases} L(x), & \text{if } x \notin [x_0 - b, x_0 + b] \\ 0, & \text{otherwise}. \end{cases}
\]

Suppose (5.7) is not true, i.e., there exists a polynomial \( P_n \in \mathbb{P}_n \) such that \( P_n(x) \geq 0, \) \( x \in [y_s + (1 - y_s)/3, 1 - (1 - y_s)/3] \) (therefore, \( P_n(x) \geq 0 \) for \( x \in [x_0 - b, x_0 + b] \)), and
\[
\|f - P_n\|_p \leq A n^{\beta} \omega^m(f, n^{-1})_p.
\]

Without loss of generality, we can assume \( \beta \geq 0. \) Note that
\[
\|f - L\|_p = \left( \int_{x_0 - b}^{x_0 + b} |L(x)|^p \, dx \right)^{1/p} \leq C b^{2+1/p}
\]
and
\[
\omega^m(f, n^{-1})_p \leq \omega^m(f - L, n^{-1})_p + \omega^m(L, n^{-1})_p
\]
\[
\leq 2^m \|f - L\|_p + n^{-m} \|L^{(m)}\|_p \leq C b^{2+1/p} + C n^{-m}.
\]

Also, by Lemma 7, we have
\[
\|P_n - L\|_p
\]
\[
\geq C n^{-1}\|P_n(x_0) - L(x_0)\| \sqrt{1-x_0^2}
\]
\[
\geq -C(Y_s) n^{-1/p} L(x_0) = C(Y_s) n^{-1/p} b^2 \prod_{j=1}^{s} (x_0 - y_j) \geq C(Y_s) n^{-1/p} b^2.
\]
Therefore,
\[
C(Y_s) n^{1/p} b^2 \leq \|P_n - L\|_p \leq \|P_n - f\|_p + \|f - L\|_p \\
\leq A n^{\beta} \omega^m(f, n^{-1})_p + C b^{2 + 1/p} n^{\beta^2 + 1/p} + C n^{-m + \beta}.
\]
This implies the inequality
\[
b^{2}(C(Y_s) n^{-1/p} - C n^{\beta} b^{1/p}) \leq C n^{-m + \beta}.
\]
Now, let \( b = cn^{-1/\beta} \) with sufficiently small \( c \), then the last inequality implies
\[
n^{m - 2 - 2\beta - 1/p} \leq C(p, Y_s).
\]
But this cannot be true for sufficiently large \( n \) since the condition on \( m, \beta \), and \( p \) in the theorem imply \( m > 2 + 2\beta + \beta + 1/p \).

**Theorem 8.** For any given \( A > 0, \varepsilon < 0, 1 \leq p < \infty \), and sufficiently large \( n \in \mathcal{N} \), there exists \( f \in L_p([-1, 1]) \) that changes sign only once at \( x = 0 \) such that for every polynomial \( P_n \in \mathcal{P}_n \) with \( P_n(2n^{-1 + \varepsilon}) \geq 0 \) the following inequality holds:
\[
\|f - P_n\|_p > A \omega^2(f, 1)_p.
\]

**Proof.** Here once again we omit details of the proof since the argument is similar to our previous counterexamples. Let \( L(x) := n/a(x - a/n) \), and let
\[
f(x) :=
\begin{cases}
L(x), & \text{if } x \notin [0, a/n] \\
0, & \text{otherwise}.
\end{cases}
\]
Then \( f \in L_p([-1, 1] \cap A^0(Y_1) \), where \( Y_1 = \{0\} \). By choosing the value of the parameter \( a > 2n^\varepsilon \) carefully one can readily prove that \( P_n(2n^{-1 + \varepsilon}) < 0 \) for any polynomial \( P_n \in \mathcal{P}_n \) with \( \|f - P_n\|_p \leq A \omega^2(f, 1)_p \) provided \( n \) is sufficiently large.

**Theorem 9.** Let \( Y_s, s \geq 1 \), be fixed. For any given \( \varepsilon < 0, A > 0, 1 \leq p < \infty \), and sufficiently large \( n \in \mathcal{N} \), there exists a function \( f \in L_p([-1, 1] \cap A^0(Y_s)) \) such that for every polynomial \( P_n \in \mathcal{P}_n \) with \( P_n(y_s + 2A_y n^\varepsilon) \geq 0 \) the following inequality holds,
\[
\|f - P_n\|_p > A \omega^m(f, n^{-1})_p, \quad (5.8)
\]
where \( m = 2 \) if \( 1 < p < \infty \), and \( m = 3 \) if \( p = 1 \). In particular, (5.8) holds for all polynomials that are strongly almost copositive with \( f \).
Remark. As in Theorem 5, the inequality (5.8) can be improved to
\[ \|f - P_n\|_p > A\beta \omega_m(f, n^{-1}) \] 
for some \( \beta > 0 \).

Proof. We use the same idea as in Theorem 5. Let \( A > 0 \) be fixed, and let \( n \geq 2^{-1/\beta} \) and \( b \) be chosen later. Denote
\[ L(x) := (x - y_s - bA_n(y_s)) \prod_{j=1}^{s-1} (x - y_j) \]
and
\[ f(x) := \begin{cases} L(x), & \text{if } x \notin (y_s - bA_n(y_s), y_s + bA_n(y_s)), \\ 0, & \text{otherwise}. \end{cases} \]
Suppose that the assertion of the theorem is not true, i.e., there exists a polynomial \( P_n \in \mathcal{P}_n \) such that \( P_n(x) \geq 0 \), where \( x = y_s + 2A_n(y_s) n^s \), and
\[ \|f - P_n\|_p \leq A\beta \omega_m(f, n^{-1})_p. \]
Note that
\[ \|f - L\|_p \leq \|L\|_{\mathcal{P}_n}(y_s - bA_n(y_s), y_s + bA_n(y_s)) \leq C(bA_n(y_s))^{1+1/p} \leq C(Y_s)(b/n)^{1+1/p}, \]
and
\[ \omega_m(f, n^{-1})_p \leq \omega_m(f - L, n^{-1})_p + \omega_m(L, n^{-1})_p \]
\[ \leq 2\|f - L\|_p + n^{-m} \|L\|_p \leq C(Y_s)(b/n)^{1+1/p} + Cn^{-m}. \]
Now, by Lemma 7, we have
\[ P_n(x) - L(x) \]
\[ \leq Cn^{1/p}\sqrt{1 - \beta^2}^{-1} \|P_n - L\|_p \]
\[ \leq C(Y_s) n^{1/p}(\|P_n - f\|_p + \|f - L\|_p) \leq C(Y_s) n^{1/p}(b/n)^{1+1/p} + n^{-m} \]
\[ \leq C(Y_s)(b^{1+1/p}n^{-1} + n^{-m} + 1/p). \]
Since
\[ L(x) = -A_n(y_s)(b - 2n^s) \prod_{j=1}^{s-1} (y_s - y_j + 2A_n(y_s) n^s) \]
\[ \leq - C(Y_s) n^{-1}(b - 2n^s), \]
we have
\[ P_d(x) \leq -C(Y) n^{-1}(b - 2n') + C(Y)(b'^{1+1/p}n^{-1} + n^{-m+1/p}) < 0 \]
if \( b > 4n', b'^{1+1/p} > n^{-m+1+1/p} \) and \( b < C'p \). This choice of \( b \) is possible if \( n \) is sufficiently large, and \( m > 1+1/p \), which is true with the choices of \( m \) and \( p \) in the theorem. This is the desired contradiction.

So far we have not mentioned anything about almost copositive approximation for \( p = \infty \). The theorem below says it reaches the same rate as the unconstrained case. The result follows from Theorem 13, their analogue for almost intertwining approximation.

**Theorem 10.** Let \( f \in C[-1, 1] \cap \mathcal{A}(Y) \), and \( m \) be a positive integer. Then
\[ E_n^0(f, \text{alm} Y_s) \leq C\omega_n(f, n^{-1}). \]
Moreover, there exists a \( P_n \in P_n \cap (\text{alm} \mathcal{A}_n(Y)) \) such that
\[ |f(x) - P_n(x)| \leq C\omega_n(f, A_n(x)). \]

The last theorem of the section shows that \( \gamma \)-modulus of any order \( m > 0 \) can be used for \( 1 < p < \infty \). This is consistent with the previous theorem since \( \gamma_n(f, t) = \omega_n(f, t) \), \( \forall t > 0 \). This theorem follows from its analogue for almost intertwining approximation (Theorem 14) again.

**Theorem 11.** Let \( f \in L_p[-1, 1] \cap \mathcal{A}(Y) \), \( 1 < p < \infty \), and \( m \) be a positive integer. Then
\[ E_n^0(f, \text{alm} Y_s) \leq C\gamma^m(f, n^{-1})_p. \]
If \( f \) also belongs to \( \mathcal{W}_p[-1, 1] \), then
\[ E_n^0(f, \text{alm} Y_s) \leq Cn^{-1}\omega^m(f', n^{-1})_p. \]

6. WEAK INTERTWINING APPROXIMATION

6.1. Almost Intertwining Approximation

We first prove the following result for almost intertwining spline approximation. We remind the reader that \( [y_j, y_j + A_d(y_j)] \).
**Theorem 12.** Let \( f \in L^p_{[-1, 1]} \), \( m > 0 \) and \( Y \) be given. Let \( T_k := \{ t_i \}_i \) be a single-knot sequence, with auxiliary knots added as before Lemma 3, that has at least \( 2(m - 1)^2 + 1 \) knots in the interior of each of \( J_j^- \) and \( J_j^+ \), \( j = 1, \ldots, s \), if they do not intersect \( J_{j-1}^- \) or \( J_{j+1}^- \), respectively. Then there exists an almost intertwining pair of splines \( \{ \tilde{S}, S \} \) of order \( m \) on the knot sequence \( T_k \) such that for \( i = 0, \ldots, k - 1 \),

\[
\| \tilde{S} - f \|_{L^p(I_i)} + \| S - f \|_{L^p(I_i)} \leq C \left( m, |I_i|, \max |J_{i+1}|, \max |J_i| \right),
\]

where \( I_i := [t_i, t_{i+1}] \), \( \mathcal{J}_i \) is an interval such that \( I_i \subset \mathcal{J}_i \subset [t_i - 6(m - 1)^2, t_i + 6(m - 1)^2] \), and \( C \) is a constant depending on \( m \) and the maximum ratios of lengths of neighboring subintervals \( I_i \) and \( I_{i+1} \).

**Proof.** Since the theorem can be easily proved from results of one-sided approximation and Beatson’s blending lemma [1, Lemma 3.2] by somehow standard techniques, (see, for example, Lemma 3, Theorem 4 and [8-13]), we only sketch the proof. We first construct overlapping local polynomials of degree \( m - 1 \) by using one-sided approximations. The adjacent local polynomials are then blended by Beatson’s Lemma. The error estimate is similar to that of Theorem 4.

From this and Theorem A, we can prove the following two theorems.

**Theorem 13.** Let \( f \in C[-1, 1], m > 0 \), and \( Y \) be given. Then

\[
\tilde{E}_d(f, \lambda M Y) \leq C \left( m, |I|, \max |J_{i+1}|, \max |J_i| \right).
\]

Moreover, there exists an almost intertwining pair of polynomials \( \{ P_n, Q_n \} \) such that

\[
|P_n(x) - f(x)| + |Q_n(x) - f(x)| \leq C \left( m, \lambda M Y \right).
\]

**Proof.** Let \( r = 2m + 1 \) and \( T_{k-2r} := \{ x_k = \cos(\pi k) \}_{k \in \mathbb{Z}} \), where \( n \leq k \leq C \left( Y, r \right) \) such that \( T_{k-2r} \) satisfies the hypothesis in Theorem 12. Then, from Theorem 12, there exists an almost intertwining pair of splines \( \{ S, \tilde{S} \} \) of order \( r \) on the knot sequence \( T_{k-2r} \) such that

\[
\| \tilde{S} - f \|_{C(I_i)} + \| S - f \|_{C(I_i)} \leq C \left( m, |I_i|, \max |J_{i+1}|, \max |J_i| \right).
\]

Moreover, we have

\[
E_{r-1}(\tilde{S}, I_i \cup I_{i+1}) \leq E_{r-1}(\tilde{S} - f, I_i \cup I_{i+1}) + E_{r-1}(f, I_i \cup I_{i+1}) \leq \| \tilde{S} - f \|_{C(I_i \cup I_{i+1})} + C \left( m, |I_i \cup I_{i+1}|, \max |J_{i+1}|, \max |J_i| \right)
\]

\[
\leq C \left( m, |I_i|, \max |J_{i+1}|, \max |J_i| \right),
\]
and, similarly,
\[ E_{r-1}(S, I_i \cup I_{i+1})_\infty \leq C \omega(f, |\mathcal{J}_i|, |\mathcal{J}_j|). \]

Now, by applying Theorem A to \( \tilde{S} \) and \( S \), respectively, we obtain intertwining pairs of polynomials \( \{\tilde{P}_1, \tilde{P}_2\} \) and \( \{P_1, P_2\} \) such that the estimate (5.2) holds for \( |\tilde{P}_1(x) - \tilde{P}_2(x)| \) and \( |P_1(x) - P_2(x)| \). Let \( x \in I_i \) and \( \psi_i(x) := |I_i|/(|x-x_i| + |I_i|) \). Since \( |\mathcal{J}_i| = |I_i| \sim A_d(x) \) and \( \sum \psi_i^2(x) < \infty \), it follows that
\[
|\tilde{P}_1(x) - \tilde{P}_2(x)| \leq C(r, s, s) (|I_i|/(|x-x_i| + |I_i|))^{s} \\
\leq C(r, s, s) \sum_{j=1}^{k-1} \omega(f, |\mathcal{J}_j|, |\mathcal{J}_j|)(\psi_j(x))^{s}\]
implies
\[ |\tilde{P}_1(x) - \tilde{P}_2(x)| \leq C(r, s, s) \omega^m(f, n^{-1})\]
and
\[ |\tilde{P}_1(x) - \tilde{P}_2(x)| \leq C(r, s, s) \omega^m(f, A_d(x))_\infty. \]

Similar inequalities hold for \( |P_1(x) - P_2(x)| \), and therefore, also for \( |\tilde{P}_1(x) - f(x)| \) and \( |f(x) - P_2(x)| \). It is easy to see \( \{\tilde{P}_1, \tilde{P}_2\} \) is the desired almost intertwining pair of polynomials for \( f \).

The proof of the following theorem is similar and thus will be omitted.

**Theorem 14.** Let \( f \in L_p[-1, 1], \ 1 \leq p < \infty, \ m > 0, \) and \( Y_s \) be given. Then
\[ E_d(f, \text{alm} Y_s)_p \leq C \omega^m(f, n^{-1})_p. \]
If \( f \) also belong to \( W^m_p[-1, 1] \), then
\[ E_d(f, \text{alm} Y_s)_p \leq C n^{-1} \omega^m(f, n^{-1})_p. \]

6.2. Nearly Intertwining Approximation

In the rest of the paper, we show that the rate of nearly intertwining approximation can not be expressed in terms of \( \tau \)- and \( \omega \)-moduli of \( f \), nor in terms of \( \|f\|_p \), even if \( f \) is infinitely continuously differentiable (Theorem 15), which is no improvement over the ordinary intertwining approximation. This is because, probably, we still require \( P - f \) and \( f - Q \) change sign simultaneously at each \( \tilde{y}_j \). If the first derivative of \( f \) is used in the bound, however, it has the optimal rate as unconstrained approximation (Theorem 17).
**Theorem 15.** For every $n \in \mathbb{N}$, $0 < p \leq \infty$, and $A > 0$, there exists a function $f \in C^\infty([-1, 1])$ such that for every $y \in [-1, 1]$ and every pair of polynomials $P$ and $Q$ \( P = f \in \mathcal{A}(\mathcal{Y}_1) \) and $f - Q \in \mathcal{A}(\mathcal{Y}_1)$, where $\mathcal{Y}_1 := \{ y \}$, the following inequality holds:

$$
\| P - Q \|_p > A \max\{ \| f \|_p, \| f, 1 \|_p \}.
$$

(6.1)

**Proof.** Let $n \in \mathbb{N}$, $0 < p \leq \infty$, and $A > 0$ be fixed, and define

$$
f(x) := \sin(bx),
$$

where $b$ is a large positive number to be chosen later. Suppose that $y \in [0, 1]$ (if $y \in [-1, 0)$, then considerations are similar). If $y \in [2k(\pi/b), (2k + 1) \pi/b]$ for some $k \in \mathbb{Z}$, we have $P(y) = 0$ and $P((2k - 1/2) \pi/b) = 1$, and, therefore, $\exists \xi \in ((2k - 1/2) \pi/b, y)$ such that

$$
|P'(|\xi|)| > \frac{2b}{3\pi}.
$$

Similarly, if $y \in ((2k + 1) \pi/b, (2k + 2) \pi/b]$, then $Q(y) = 0$ and $Q((2k + 1/2) \pi/b) = 1$, and, thus, $\exists \xi \in ((2k + 1/2) \pi/b, y)$ such that

$$
|Q'(|\xi|)| > \frac{2b}{3\pi}.
$$

This implies that

$$
\| P \|_\infty + \| Q \|_\infty \geq \frac{2b}{3\pi}.
$$

Suppose now that $P$ and $Q$ satisfy the inequality

$$
\| P - Q \|_p \leq A \max\{ \| f \|_p, \| f, 1 \|_p \}.
$$

Taking into account that $\| f \|_p \leq 2^{1/p}$ and $\| f, 1 \|_p \leq 2^{1+1/p}$ we have the following estimates

$$
\frac{2b}{3\pi} \leq \| P' \|_\infty + \| Q' \|_\infty \leq n^2(\| P \|_\infty + \| Q \|_\infty)
$$

$$
\leq M(\| P \|_p + \| Q \|_p) \leq M(\| P - f \|_p + \| Q - f \|_p + 2 \| f \|_p)
$$

$$
\leq 2M(\| P - Q \|_p + \| f \|_p) \leq 2^{1+1/p}M(A + 1),
$$

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where the constant $M$ depends on $n$ and $p$ but not on $b$. Choosing $b > 3 \cdot 2^{1+1/2p}M(A+1)$ gives the desired contradiction.

The key to the proof of Theorem 17 is the lemma below. The proof of the theorem itself is then somehow standard (see the proofs of Theorems 4 and 12) and thus will be omitted.

**Lemma 16.** Let $f \in W^1_1[-1, 1]$, and let $I := [a, b]$ and $\bar{I} \subseteq I$ be two sub-intervals of $[-1, 1]$. Then there exist $y_1 \in \bar{I}$ and two polynomials $p_1$ and $p_2$ of degree $m \geq 1$ such that $\{p_1, p_2\}$ is an intertwining pair for $f$ on $I$ with respect to $Y_1 := \{y_1\}$ that satisfies

$$\|p_1 - p_2\|_{L^p(I)} \leq C \omega^m(f, \|I\|, I)_p,$$

where $C$ depends on $m$ and the ratio $\|I\|/|\bar{I}|$.

**Proof.** It is well-known that for any integrable function $g$ on $(-\infty, \infty)$,

$$|\{x: Mg(x) > t\}| \leq ct^{-1} \int_{-\infty}^{\infty} |g|, \quad t > 0,$$

where $Mg(x) := \sup_{J \ni x} |J|^{-1} \int_J |g|$ is the Hardy–Littlewood maximal operator. Let

$$F(x) := \begin{cases} f(a), & x < a \\ f(x), & x \in [a, b] \\ f(b), & x > b, \end{cases}$$

and $t = 2c |\bar{I}|^{-1} \int_{\bar{I}} |f'|$, and apply the maximal operator to $F'$, we have

$$|\{x: M(F')(x) > t\}| \leq ct^{-1} \int_{-\infty}^{\infty} |F'| = ct^{-1} \int_{\bar{I}} |F'| = ct^{-1} \int_{\bar{I}} |f'| = \frac{1}{2} |\bar{I}|.$$

Thus there exists $y_1 \in \bar{I}$ such that $M(F')(y_1) \leq t$. Let $l_1(x) = f(y_1) + t(x - y_1)$ and $l_2(x) = f(y_1) - t(x - y_1)$, then they form a intertwining pair of $f$ on $I$ with respect to $Y_1 = \{y_1\}$. This is because we have, for $l_1(x), y_1 \leq x \leq b$, for example,

$$l_1(x) = f(y_1) + t(x - y_1) \geq f(y_1) + (x - y_1) M(F')(y_1) \geq f(y_1) - \int_{y_1}^{x} |F'| = f(y_1) + \int_{y_1}^{x} |f'| \geq f(x).$$
For an error estimate of \( l_1 \) and \( l_2 \), we first note for \( x \in I \),
\[
|l_1(x) - l_2(x)| = 2r |x - y_1| = 4e |\bar{T}|^{-1} |x - y_1| \int_{\bar{T}} |f'| \leq C \int_{\bar{T}} |f'|
\]
\[
\leq C |I|^{1/q} \|f'\|_{L_p(I)},
\]
where \( 1/p + 1/q = 1 \). It follows that
\[
\|l_1 - l_2\|_{L_p(I)} \leq C |I| \|f'\|_{L_p(I)}.
\] (6.3)

Let \( P \) be a best polynomial approximation to \( f \) on \( I \) of degree \( m - 1 \), and \( P := \int_{a}^{b} P(t) \ dt \). To prove (6.2), we apply (6.3) to \( f - P \), then \( \{l_1, l_2\} \) is an intertwining pair for \( f - P \). Define \( p_i := l_i + P, i = 1, 2 \). Obviously \( \{p_1, p_2\} \) is an intertwining pair of polynomials of degree \( m \) for \( f \) on \( I \) with
\[
\|p_1 - p_2\|_{L_p(I)} = \|l_1 - l_2\|_{L_p(I)} \leq C |I| \|f' - P'\|_{L_p(I)} \leq C |I| \omega^m(f^*, |I|, I)_p,
\]
where Whitney's Theorem has been used.

**Theorem 17.** Suppose \( f \in W^k_{[L_1, L_2]} \), \( k > 0 \) and \( n > m > 0 \).

(i) If \( T_k \) is a single-knot sequence, with auxiliary knots added as before Lemma 3, that has at least \( 2(r - 1)^2 + 1 \) knots in each of \( (y_j, y_j) \), \( j = 2, ..., s \), then there exists a nearly intertwining pair \( \{S_1, S_2\} \) of splines of order \( m + 1 \) on \( T_k \) for \( f \) with respect to \( Y_s \) satisfying
\[
\|S_1 - S_2\|_p \leq C \omega_m(f^*, \Delta T_k)_p,
\] (6.4)
where \( C \) depends on \( m \) and on ratios \( \Delta_t / \Delta_{t+1} \) of lengths of neighboring subintervals \( I_t \) and \( I_{t+1} \). It also depends on ratios \( \Delta_t / \Delta_s(y_j) \) if \( y_j \in I_t \) and \( \Delta_s(y_j) \ll \Delta_t = |I_t| \).

(ii) There exists a nearly intertwining pair \( \{P_1, P_2\} \) of polynomials of degree \( C_n \) for \( f \) with respect to \( Y_s \) satisfying
\[
\|P_1 - P_2\|_p \leq C n^{-1} \omega_m(f^*, n^{-1})_p,
\]
where \( C \) depends only on \( m \) while \( C \) depends on \( m \) and \( Y_s \).

**REFERENCES**


