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Gorensteinness of Invariant Subrings of Quantum Algebras

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We prove Auslander–Gorenstein and GKdim–Macaulay properties for certain invariant subrings of some quantum algebras, the Weyl algebras, and the universal enveloping algebras of finite-dimensional Lie algebras. © 1999 Academic Press

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0. INTRODUCTION

Given a noncommutative algebra it is generally difficult to determine its homological properties such as global dimension and injective dimension. In this paper we use the noncommutative version of Watanabe theorem proved in [JoZ, 3.3] to give some simple sufficient conditions for certain classes of invariant rings having some good homological properties.

Let k be a base field. Vector spaces, algebras, etc. are over k. Suppose G is a finite group of automorphisms of an algebra A. Then the *invariant*

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subring is defined to be

$$A^G = \{x \in A \mid g(x) = x \quad \text{for all } g \in G\}.$$

Let A be a filtered ring with a filtration $\{F_i \mid i \geq 0\}$ such that $F_0 = k$. The associated graded ring is defined to be Gr $A = \bigoplus_n F_n/F_{n-1}$. A *filtered (or graded) automorphism* of A (or Gr A) is an automorphism preserving the filtration (or the grading).

Let $\{p_{ij} | 1 \leq i < j \leq n\}$ be a set of nonzero elements in k. The skew polynomial algebra $k_{p_{ij}}[x_1, \ldots, x_n]$ is generated by $\{x_1, \ldots, x_n\}$ with relations $x_i x_j = p_{ij} x_j x_i$ for all $i < j$. We always think $k_{p_{ij}}[x_1, \ldots, x_n]$ as a connected graded ring with the degree of x_i not necessarily 1. We identify the graded vector space $V = \bigoplus_{i=1}^n kx_i$ with the quotient space $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal graded ideal of $k_{p_{ij}}[x_1, \ldots, x_n]$. Any filtered or graded automorphism g induces a linear automorphism of V, and $\det g|_V$ denotes the usual determinant of the linear map $g: V \to V$.

The following is a noncommutative version of [Ben, 4.6.2].

THEOREM 0.1. *Suppose A is a filtered ring such that the associated graded ring* Gr A *is isomorphic to a skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$, where $p_{ij} \neq p_{kl}^{\pm 1}$ for all $(i, j) \neq (k, l)$. Let G be a finite group of filtered automorphisms of A with $|G| \neq 0$ in k. If* $\det g|_{(\oplus_{i=1}^n kx_i)} = 1$ *for all $g \in G$, then A^G is Auslander–Gorenstein and* GKdim–*Macaulay.*

The definitions of Auslander–Gorenstein and GKdim–Macaulay are given in Definition 0.6. The condition on p_{ij} in Theorem 0.1 is needed as shown by Example 2.9. The hypothesis on $\det g$ is necessary, as shown by Example 3.6.

A general version of Theorem 0.1 holds when we replace $\det g$ by the homological determinant [Theorems 3.3 and 3.5]. The notion of homological determinant was introduced in [JoZ, 2.3] (see Section 2).

We can extend the action of g from V to the exterior algebra $\Lambda(V)$. Therefore g acts on the tensor product $A \otimes \Lambda(V)$ if A is as in Theorem 0.1. The following is a noncommutative version of [Ben, 5.3.2].

THEOREM 0.2. *Suppose A is a filtered ring such that the associated graded ring* Gr A *is isomorphic to the skew polynomial ring $k_{p_{ij}}[x_1, \ldots, x_n]$, where $p_{ij} \neq p_{kl}^{\pm 1}$ for $(i, j) \neq (k, l)$. Let G be a finite group of filtered automorphisms of A with $|G| \neq 0$ in k. Then $(A \otimes \Lambda(V))^G$ is Auslander–Gorenstein and* GKdim–*Macaulay where $V = \bigoplus_{i=1}^n kx_i$.*

In general, $(A \otimes \Lambda(V))^G$ is not prime and hence is not Auslander-regular.

Theorems 0.1 and 0.2 also hold if Gr A is the 4-dimensional Sklyanin algebra (by Example 2.2 and Theorem 3.5) or the multiparameter quantum matrix algebra (by Example 2.6 and Theorem 3.5). Note that the 4-dimensional Sklyanin algebra and the multiparameter quantum matrix algebra satisfy the SSC of [Zh, p. 398] (by [Zh, 2.3(2) and 2.4]) and hence satisfy the hypothesis $\operatorname{Kdim} M = \operatorname{GKdim} M$ in Theorem 3.5(3) (by [Zh, 3.1]).

The Weyl algebras and the universal enveloping algebras of finite-dimensional Lie algebras have standard filtrations, and their associated graded rings are commutative polynomial rings. We prove the following theorem by using similar ideas.

THEOREM 0.3. *Let A_n be the nth Weyl algebra with the standard filtration. Let G be a finite group of filtered automorphisms of A_n. If $|G| \neq 0$ in k, then A_n^G is Auslander–Gorenstein and* GKdim–*Macaulay. Moreover if* char $k = 0$, *then A_n^G is Auslander-regular.*

A special case of Theorem 0.3 was proved by Levasseur, see [L1, 3.2]. If char $k = 0$, it is well known that A_n^G has finite global dimension because A_n is simple (see Remark 4.2). A quantum version of Theorem 0.3 is the following. We say that q is *generic with respect to* $\{p_{ij}\}$ if q^n ($n > 0$) are not in the the multiplicative subgroup generated by $\{p_{ij}\}$.

THEOREM 0.4. *Let $A_n(q, p_{ij})$ be the nth quantum Weyl algebra defined in* [GZ, (2.3)] *and let G be a finite group of filtered automorphisms of $A_n(q, p_{ij})$. Suppose that q is generic with respect to $\{p_{ij}\}$. If $|G| \neq 0$ in k, then $A_n(q, p_{ij})^G$ is Auslander–Gorenstein and* GKdim–*Macaulay.*

THEOREM 0.5. *Let k be an algebraically closed field of characteristic zero, let L be a finite-dimensional Lie algebra over k, and let $U(L)$ be the universal enveloping algebra of L. Let G be a finite group of filtered automorphisms of $U(L)$. Then $U(L)^G$ is Auslander–Gorenstein and* GKdim–*Macaulay if either of the two following conditions holds*:

(1) *Elements of G act as inner automorphisms of the Lie algebra L.*

(2) *L is one of the following simple Lie algebras*:

  (a) A_n, *for* $n = 4m$ *or* $n = 4m + 1$.

  (b) B_n.

  (c) C_n.

  (d) E_6, E_7, E_8, F_4, G_2.

Theorem 0.5(1) was essentially known by Kraft and Small [KS, Proposition 4(5)] (see Remark 6.7).

DEFINITION 0.6. Let A be a noetherian ring.

(1) The *grade* of an A-module M is defined to be

$$j(M) = \min\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}$$

or ∞ if no such i exists.

(2) We say A is *Auslander–Gorenstein* if the following two conditions hold:

(a) (Gorenstein property) A has finite left and right injective dimension.

(b) (Auslander property) For every noetherian A-module M and for all $i \geq 0$ and all submodule $N \subset \mathrm{Ext}^i_A(M, A)$, $j(N) \geq i$.

We say A is *Auslander-regular* if A is Auslander–Gorenstein and A has finite global dimension.

(3) We say A is GKdim–*Macaulay* (where GKdim denotes the Gelfand–Kirillov dimension) if

$$\mathrm{GKdim}\, M + j(M) = \mathrm{GKdim}\, A < \infty$$

for every noetherian A-module M.

Note that GKdim–Macaulay is called Cohen–Macaulay by some authors. Auslander–Gorenstein and GKdim–Macaulay properties are considered as nice homological properties. Such rings have been studied by several researchers since the 1980s (see [ASZ], [B1], [B2], [BE], [Ek], [L1], [L2], and so on). The skew polynomial rings, the Weyl algebras, and the universal enveloping algebras are Auslander–regular and GKdim–Macaulay. Using the above results one can construct more examples of Auslander–Gorenstein and GKdim–Macaulay rings.

1. PRELIMINARY

An old and basic tool for filtered rings is to use their Rees rings and the associated graded rings. In this section we recall some definitions and properties about filtered rings and graded rings.

A *filtered algebra* is an algebra A with a filtration $F = \{F_i \mid i \geq 0\}$ satisfying the conditions (F0) $F_0 = k$, (F1) $F_i \subset F_j$ if $i \geq j$, (F2) $F_iF_j \subset F_{i+j}$, and (F3) $A = \bigcup_i F_i$. Given a filtered algebra A, the Rees ring is defined to be

$$\mathrm{Rees}\, A = \bigoplus_n F_n t^n.$$

The elements $t(=1t)$ and $t-1$ are central elements of Rees A and

$$\text{Rees } A/(t-1) = A, \qquad \text{and} \qquad \text{Rees } A/(t) = \text{Gr } A.$$

Both Rees A and Gr A are *connected graded* in the sense that the degree zero part is k. By [L2, 3.5], Rees A is noetherian if and only if Gr A is noetherian. We call a filtered ring A *filtered noetherian* if Gr A is noetherian.

We will adopt the basic notations about graded rings and graded modules from [JiZ], [JoZ], and [YZ]. For example, the opposite ring of A is denoted by A^{op}.

Let A be a connected graded ring. The maximal graded ideal $A_{\geq 1}$ of A is denoted by $\mathfrak{m}$, and the trivial module $A/\mathfrak{m}$ is denoted by k. The graded Ext-group is denoted by $\underline{\text{Ext}}$.

DEFINITION 1.1. Let A be a noetherian connected graded ring.

(1) A is called *AS–Gorenstein* (here AS stands for Artin–Schelter) if

(a) A is Gorenstein with injective dimension d, and

(b) there is an integer l such that

$$\underline{\text{Ext}}^i_A(k, A) = \underline{\text{Ext}}^i_{A^{op}}(k, A) = \begin{cases} 0 & \text{for } i \neq d \\ k(l) & \text{for } i = d, \end{cases}$$

where $k(l)$ is the lth degree shift of the trivial module k.

(2) A is called *AS-regular* if it is AS–Gorenstein and has finite global dimension.

Condition 1.1(1b) follows from 1.1(1a) if A has enough normal elements [Zh, 0.2]. For example, the commutative polynomial rings are AS-regular. By [L2, 6.3], a noetherian, connected graded, Auslander–Gorenstein ring is AS–Gorenstein.

Given a connected graded ring A and any graded right A-module M, we define the local cohomology of M to be

$$H^i_{\mathfrak{m}}(M) = \varinjlim_n \underline{\text{Ext}}^i_A(A/A_{\geq n}, M).$$

If A is AS–Gorenstein, then $H^i_{\mathfrak{m}}(A)$ is 0 if $i \neq d$ and is equal to $A'(l)$ if $i = d$. Here $(-)'$ is the graded vector space dual of $-$.

The Auslander property is also defined for non-Gorenstein ring if the ring has a (rigid or balanced) dualizing complex [YZ, 2.1]. In the case where Gr A is AS–Gorenstein, A has the Auslander property [Definition 0.6(2b)] is equivalent to A has an Auslander, rigid dualizing complex [YZ, 2.1 and 3.1]. We collect some results below.

THEOREM 1.2. (1) (*Rees' lemma*) *If A is a noetherian connected graded ring with a regular normal element x of positive degree, then A is AS–Gorenstein if and only if $A/(x)$ is.*

(2) *If A is as in* (1) *and x is of degree* 1; *then A is AS-regular if and only if $A/(x)$ is.*

(3) *If A is a noetherian filtered algebra, then* Rees A *is AS-regular* (*respectively AS–Gorenstein*) *if and only if* Gr A *is.*

(4) [L2, 3.6] *Let A be a filtered noetherian algebra. Then* Rees A *is Auslander–Gorenstein* (*or Auslander-regular*) *if and only if* Gr A *is.*

(5) [Ek, 0.1; L2, 3.6, 5.8, *and* 5.10]*Let A be a filtered noetherian algebra. If* Gr A *is Auslander–Gorenstein* (*respectively* GKdim–*Macaulay*) *then so is A.*

We make the following definition.

DEFINITION 1.3. Suppose P is a property such that Rees A has P if and only if Gr A has P. We say a filtered algebra A has *filtered P* if Rees A (or Gr A) has P.

2. HOMOLOGICAL DETERMINANT

First we recall some definitions in the graded case from [JiZ] and [JoZ]. Let $g \in \mathrm{GrAut}(A)$, the group of graded algebra automorphisms of A. For any A-module M we define the g-twisted module M^g such that $M^g = M$ as a vector space and the action is

$$m * a = mg(a)$$

for all $m \in M$ and $a \in A$. Let f be a k-linear homomorphism from A-module M to A-module N. We say f is g-linear if

$$f(ma) = f(m)g(a)$$

for all $m \in M$ and $a \in A$. Then f is a g-linear map if and only if f is an A-module homomorphism from M to N^g. Therefore g-linear maps can be extended to injective resolutions (or projective resolutions). Moreover we can apply the local cohomology functor $H^*_{\mathfrak{m}}$ to g-linear maps. Now let A be a graded AS–Gorenstein ring with injective dimension d. By [JoZ, 2.2 and 2.3], $g\colon A \to A$ induces a g-linear map $H^d_{\mathfrak{m}}(g)\colon A'(l) \to A'(l)$, where l is the integer in Definition 1.1(1b), $'$ is the graded vector space dual, and $H^d_{\mathfrak{m}}(g) = c(g^{-1})'$. The constant c^{-1} is called the *homological determinant* of g, and we denote this by $\mathrm{hdet}_A\, g = c^{-1}$. By [JoZ, 2.5], hdet_A defines a group homomorphism $\mathrm{GrAut}(A) \to k^{\times}$.

The *trace* of g on A is defined to be

$$Tr_A(g, t) = \sum_{n\geq 0} \operatorname{tr}(g|_{A_n})t^n.$$

See [JiZ] for more details. The Hilbert series of A is the trace of the identity map, namely,

$$H_A(t) = Tr_A(id, t) = \sum_{n\geq 0} \dim A_n\, t^n.$$

By [JoZ, 2.6], if A is AS–Gorenstein and g is rational in the sense of of [JoZ, 1.3] then

$$Tr_A(g, t) = (-1)^d(\operatorname{hdet}_A g)^{-1}t^{-l} + \text{lower terms},$$

when we expand the trace function as a Laurent series in t^{-1}. Here d and l are given in Definition 1.1(1b). The rationality of g is automatic for AS-regular algebras [JoZ, 4.2].

LEMMA 2.1. (1) *If A is the commutative polynomial ring $k[V]$ and g is a graded automorphism of $k[V]$, then* $\operatorname{hdet}_A g = \det g|_V$, *where* det *is the usual determinant of a k-linear map.*

(2) *If A is the exterior algebra $\Lambda(V)$ and g is a graded automorphism of A, then* $\operatorname{hdet}_A g = (\det g|_V)^{-1}$.

Proof. (1) It follows by [Ben, 2.5.1] that $Tr_A(g, t) = (\det\,(1 - g|_V t))^{-1}$. (Note that the action of g in [Ben] is defined via g^{-1} (see [Ben, p. 1], so we change g^{-1} in [Ben, 2.5.1] to g.) Expanding $(\det\,(1 - gt)|_V)^{-1}$ as a Laurent series in t^{-1} we have

$$(\det(1 - gt)|_V)^{-1} = (-1)^n(\det g|_V)^{-1}t^{-n} + \text{lower terms}.$$

Therefore the result follows by [JoZ, 2.6].

(2) It is easy to see that $Tr_A(g, t) = \det\,(1 + g|_V t) = (\det\, g|_V)t^n +$ lower terms [Ben, 5.2.1]. Therefore the result follows by [JoZ, 2.6]. ∎

Lemma 2.1(2) shows that the homological determinant of g may not be equal to the determinant of $g|_V$ (also see Examples 2.8 and 2.9). Lemma 2.1(1) shows that the homological determinant of g is equal to the determinant of $g|_V$ when A is the commutative polynomial ring. The next example shows that Lemma 2.1(1) also holds for the 4-dimensional Sklyanin algebra.

EXAMPLE 2.2. Let k be the field $\mathbb{C}$ of complex numbers. Then the automorphisms of the 4-dimensional Sklyanin algebra S are classified in [SS, Sect. 2] and the generators of the automorphism group are also listed there. So one can check $\det g|_V$ easily, where $V = S_1$ is the degree 1 part of S.

By [JoZ, 2.6], the homological determinant of g can be computed by the trace of g. The trace of generators of the automorphism group is listed in [JiZ, Sect. 4]. Using these facts one can check that $\operatorname{hdet}_S g = \det g|_V$ for all $g \in \operatorname{GrAut}(S)$.

We now extend the definition of the homological determinant to the filtered case. Let $\operatorname{FilAut}(A)$ be the group of automorphisms of A preserving the given filtration F. Hence for any $g \in \operatorname{FilAut}(A)$ one can extend g to a graded automorphism of Rees A by sending t to t. Also g induces a graded automorphism of Gr A. For simplicity we still use g for both these graded automorphisms. The next lemma is clear.

LEMMA 2.3. *Let B be a connected graded ring. Then $B =$ Rees A for some filtered algebra A if and only if there is a regular central element in B_1.*

The next proposition is useful for computing the homological determinant when A has normal elements.

PROPOSITION 2.4. *Let B be a noetherian graded AS–Gorenstein ring and let x be a regular normal element of positive degree. Suppose g is a graded automorphism of B such that $g(x) = \lambda x$. Then $\operatorname{hdet}_B g = \lambda \operatorname{hdet}_A g$ where $A = B/(x)$.*

Proof. Suppose d is the injective dimension of B. Then $d-1$ is the injective dimension of A. Let $l_x : B(-s) \to B$ be the left multiplication by x where $s = \deg x$. Applying $H^*_{\mathfrak{m}}$ to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B(-s) & \xrightarrow{l_x} & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & B(-s) & \xrightarrow{l_x} & B & \longrightarrow & A & \longrightarrow & 0,
\end{array}$$

we obtain that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(A) & \longrightarrow & H^{d}_{\mathfrak{m}}(B)(-s) & \longrightarrow & H^{d}_{\mathfrak{m}}(B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\operatorname{hdet}_A g)^{-1}(g^{-1})'} & & \downarrow{\scriptstyle \lambda(\operatorname{hdet}_B g)^{-1}(g^{-1})'} & & \downarrow & & \\
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(A) & \longrightarrow & H^{d}_{\mathfrak{m}}(B)(-s) & \longrightarrow & H^{d}_{\mathfrak{m}}(B) & \longrightarrow & 0
\end{array}$$

The result follows by comparing the first two vertical maps in the highest nonzero homogeneous component. ∎

Let A be a filtered noetherian AS–Gorenstein ring. If g is a filtered automorphism of A, then $g(t) = t$ for $t \in \text{Rees } A$ and, by Proposition 2.4, $\text{hdet}_{\text{Gr } A} g = \text{hdet}_{\text{Rees } A} g$. We define the *homological determinant* of g on A to be

$$\text{hdet}_A g = \text{hdet}_{\text{Gr } A} g.$$

In the rest of this section we compute some homological determinants.

LEMMA 2.5. *Let A be a filtered algebra with* $\text{Gr } A = k_{p_{ij}}[x_1, \ldots, x_n]$, *where $p_{ij} \neq p_{kl}^{\pm 1}$ for all $(i, j) \neq (k, l)$. Let g be a filtered automorphism of A. Then* $\text{hdet}_A g = \det g|_V$ *where $V = \bigoplus_{i=1}^n kx_i$.*

Proof. First we assume that A is $\mathbb{N}$-graded and $\deg x_i = 1$ for all i.

Under the hypothesis on p_{ij} it is routine to check that if $\sum_{i=1}^n a_i x_i$ is a normal element then it must be $a_i x_i$ for some i. Since $g(x_i)$ is normal, we can write $g(x_i) = b_i x_{\sigma(i)}$ for some permutation $\sigma \in S_n$ and $b_i \neq 0$. Applying g to the relations $x_i x_j = p_{ij} x_j x_i$ we see that $p_{ij} = p_{\sigma(i), \sigma(j)}$. Therefore σ is the identity by the hypothesis.

Now we consider the general case. By definition of hdet we can pass to the graded case. Let g be a graded automorphism of $A = k_{p_{ij}}[x_1, \ldots, x_n]$ with

$$\deg x_1 \geq \cdots \geq \deg x_n \geq 1.$$

Let w be the minimal index such that $\deg x_w = \deg x_n$. Then g preserves $\bigoplus_{i=w}^n kx_i$. As an automorphism of $k_{p_{ij}}[x_w, \ldots, x_n]$, $g(x_n) = b_n x_n$ for some $b_n \neq 0$ by the previous argument. By induction, $g(x_i) = b_i x_i + f_i(x_{i+1}, \ldots, x_n)$ for some polynomial f_i. By Proposition 2.4, $\text{hdet}_A g = b_n \text{hdet}_{A/(x_n)} g$. Therefore $\text{hdet}_A g = \prod_i b_i = \det g|_V$. ∎

EXAMPLE 2.6. Let $M_{q, p_{ij}}(n)$ be the multiparameter quantum matrix algebra. It is a connected graded algebra generated by $\{x_{ij} \mid i, j = 1, \ldots, n\}$ with $\binom{n^2}{2}$ relations (see [AST] for a list of relations), where the structure constants $\{q, p_{ij}\}$ are generic. Similar to the proof of Lemma 2.5 with more careful computations we check that $\text{hdet}_{M_{q, p_{ij}}(n)} g = \det g|_V$, where $V = \bigoplus_{i, j=1}^n kx_{ij}$. Details are left to interested readers.

PROPOSITION 2.7. *Let A and B be two filtered AS–Gorenstein rings and assume that $A \otimes B$ is filtered noetherian. If g (respectively h) is a filtered automorphism of A (respectively B), then* $\text{hdet}_{(A \otimes B)}(g \otimes h) = (\text{hdet}_A g)(\text{hdet}_B h)$.

Proof. Let $\{F_i \mid i \geq 0\}$ (respectively $\{W_i \mid i \geq 0\}$) be the filtration of A (respectively B). The filtration of $A \otimes B$ is defined by $Z_n = \bigcup_{i+j=n} F_i \otimes W_j$. Then $\text{Gr } (A \otimes B) = (\text{Gr } A) \otimes (\text{Gr } B)$. By the definition of hdet in the filtered case, we may work with the associated graded rings, or equivalently we may assume g and h are graded automorphisms of graded AS–Gorenstein rings A and B, respectively. Since systems $\{(A \otimes B)_{\geq n}\}$ and $\{A_{\geq n} \otimes B + A \otimes B_{\geq n}\}$ are cofinal, we have

$$H^n_{\mathfrak{m}_{A\otimes B}}(M_A \otimes N_B) = \bigoplus_{i+j=n} H^i_{\mathfrak{m}_A}(M_A) \otimes H^j_{\mathfrak{m}_B}(N_B).$$

In particular, if E_A and F_B are graded injective modules over A and B, respectively, then $E_A \otimes F_B$ is $H^*_{\mathfrak{m}_{A\otimes B}}$-acyclic. Let E^* and F^* be injective resolutions of A and B, respectively. We can compute $H^*_{\mathfrak{m}}(A \otimes B)$ by the resolution $E^* \otimes F^*$. Since g gives a g-linear map of E and h gives an h-linear of F, $g \otimes h$ gives a $g \otimes h$-linear map of $E^* \otimes F^*$. Therefore

$$H^{d+e}_{\mathfrak{m}_{A\otimes B}}(g \otimes h) = H^d_{\mathfrak{m}_A}(g) \otimes H^e_{\mathfrak{m}_B}(h),$$

where $d = \text{injdim } A$ and $e = \text{injdim } B$. By the definition of the homological determinant in the graded case, $\text{hdet}_{A\otimes B}(g \otimes h) = (\text{hdet}_A g)(\text{hdet}_B h)$. ∎

Finally we give two examples where $\text{hdet}_A g$, is not equal to $\det g|V$, where V is the minimal generating space.

EXAMPLE 2.8. Let A be the commutative graded ring $k[x_1, x_2]/(x_1^3)$. Let g be the graded automorphism of A sending x_i to $-x_i$ for $i = 1, 2$. Then $V = kx_1 \oplus kx_2$ is the minimal generating space and $\det g|V = 1$. Also $g^2 = id$. Since $g(x_1^3) = -x_1^3$, by Proposition 2.4, $\text{hdet}_A g = (-1)\text{hdet}_{k[x_1, x_2]} g = -1$. Hence $\text{hdet}_A g$ is not equal to $\det g|_V$ or $(\det g|_V)^{-1}$. It is also easy to check that A^G is not Gorenstein with $G = \{g, id\}$.

EXAMPLE 2.9. Let A be the skew polynomial ring generated by x, y, z with relations $xy = -yx,\ xz = zx,\ yz = zy$. Let g be the graded automorphism of A such that $g(x) = -y,\ g(y) = -x,\ g(z) = -z$. Then $g^2 = 1$ and $\det g|_V = 1$. The Koszul dual is $A^! = k\langle x, y, z\rangle/I$, where I is generated by $x^2 = y^2 = z^2 = 0,\ xy = yx,\ yz = -zy,\ xz = -zx$. The induced automorphism g^τ of $A^!$ is given by $x \to -y,\ y \to -x,\ z \to -z$. It follows from [JiZ, 3.4] that

$$Tr_A(g, t) = \frac{1}{Tr_{A^!}(q^\tau, -t)} = \frac{1}{1 + t + t^2 + t^3}.$$

Therefore $\text{hdet}_A g = -1$ [JoZ, 2.6]. Let $G = \{1, g\}$. Then the Hilbert series of A^G is

$$H_{A^G}(t) = \tfrac{1}{2}(Tr_A(1, t) + Tr_A(g, t)) \neq \pm t^m H_{A^G}(t^{-1})$$

for any $m \in \mathbb{Z}$. So A^G is not Auslander–Gorenstein [Ei, Ex. 21.17(c)]. This gives another example of $\text{hdet} g \neq \det g|_V$ and shows that Theorem 0.1 fails without the extra hypothesis on p_{ij}.

3. INVARIANT SUBRING OF FILTERED RINGS

In this section we prove Theorems 0.1 and 0.2. The following lemma is clear.

LEMMA 3.1. *Let A be a filtered ring and let G be a subgroup of* FilAut(A). *Then*

(1) Rees $A^G = (\text{Rees } A)^G$.

(2) *If* $|G| \neq 0$ *in* k, *then* Gr $A^G = (\text{Gr } A)^G$.

If $|G| = 0$ in k, then Lemma 3.1(2) may fail.

EXAMPLE 3.2. Let k be a field of characteristic 2 and let g be a filtered automorphism of the polynomial ring $k[x]$ determined by $g(x) = x + 1$. Hence $g^2 = id$. Let $G = \{id, g\}$. Consider $k[x]$ as a filtered ring and G as a subgroup of filtered automorphisms. Then the induced graded automorphism of g is the identity on $k[x] = \text{Gr } k[x]$. Hence $k[x^2] = \text{Gr } A^G \subsetneq (\text{Gr } A)^G = k[x]$.

We now recall results from [JoZ] and [YZ].

THEOREM 3.3. *Let A be a noetherian connected graded ring and let G be a finite subgroup of* GrAut(A) *with* $|G| \neq 0$ *in* k. *Suppose that* $\text{hdet}_A g = 1$ *for all* $g \in G$.

(1) [JoZ, 3.3] *If A is AS–Gorenstein, then so is A^G.*

(2) [YZ, 4.20] *If A is Auslander–Gorenstein, then so is A^G.*

(3) [YZ, 5.13, *and* 5.14] *If A^G is AS–Gorenstein and A^G either is PI or has enough normal elements, then A^G is Auslander–Gorenstein and* GKdim–*Macaulay.*

We now prove a slightly more general version of Theorem 3.3(3) in Theorem 3.5. Let Kdim denote the Krull dimension.

LEMMA 3.4. *Let $B \subset A$ be two noetherian connected graded rings such that A is finitely generated over B on both sides and B is a direct summand of A as graded B-bimodules. Suppose* GKdim M = Kdim $M < \infty$ *for all finitely generated graded A-module M. Then* GKdim N = Kdim N = Kdim $N \otimes_B A$ *for all finitely generated graded B-module N.*

If G is a finite subgroup of GrAut(A) with $|G| \neq 0$ in k, then $B := A^G \subset A$ is a direct summand of A. If moreover A is noetherian, then A is finitely generated over B on both sides [Mo, 2.1.1 and 1.12.1].

Proof. Since B is a direct summand of A as B-bimodule, N_B is a direct summand of $N \otimes_B A_B$. Hence $\operatorname{GKdim} N \leq \operatorname{GKdim} N \otimes_B A_B$ and $\operatorname{Kdim} N \leq \operatorname{Kdim} N \otimes_B A_B$.

By [MR 8.3.14(iii)], $\operatorname{GKdim} N \geq \operatorname{GKdim} N \otimes_B A_A$ and $\operatorname{GKdim} M_A \geq \operatorname{GKdim} M_B$ for any A-module M. Hence

$$\operatorname{GKdim} N \otimes_B A_A = \operatorname{GKdim} N \otimes_B A_B = \operatorname{GKdim} N.$$

It remains to show that $\operatorname{Kdim} N = \operatorname{Kdim} N \otimes_B A$. For every submodule $L \subset N$, let ϕ be the map $L \otimes_B A \to N \otimes_B A$. Then $L \to \phi(L \otimes_B A)$ is an order-preserving map from the lattice of B-submodules of N to the lattice of A-submodules of $N \otimes_B A$. Since $A = B \oplus C$ for some C as B-bimodule, this map preserves proper containment. Therefore

$$\operatorname{Kdim} N \leq \operatorname{Kdim} N \otimes_B A.$$

We now prove $\operatorname{Kdim} N \geq \operatorname{Kdim} N \otimes_B A (= \operatorname{GKdim} N)$ by induction on $\operatorname{Kdim} N$. By noetherian induction we may assume N is Kdim-critical. Let $s = \operatorname{Kdim} N \otimes_B A = \operatorname{GKdim} N < \infty$. Pick an A-submodule $M \subset N \otimes_B A$ such that $(N \otimes_B A)/M$ is critical of Krull (and GK) dimension $s - 1$ and let $\bar{N}$ be the image of the composition

$$N \to N \otimes_B A \to (N \otimes_B A)/M.$$

Then

$$\operatorname{GKdim} \bar{N} \leq \operatorname{GKdim} \bar{N} A_B \leq \operatorname{GKdim} \bar{N} \otimes_B A_B = \operatorname{GKdim} \bar{N}.$$

This implies the first $=$ of

$$\begin{aligned} \operatorname{GKdim} \bar{N} A_A \geq \operatorname{GKdim} \bar{N} A_B = \operatorname{GKdim} \bar{N} = \operatorname{GKdim} \bar{N} \otimes_B A_A \\ \geq \operatorname{GKdim} \bar{N} A_A. \end{aligned}$$

∎

Therefore

$$\operatorname{GKdim} \bar{N} = \operatorname{GKdim} \bar{N} A_A = \operatorname{GKdim} (N \otimes_B A)/M = s - 1.$$

Hence $\bar{N}$ is a proper quotient of N, and by the induction hypothesis $\operatorname{Kdim} \bar{N} \geq \operatorname{GKdim} \bar{N} = s - 1$. Since N is critical, $\operatorname{Kdim} N \geq \operatorname{Kdim} \bar{N} + 1 \geq s$, as required. ∎

By [Zh, 2.3, 2.4, and 3.1], if A has enough normal elements or if A is the 4-dimensional Sklyanin algebra (or more generally if A satisfies SSC) then

$$\operatorname{Kdim} M = \operatorname{GKdim} M$$

for all finitely generated graded A-module M.

THEOREM 3.5. *Let A be a noetherian filtered ring and let G be a finite subgroup of* $\operatorname{FilAut}(A)$. *Suppose that $|G| \neq 0$ in k and that* $\operatorname{hdet}_A g = 1$ *for all $g \in G$.*

(1) *If A is filtered AS–Gorenstein, then so is A^G.*

(2) *If A is filtered Auslander–Gorenstein, then so is A^G.*

(3) *Suppose* $\operatorname{GKdim} M = \operatorname{Kdim} M$ *for all finitely generated graded* Gr *A-module M. If A is filtered Auslander–Gorenstein and* GKdim–*Macaulay, then so is A^G.*

Proof. (1) By definition we only need to show Gr A^G is AS–Gorenstein. Now the result follows by Theorem 3.3(1).

(2) It suffices to show Gr A^G is Auslander–Gorenstein, which follows by Theorem 3.3(2).

(3) We can pass to the graded case and assume that A is graded. By (2) A^G is Auslander–Gorenstein. It remains to show the GKdim–Macaulay property. Let Cdim be the canonical dimension defined in [YZ, 2.9]. Since A is GKdim–Macaulay, $\operatorname{GKdim} M = \operatorname{Cdim} M$ for all finitely generated graded A-module M. Let B denote A^G. For every finitely generated graded B-module N, it follows from Lemma 3.4 that

$$\operatorname{Kdim} N = \operatorname{GKdim} N = \operatorname{GKdim} N \otimes_B A_A.$$

By [YZ, 4.14], $\operatorname{Kdim} N \leq \operatorname{Cdim} N$, and it is clear that

$$\operatorname{Cdim} N \leq \operatorname{Cdim} N \otimes_B A_B = \operatorname{Cdim} N \otimes_B A_A = \operatorname{GKdim} N \otimes_B A_A,$$

where the first $=$ is [YZ, 4.17(3)]. Combining these (in)equalities we obtain $\operatorname{Kdim} N = \operatorname{GKdim} N = \operatorname{Cdim} N$. Therefore A^G is graded GKdim–Macaulay. By [L2, 5.8], A^G is (ungraded) GKdim–Macaulay as required. ∎

The following example is well known; it shows that the hypothesis $\operatorname{hdet}_A g = 1$ cannot be removed from Theorem 3.5.

EXAMPLE 3.6. Let A be the skew polynomial ring $\mathbb{C}_{p_{ij}}[x, y, z]$, where $p_{ij} \neq 0 \in \mathbb{C}$, and g is a graded automorphism sending $x \to -x$, $y \to -y$, and $z \to -z$. Then $g^2 = id$ and $\det g = -1$. By Proposition 2.4, $\operatorname{hdet} g = -1$, and it is easy to check that $Tr(g, t) = 1/(1+t)^3$. Let $G = \{id, g\}$. Then the Hilbert series of A^G is

$$H_{A^G}(t) = \frac{1}{2}\left(\frac{1}{(1-t)^3} + \frac{1}{(1+t)^3}\right) = \frac{1+3t^2}{(1-t^2)^3}.$$

So $H_{A^G}(t^{-1}) \neq \pm t^m H_{A^G}(t)$ for any $m \in \mathbb{Z}$. Therefore A^G is not Gorenstein by [Ei, Exercise 21.17(c)].

To apply Theorem 3.5, one needs to show hdet$g = 1$ for all $g \in G$. To show that hdet$g = 1$ it suffices to show that the only group homomorphism $G \to k^\times := k - \{0\}$ is trivial. Here are some easy cases.

LEMMA 3.7. [JoZ, 3.4]. *The homomorphism $G \to k^\times$ is trivial if one of the following holds*:

(a) $G = [G, G]$.

(b) $|G|$ *is odd and* $k = \mathbb{Q}$.

(c) $k = \mathbb{Z}/2\mathbb{Z}$.

One can easily get some corollaries by combining Lemma 3.7 and Theorem 3.5. We now prove Theorems 0.1 and 0.2.

Proof of Theorem 0.1 By [Zh, 2.3(2) and 3.1], GKdim M = Kdim M holds for all finitely generated graded $k_{p_{ij}}[x_1, \ldots, x_n]$-module M. So the result follows from Lemma 2.5 and Theorem 3.5(3). ∎

Proof of Theorem 0.2 Again by [Zh, 2.3(2) and 3.1], GKdim M = Kdim M holds for all finitely generated graded $k_{p_{ij}}[x_1, \ldots, x_n] \otimes \Lambda(V)$-module M.

Since A is a filtered algebra, $A \otimes \Lambda(V)$ is a filtered algebra with the filtration induced by the filtration of A and $\mathrm{Gr}\,(A \otimes \Lambda(V)) = (\mathrm{Gr}\,A) \otimes \Lambda(V) = k_{p_{ij}}[x_1, \ldots, x_n] \otimes \Lambda(V)$. By [Zh, 0.2] and Theorem 1.2(5), $A \otimes \Lambda(V)$ is Auslander–Gorenstein and GKdim–Macaulay. Furthermore, $A \otimes \Lambda(V)$ and $(A \otimes \Lambda(V))^G$ are filtered noetherian.

By Proposition 2.7 and Lemmas 2.5 and 2.1(2),

$$\mathrm{hdet}_{A\otimes\Lambda(V)}\, g = \mathrm{hdet}_A\, g \cdot \mathrm{hdet}_{\Lambda(V)}\, g = \det g|_V \cdot (\det g|_V)^{-1} = 1$$

for all $g \in G$. Hence the result follows from Theorem 3.5(3). ∎

The following is an immediate consequence of Theorems 0.1–0.2 and [YZ, 6.23].

COROLLARY 3.8. *Let A^G be as in Theorem* 0.1, $(A \otimes \Lambda(V))^G$ *as in Theorem* 0.2. *Then they have quasi-Frobenius rings of fractions*.

Another method for checking whether A^G is Gorenstein is to use a noncommutative version of Stanley's theorem [JoZ, Sect. 6]. For example, if Gr A (or Rees A) is Auslander-regular and char $k = 0$, then A^G is Auslander–Gorenstein if the Hilbert series of Gr A^G satisfies

$$H_{\mathrm{Gr}\,A^G}(t^{-1}) = \pm t^m H_{\mathrm{Gr}\,A^G}(t)$$

for some $m \in \mathbb{Z}$ (see [JoZ, 6.2 and 6.4]).

4. INVARIANT SUBRINGS OF THE WEYL ALGEBRAS

Let A_n be the nth Weyl algebra and G a finite group of automorphisms of A_n. If char $k = 0$, then it is well-known that A_n^G has global dimension n for any G. We start with some basic facts below.

THEOREM 4.1. *Let* char $k = 0$ *and let G be a finite group of automorphisms of the nth Weyl algebra A_n. Then A_n^G is a noetherian simple domain of global dimension and Krull dimension n and Gelfand–Kirillov dimension* $2n$.

Proof. By [Mo, 1.12.1], A_n^G is noetherian. It is clear that A_n^G is a domain.

Recall that an automorphism g is inner if there is a unit $u \in A$ such that $g(x) = uxu^{-1}$. Since units of A_n are elements of $k^\times$, every automorphism of A_n is outer (namely, not inner). By [Mo, 2.6] and [Mo, 2.4], A_n^G is simple and A_n is a finitely generated projective A_n^G-module. Hence the standard spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_{A_n}(\operatorname{Tor}_q^{A_n^G}(N, A_n), M) \Rightarrow \operatorname{Ext}^{p+q}_{A_n^G}(N, M)$$

collapses to the isomorphisms

$$\operatorname{Ext}^p_{A_n}(N \otimes_{A_n^G} A_n, M) = \operatorname{Ext}^p_{A_n^G}(N, M)$$

for all right A_n-module M and right A_n^G-module N. Therefore $\operatorname{Ext}^p_{A_n^G} \cdot (N, M) = 0$ for all $p > n$. For any right A_n^G-module M_0, let $M = M_0 \otimes_{A_n^G} A_n$. Since A_n^G is an (A_n^G, A_n^G)-bimodule direct summand of A_n [Mo, 2.1.1], M_0 is a direct summand of A_n^G-module M. Thus $\operatorname{Ext}^p_{A_n^G}(N, M_0) = 0$ for all right A_n^G-modules N and M_0. This implies that $\operatorname{gldim} A_n^G \le n$. Pick a simple A_n-module N with $\operatorname{GKdim} n$, then $\operatorname{GKdim}_{A_n} N \otimes_{A_n^G} A_n = n$ [MR, 8.3.14(iii)], thus $\operatorname{Ext}^n_{A_n}(N \otimes_{A_n^G} A_n, A_n) \neq 0$. Consequently, $\operatorname{Ext}^n_{A_n^G}(N, A_n) \neq 0$ and $\operatorname{gldim} A_n^G \ge n$. So $\operatorname{gldim} A_n^G = n$.

For the GK-dimension we have

$$2n = \operatorname{GKdim} A_n = \operatorname{GKdim} (A_n^G \otimes_{A_n^G} A_n) \le \operatorname{GKdim} A_n^G \le \operatorname{GKdim} A_n,$$

where the first $\le$ is [MR, 8.3.14(iii)]. Therefore $\operatorname{GKdim} A_n^G = 2n$.

Since A_n is projective over A_n^G and contains A_n^G as a direct summand, A_n is faithfully flat over A_n^G. Thus Kdim $A_n^G \le$ Kdim A_n. For any group action one can form a skew group ring $A_n * G$ which is free (hence faithfully flat) over A_n. Thus Kdim $A_n \le$ Kdim $A_n * G$. By [Mo, 2.5], A_n^G and $A_n * G$ are Morita equivalent. Therefore Kdim $A_n^G =$ Kdim $A_n * G =$ Kdim $A_n = n$. ∎

Remark 4.2. The above proof shows the following known result: If A is noetherian and simple with finite global dimension and G is a finite group of (outer) automorphisms of A with $|G| \neq 0$ in k, then

(a) A^G is noetherian and simple.
(b) gldim $A^G =$ gldim A.
(c) Kdim $A^G =$ Kdim A.
(d) GKdim $A^G =$ GKdim A.

It is not clear from the above proof that A_n^G satisfies Auslander and GKdim–Macaulay properties (which we believe to be true). Next we are going to show for a certain automorphism group G (namely for a filtered automorphism group) that A_n^G satisfies the Auslander and GKdim–Macaulay properties without any restriction on char k.

Let A_n be generated by $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ subject to the relations

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [y_i, x_j] = \delta_{ij}, \qquad \text{for all } i, j.$$

This is a filtered algebra with the standard filtration determined by $F_1 = k + \sum_i kx_i + \sum_j ky_j$. The associated graded ring is the commutative polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

Let V_n be the vector space generated by these $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. We say a linear map $\sigma: V_n \to V_n$ is a $[-, -]$-map if

$$[\sigma(x_i), \sigma(x_j)] = [\sigma(y_i), \sigma(y_j)] = 0, \quad [\sigma(y_i), \sigma(x_j)] = \delta_{ij}, \qquad \text{for all } i, j.$$

We call another basis $\{x'_1, \ldots, x'_n, y'_1, \ldots, y'_n\}$ of V_n a $[-, -]$-basis if the map $\sigma: x_i \to x'_i,\ y_i \to y'_i$ is a $[-, -]$-map.

LEMMA 4.3. *If g is a filtered endomorphism of A_n, then there is a $[-, -]$-map $\sigma: V_n \to V_n$ and a linear map $\epsilon: V_n \to k$ such that $g|_{V_n} = \sigma + \epsilon$. Furthermore, if σ is an automorphism, then g is a filtered automorphism of A_n and* $\operatorname{hdet}_{A_n} g = \det \sigma|_{V_n}$.

Proof. Since g maps $F_1 = k \oplus V_n$ to F_1, the restriction $g|_{V_n}$ decomposes into two parts σ and ϵ. For every $a, b \in V_n$, $[a, b] \in k$, and hence

$$\begin{aligned} [a, b] = g([a, b]) &= [g(a), g(b)] = [\sigma(a) + \epsilon(a), \sigma(b) + \epsilon(b)] \\ &= [\sigma(a), \sigma(b)] \end{aligned}$$

because $\epsilon(a)$ commutes with all elements. Therefore σ is a $[-, -]$-map. If σ is an automorphism then g is an automorphism of F_1 and hence a filtered automorphism of A_n. By Lemma 2.1(1),

$$\operatorname{hdet}_{A_n} g = \operatorname{hdet}_{k[V_n]} g = \det g|_{V_n} = \det \sigma|_{V_n}.$$

∎

LEMMA 4.4. *If $\sigma: V_n \to V_n$ is a $[-, -]$-map, then* $\det\sigma = 1$. *As a consequence, σ is an automorphism.*

Proof. Replacing k by its algebraic closure will not change the determinant. So we may assume that k is algebraically closed. Let x be an eigenvector of σ. So $\sigma(x) = sx$ for some $s \in k$. Write $x = \sum a_i x_i + \sum b_i y_i$. We may assume some $a_i \neq 0$. We will make several base changes such that $x = x_n$. Apparently we will require that all base changes preserve the bracket relations. By changing x_i and then changing y_i properly (to keep the $[-,-]$), we have another $[-,-]$-basis so that $x = x_n + \sum b_i y_i$. By changing $\{y_1, \ldots, y_{n-1}\}$ we may assume $x = x_n + b_{n-1} + b_n y_n$. Changing basis within $\{x_n, y_n\}$, we have $x = x_n + b_{n-1} y_{n-1}$. Exchanging x_{n-1} and $-y_{n-1}$, we have $x = x_n + b_{n-1} x_{n-1}$, Finally we change x_i so that $x = x_n$. Since σ is a $[-,-]$-map,

$$1 = [y_n, x_n] = [\sigma(y_n), \sigma(x_n)] = [\sigma(y_n), s x_n].$$

Thus $s \neq 0$ and $\sigma(y_n) = s^{-1} y_n + \sum_i a_i x_i + \sum_{i<n} b_i y_i$. Let $T = \bigoplus_{i=1}^{n} k x_i + \bigoplus_{i<n} k y_i$. Then $T = \{x \in V_n \mid [x_n, x] = 0\}$. Since σ is a $[-,-]$-map, T is σ-invariant, namely, $\sigma(T) \subset T$. Write $T = V_{(n-1)} \oplus k y_n$. We may decompose the restriction $\sigma|_{V_{(n-1)}}$ into two linear maps $\sigma|_{V_{(n-1)}} = \theta + \eta$, where $\eta : V_{(n-1)} \to k y_n$ is a linear map and $\theta : V_{(n-1)} \to V_{(n-1)}$ is a $[-,-]$-map because y_n commutes with $V_{(n-1)}$. By induction $\det\theta = 1$, clearly

$$\det \sigma = s \times s^{-1} \times \det \theta = 1.$$

As a consequence σ is invertible. ∎

By Lemmas 4.3 and 4.4 the following is clear.

COROLLARY 4.5. *Every filtered endomorphism of the Weyl algebra A_n is an automorphism.*

THEOREM 4.6. *Let G be a finite group of filtered automorphisms of the Weyl algebra A_n. If $|G| \neq 0$ in k, then A_n^G is filtered Auslander–Gorenstein and* GKdim–*Macaulay.*

Proof. This is a consequence of Theorem 0.1 and Lemmas 4.3 and 4.4. ∎

Theorem 0.3 is an immediate consequence of Theorems 4.1 and 4.6.

For the first Weyl algebra A_1 over the complex numbers $\mathbb{C}$, it is known that every finite subgroup of $\mathrm{Aut}(A_1)$ is conjugate to a subgroup of $SL(2,\mathbb{C}) \subset \mathrm{FilAut}(A_1)$ (see [AHV]). A complete list of the finite subgroup of $SL(2,\mathbb{C})$ is also listed in [AHV]. Therefore we have the following.

COROLLARY 4.7. *Let A_1 be the first Weyl algebra over $\mathbb{C}$ and let G be a finite group of automorphisms of A_1. Then A_1 is Auslander-regular and* GKdim–*Macaulay.*

5. INVARIANT SUBRINGS OF QUANTUM WEYL ALGEBRAS

In this section we look at a family of quantum Weyl algebras studied in [GZ] and prove similar statements as in Section 4.

Let $\{q\} \cup \{p_{ij} \mid i, j = 1, \ldots, n\}$ be a set of nonzero elements in k with $p_{ii} = 1$ and $p_{ij} = p_{ji}^{-1}$. The quantum Weyl algebra $A_n(q, p_{ij})$ is generated by the elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the relations

$$\begin{aligned} x_i x_j &= p_{ij} q x_j x_i, && \text{for all } i < j \\ y_i y_j &= p_{ij} q^{-1} y_j y_i, && \text{for all } i < j \\ y_i x_j &= p_{ij}^{-1} q x_j y_i, && \text{for all } i \neq j \\ y_i x_i &= 1 + q^2 x_i y_i + (q^2 - 1) \sum_{j>i} x_j y_j, && \text{for all } i. \end{aligned} \tag{E1}$$

By [GZ, 3.11(1)], $A_n(q, p_{ij})$ is a noetherian, Auslander-regular, and GKdim–Macaulay domain of GK-dimension $2n$. It is easy to check that the associated graded ring has enough normal elements. Hence it follows from [Zh, 2.3(2) and 3.1] that the hypothesis $\operatorname{GKdim} M = \operatorname{Kdim} M$ in Theorem 3.5(3) is satisfied. If $q^2 = 1$ and char $k = 0$, then it is simple and has global dimension and Krull dimension n [FKK]. If $q^2 \neq 1$ or char $k \neq 0$, then A is not simple and has global dimension and Krull dimension $2n$ [GZ, 3.11(3)]. If $q^n \neq 1$ for all $n > 0$, then it is primitive [GZ, 3.2].

LEMMA 5.1. *Suppose that $q^4 \neq 1$, $qp_{ij} \neq 1$, $q^3 p_{ij} \neq 1$ for all $i \neq j$. Then*

(1) *Every filtered automorphism g of $A_n(q, p_{ij})$ has the form*

$$g(x_i) = \alpha_i x_i + \sum_{j>i} a_{ij} x_j + \sum_{j>i} b_{ij} y_j$$

$$g(y_i) = \alpha_i^{-1} y_i + \sum_{j>i} c_{ij} x_j + \sum_{j>i} d_{ij} y_j$$

for $\alpha_i \neq 0, a_{ij}, b_{ij}, c_{ij}, d_{ij} \in k$.

(2) $\det g|_{V_n} = 1$, *where* $V_n = \bigoplus_{i=1}^n kx_i \oplus \bigoplus_{i=1}^n ky_i$.

Proof. (2) It follows from (1) immediately.

(1) By [GZ, 1.5] and the relations above, the ordered monomials in $x_1, \ldots, x_n, y_1, \ldots, y_n$ are the basis of $A_n(q, p_{ij})$. Write

$$g(x_n) = \sum_j a_j x_j + \sum_j b_j y_j + t$$

$$g(y_n) = \sum_j c_j x_j + \sum_j d_j y_j + s.$$

Using the relations listed in (E1) and

$$g(y_n)g(x_n) = 1 + q^2 g(x_n)g(y_n) \tag{E2}$$

we obtain $a_i c_i = q^2 a_i c_i$ by comparing the coefficients in x_i^2 terms. If $a_{i_0} \neq 0$ for some i_0 then $c_{i_0} = 0$ since $q^2 \neq 1$. By comparing the coefficients of $x_{i_0} x_j$ terms for $j \neq i_0$, we obtain $c_j a_{i_0} p_{ji_0} q^{-1} = a_{i_0} c_j q^2$. Thus $c_j = 0$ since $q^3 p_{i_0 j} \neq 1$. Since g is an algebra automorphism $d_j \neq 0$ for some j. In a similar argument, by exchanging $x's$ and $y's$ we see that $b_j = 0$ for all j. Thus (E2) implies that $s = t = 0$. If there are $a_i \neq 0$ and $d_j \neq 0$ for $j \neq i$ then $a_i d_j = d_j a_i p_{ij} q$ by a comparison of the $x_i y_j$ terms. This contradicts with $p_{ij} q \neq 1$. Thus $a_j = d_j = 0$ for all $j \neq i_0$. Now if we compare (E2) with (E1) we see that $g(x_n) = a_n x_n$ and $g(y_n) = a_n^{-1} y_n$ and write $\alpha_n = a_n$. In the other case when $a_i = 0$ for i, we can show that this is impossible, which is similar to the above argument (the condition $q^4 \neq 1$ will be used in this case). Thus

$$g(x_n) = \alpha_n x_n, \qquad g(y_n) = \alpha_n^{-1} y_n.$$

Now we decompose the vector space V_n into $V_{n-1} \bigoplus (kx_n \oplus ky_n)$ and decompose $g|_{V_n}$ into

$$g|_{V_{n-1}} = g'|_{(V_{n-1} \to V_{n-1} \oplus k)} + \epsilon|_{(V_{n-1} \to kx_n \oplus ky_n)}.$$

It is not hard to see that g' can be extended to an algebra automorphism of $A_{n-1}(q, p_{ij})$. By induction g' has the required form, and the statement follows by induction. ∎

Proof of Theorem 0.4 If q is generic, then the hypothesis of Lemma 5.1 holds. The result follows from Theorem 3.5(3) and Lemma 5.1. ∎

Remark 5.2 Theorem 0.4 should also hold even if q is not generic and holds for other nice quantum Weyl algebras studied in [GZ].

6. INVARIANT SUBRINGS OF THE ENVELOPING ALGEBRAS

Let L be a finite-dimensional Lie algebra over k and let $U(L)$ be the universal enveloping algebra. There is a standard filtration given by $F_i = (k + L)^i$ with the associated graded ring isomorphic to the commutative polynomial ring $k[L]$.

LEMMA 6.1. *If g is a filtered automorphism of $U(L)$, there is a Lie algebra automorphism $\sigma: L \to L$ and a linear map $\epsilon: L \to k$ with $\epsilon([L, L]) = 0$ such that $g|_L = \sigma + \epsilon$. Also* $\operatorname{hdet}_{U(L)} g = \det \sigma|_L$.

Proof. Since g sends L to $L \oplus k$, g decomposes as $\sigma + \epsilon$. For every $a, b \in L$,

$$\sigma([a,b]) = g([a,b]) - \epsilon([a,b]) = [\sigma(a) + \epsilon(a), \sigma(b) + \epsilon(b)] - \epsilon([a,b])$$
$$= [\sigma(a), \sigma(b)] - \epsilon([a,b]).$$

Therefore σ is an endomorphism (and hence automorphism) of Lie algebra L and $\epsilon([L,L]) = 0$. By the definition and Lemma 2.1(1), $\operatorname{hdet}_{U(L)} g = \det \sigma|_L$. ∎

In the rest of this section we study the cases where the determinant is 1. We assume that char $k = 0$. The next lemma is elementary.

LEMMA 6.2. *Let ν be a nilpotent linear map of a finite-dimensional space V; then the linear map*

$$\exp(\nu) := \sum_{n=0}^{\infty} \frac{\nu^n}{n!}$$

is a linear map of V with determinant 1.

The above lemma is also true if the trace of ν is zero. An automorphism σ of the Lie algebra L is called *inner* if $\sigma = \exp(\operatorname{ad} x)$ for some $x \in L$ and ad x is nilpotent. The *inner automorphism group* of L, denoted by Int L, is generated by all inner automorphisms of L. By [Ja, Chapter IX], Int L is a normal subgroup of the automorphism group Aut(L) of L.

COROLLARY 6.3. *Let L be a Lie algebra over k with* char $k = 0$.

(1) *If $g|_L \in$* Int L, *then* $\operatorname{hdet}_{U(L)} g = 1$.

(2) *If G is a finite subgroup of* Int L, *then $U(L)^G$ is Auslander–Gorenstein and* GKdim–*Macaulay.*

Proof. (1) A consequence of Lemmas 6.1 and 6.2.

(2) A consequence of (1) and Theorem 3.5(3). ∎

The following classification of automorphism groups of simple Lie algebra is standard (see [Ja], [Hu], [Ka]). Let k be an algebraically closed field of characteristic 0. Let $\Gamma L = \operatorname{Aut} L / \operatorname{Int} L$ [Hu, p. 66]. Then

(a) ΓL is the graph automorphism of the Dynkin diagram.

(b) $\Gamma L = 1$ (namely, Aut$L$ = Int L) for the following types: B_n, C_n, E_7, E_8, F_4, G_2.

(c) Γ of other types are

$\Gamma A_n = \mathbb{Z}/2\mathbb{Z} (n \geq 2)$, and the outer automorphism is given by

$$g\colon\ e_i \to e_{n+1-i}, \quad f_i \to f_{n+1-i}, \qquad i = 1, \ldots, n.$$

$\Gamma D_4 = S_3$, which is generated by the following two elements:

$$x\colon e_1 \to e_3, \qquad e_3 \to e_4, \qquad e_4 \to e_1, \qquad e_2 \to e_2$$
$$y\colon e_3 \to e_4, \qquad e_4 \to e_3, \qquad e_1 \to e_1, \qquad e_2 \to e_2.$$

$\Gamma D_l = \mathbb{Z}/2\mathbb{Z}(n > 4)$, and the outer automorphism is given by

$$g\colon\ e_{n-1} \to e_n, \qquad e_n \to e_{n-1}, \qquad e_i \to e_i.$$

$\Gamma E_6 = \mathbb{Z}/2\mathbb{Z}$, and the outer automorphism is given by

$$g\colon e_1 \to e_6, \qquad e_2 \to e_5, \qquad e_2 \to e_2, \qquad e_4 \to e_4.$$

In the above we assume that $\{e_i, f_i, h_i\}$ are the Serre generators of the simple Lie algebra. We assume the similar formulas for the action of the outer automorphism on f_i and h_i.

COROLLARY 6.4. *Let L be a simple Lie algebra of the following types over an algebraically closed field k with* char $k = 0$, *and let G be a finite group of filtered automorphisms of $U(L)$*:

(a) A_n, $n = 4m$ *or* $n = 4m + 1$.

(b) B_n.

(c) C_n.

(d) E_6, E_7, E_8, F_4 *and* G_2.

then $U(L)^G$ is Auslander–Gorenstein and GKdim–*Macaulay.*

Proof. By Corollary 6.3 and the above description of $\mathrm{Aut}(L)$ it suffices that relevant outer automorphisms are of determinant 1.

Let g be the outer automorphism given above (in the case of D_4, $g = x$ or y). We compute directly that

(1) $\det g = (-1)^{n(n+3)/3}$ for A_n.

(2) $\det x = 1$, $\det y = -1$ for D_4.

(3) $\det g = -1$ for D_n.

(4) $\det g = 1$ for E_6.

Then the result follows. ∎

Remark 6.5 In the case of D_4, if $G/(\mathrm{Int}\,(L) \cap G) \subset C_3 = \langle x \rangle$, then the result still holds.

Theorem 0.5 follows from Corollaries 6.3 and 6.4.

Remark 6.6 Alev and Polo showed that if L is a semisimple Lie algebra over an algebraically closed field of characteristic zero and if G is a non-trivial finite group of automorphisms of $U(L)$, then $U(L)^G$ is not isomorphic to any enveloping algebra $U(L')$ [AP, Theorem 1]. A similar result holds for Weyl algebras [AP, Theorem 2].

Remark 6.7 Kraft and Small have informed us that their result [KS, Proposition 4(5)] needs some extra hypothesis, such as the automorphisms of the Lie algebra are inner. Their method is similar to ours, namely, lifting the Gorenstein property from the associated graded ring to the universal enveloping algebra. Note that the invariant subring of the associated graded ring need not be Gorenstein if the automorphisms are not inner, as the next example shows.

EXAMPLE 6.8. Let $L = sl_3$ and g is the outer automorphism described before Corollary 6.4. Let $G = \{id, g\}$. Then Gr $U(L)^G$ is not Gorenstein. In fact we compute that

$$H_{\mathrm{Gr}\ U(L)^G}(t) = \frac{1}{2}\left(\frac{1}{(1-t^2)^3} + \frac{1}{(1-t)^6}\right) = \frac{1+3t^2}{(1-t)^3(1-t^2)^3}.$$

There is no integer m such that $H_{\mathrm{Gr}\ U(L)^G}(t^{-1}) = \pm t^m H_{\mathrm{Gr}\ U(L)^G}(t)$, so Gr $U(L)^G$ is not Gorenstein ([Ei, Ex. 21.17(c)]).

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