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Maximum principle for quasi-linear backward stochastic partial differential equations ☆

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Abstract

In this paper we are concerned with the maximum principle for quasi-linear backward stochastic partial differential equations (BSPDEs for short) of parabolic type. We first prove the existence and uniqueness of the weak solution to quasi-linear BSPDEs with the null Dirichlet condition on the lateral boundary. Then using the De Giorgi iteration scheme, we establish the maximum estimates and the global maximum principle for quasi-linear BSPDEs. To study the local regularity of weak solutions, we also prove a local maximum principle for the backward stochastic parabolic De Giorgi class.

Keywords: Stochastic partial differential equation; Backward stochastic partial differential equation; De Giorgi iteration; Maximum principle

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1. Introduction

In this paper we investigate the following quasi-linear BSPDE:

$$\begin{cases} -du(t,x) = \big[\partial_{x_j}\big(a^{ij}(t,x)\partial_{x_i}u(t,x) + \sigma^{jr}(t,x)v^r(t,x)\big) + b^j(t,x)\partial_{x_j}u(t,x) \\ \qquad + c(t,x)u(t,x) + \varsigma^r(t,x)v^r(t,x) + g\big(t,x,u(t,x),\nabla u(t,x),v(t,x)\big) \\ \qquad + \partial_{x_j} f^j\big(t,x,u(t,x),\nabla u(t,x),v(t,x)\big)\big]\,dt \\ \qquad - v^r(t,x)\,dW_t^r, \quad (t,x) \in Q := [0,T]\times\mathcal{O}; \\ u(T,x) = G(x), \quad x \in \mathcal{O}. \end{cases} \tag{1.1}$$

Here and in the following we use Einstein's summation convention, $T \in (0,\infty)$ is a fixed deterministic terminal time, $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with $\partial\mathcal{O} \in C^1$, $\nabla = (\partial_{x_1},\dots,\partial_{x_n})$ is the gradient operator in $\mathbb{R}^n$ and $W_t := (W_t^1,\dots,W_t^m)$, $t \in [0,T]$ is an m-dimensional standard Brownian motion in the filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant 0},P)$. A solution of BSPDE (1.1) is a pair of random fields (u,v) defined on $\Omega\times[0,T]\times\mathcal{O}$ such that (1.1) holds in a weak sense (see Definition 2.2).

The study of backward stochastic partial differential equations (BSPDEs) can be dated back about thirty years ago (see Bensoussan [2] and Pardoux [18]). They arise in many applications of probability theory and stochastic processes, for instance in the nonlinear filtering and stochastic control theory for processes with incomplete information, as an adjoint equation of the Duncan–Mortensen–Zakai filtration equation (for instance, see [2,13,14,22,25,26]). In the dynamic programming theory, some nonlinear BSPDEs as the so-called backward stochastic Hamilton–Jacobi–Bellman equations, are also introduced in the study of non-Markovian control problems (see Peng [19] and Englezos and Karatzas [11]).

Using the technique of Moser's iteration, Aronson and Serrin proved the maximum principle and local bound of weak solutions for deterministic quasi-linear parabolic equations (see [1, Theorems 1 and 2]), which are stated in the backward form as the following two theorems.

Theorem 1.1. *Let u be a weak solution of a quasi-linear parabolic equation*

$$-\partial_t u = \partial_{x_i}\mathscr{A}_i(t,x,u,\nabla u) + \mathscr{B}(t,x,u,\nabla u) \tag{1.2}$$

in the bounded cylinder $Q = (0,T)\times\mathcal{O} \subset \mathbb{R}^{1+n}$ such that $u \leqslant M$ on the parabolic boundary $((0,T]\times\partial\mathcal{O})\cup(\{T\}\times\mathcal{O})$. Then almost everywhere in Q

$$u \leqslant M + C\,\Xi(\mathscr{A},\mathscr{B})$$

where the constant C depends only on T, $|\mathcal{O}|$ and the structure terms of the equation, while $\Xi(\mathscr{A},\mathscr{B})$ is expressed in terms of some quantities related to the coefficients $\mathscr{A}$ and $\mathscr{B}$.

Theorem 1.2. *Let u be a weak solution of* (1.2) *in Q. Suppose that the set $Q_{3\rho}$ is contained in Q. Then almost everywhere in Q_ρ we have*

$$\big|u(t,x)\big| \leqslant C\big(\rho^{-(n+2)/2}\|u\|_{W^2(Q(3\rho))} + \rho^\theta\,\Xi_1(\mathscr{A},\mathscr{B})\big)$$

where the constant C depends only on ρ and the structure terms of (1.2), $Q_\rho := (\bar{t}, \bar{t} + \rho^2) \times B_\rho(\bar{x})$, $\theta \in (0, 1)$ *is one of the structure terms of* (1.2) *and* $\Xi_1(\mathscr{A}, \mathscr{B})$ *is expressed in terms of some quantities related to the coefficients* $\mathscr{A}$ *and* $\mathscr{B}$. *In particular, weak solutions of* (1.2) *must be locally essentially bounded.*

The maximum principle is a powerful tool to study the regularity of solutions, and constitutes a beautiful chapter of the classical theory of deterministic second-order elliptic and parabolic partial differential equations. In contrast to the deterministic one, the stochastic maximum principle has just caused an attention only recently. We note that Denis and Matoussi [6], and Denis, Matoussi, and Stoica [7] gave a stochastic version of Aronson and Serrin's above results, and obtained via Moser's iteration scheme a stochastic maximum principle, which claims an L^p estimate for the time and space maximal norm of weak solutions to *forward* quasi-linear stochastic partial differential equations (SPDEs). Any stochastic maximum principle seems to be lacking for *backward* ones in the literature, which then becomes quite interesting to know.

In this paper, we concern the maximum principle of a weak solution to BSPDE (1.1). Using the De Giorgi iteration scheme, we establish the global maximum principle and the local boundedness theorem for quasi-linear BSPDE (1.1), which include the above two theorems as particular cases. As highlighted by the classical theory of deterministic parabolic PDEs, our stochastic maximum principle for BSPDEs is expected to be used in the study of Hölder continuity of the solutions of BSPDEs and further in the study of more general quasi-linear BSPDEs.

It is worth noting that our estimates for weak solutions are uniform with respect to $w \in \Omega$. In contrast to Denis, Matoussi, and Stoica's L^p estimate ($p \in (2, \infty)$) for the time and space maximal norm of weak solutions of (*forward*) quasi-linear SPDEs, we prove an L^∞ estimate for that of quasi-linear BSPDE (1.1). This distinction comes from the essential difference between SPDEs and BSPDEs: the diffusion v in BSPDE (1.1) is endogenous, while the diffusion in the SPDEs is exogenous, which makes impossible any L^∞ estimate for a forward SPDE due to the active white noise. On the other hand, indeed, the technique of Moser's iteration can also be used to study the behavior of weak solutions of BSPDE (1.1) and to obtain the global and local maximum principles. However, as the De Giorgi iteration scheme works for the degenerate parabolic case, we prefer De Giorgi's method in this paper and leave the application of Moser's method as an exercise to the interested reader.

Many works have been devoted to the linear and semi-linear BSPDEs either in the whole space or in a domain (see, for instance, [8,10,9,13,23,25,26]). A theory of solvability of quasi-linear BSPDEs is recently established in an abstract framework in Qiu and Tang [21]. However, it is prevailing in these works to assume that the coefficients b, c and ς are essentially bounded. To inherit in our stochastic maximum principle the general structure of admitting the unbounded coefficients b and c in the deterministic maximum principle, we prove by approximation in Section 4 the existence and uniqueness result (Theorem 4.1) for the weak solution to the quasi-linear BSPDE (1.1) with the null Dirichlet condition on the lateral boundary, under a new rather general framework. This result is invoked to prove Proposition 4.3 as the Itô's formula for the composition of solutions of BSDEs into a class of time–space smooth functions, which is the starting point of the De Giorgi scheme in the proof of subsequent stochastic maximum principles.

This paper is organized as follows. In Section 2, we set notations, hypotheses and the notion of the weak solution to BSPDE (1.1). In Section 3, we prepare several auxiliary results, including a generalized Itô formula, which will be used to establish Proposition 4.3 below as a key step in the study of our stochastic maximum principle. In Section 4 we prove the existence and uniqueness of the weak solution to BSPDE (1.1). Finally, in Section 5, we establish the maximum principles for quasi-linear BSPDEs. In the first subsection, we use the De Giorgi iteration scheme to obtain the global maximum principles for BSPDEs (1.1) and in the second subsection, we prove the local maximum principle for our backward stochastic parabolic De Giorgi class.

2. Preliminaries

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete filtered probability space on which is defined an m-dimensional standard Brownian motion $W = \{W_t\colon t \in [0, \infty)\}$ such that $\{\mathscr{F}_t\}_{t\geqslant 0}$ is the natural filtration generated by W and augmented by all the $\mathbb{P}$-null sets in $\mathscr{F}$. We denote by $\mathscr{P}$ the σ-algebra of the predictable sets on $\Omega \times [0, T]$ associated with $\{\mathscr{F}_t\}_{t\geqslant 0}$.

Denote by $\mathbb{Z}$ the set of all the integers and by $\mathbb{N}$ the set of all the positive integers. Denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$ the norm and scalar product in a finite-dimension Hilbert space. Like in $\mathbb{R}, \mathbb{R}^k, \mathbb{R}^{k\times l}$ with $k, l \in \mathbb{N}$, we have defined

$$|x| := \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}} \quad \text{and} \quad |y| := \left(\sum_{i=1}^{k}\sum_{j=1}^{l} y_{ij}^2\right)^{\frac{1}{2}} \quad \text{for } (x, y) \in \mathbb{R}^k \times \mathbb{R}^{k\times l}.$$

For the sake of convenience, we denote

$$\partial_s := \frac{\partial}{\partial s} \quad \text{and} \quad \partial_{st} := \frac{\partial^2}{\partial s \partial t}.$$

Let V be a Banach space equipped with norm $\|\cdot\|_V$. For real $p \in (0, \infty)$, $\mathcal{S}^p(V)$ is the set of all the V-valued, adapted and càdlàg processes $(X_t)_{t\in[0,T]}$ such that

$$\|X\|_{\mathcal{S}^p(V)} := \left(E\Big[\sup_{t\in[0,T]} \|X_t\|_V^p\Big]\right)^{1\wedge\frac{1}{p}} < \infty.$$

It is worth noting that $(\mathcal{S}^p(V), \|\cdot\|_{\mathcal{S}^p(V)})$ is a Banach space for $p \in [1, \infty)$ and for $p \in (0, 1)$, $dis(X, X') := \|X - X'\|_{\mathcal{S}^p(V)}$ is a metric of $\mathcal{S}^p(V)$ under which $\mathcal{S}^p(V)$ is complete.

Define the parabolic distance in $\mathbb{R}^{1+n}$ as follows:

$$\delta(X, Y) := \max\left\{|t - s|^{1/2}, |x - y|\right\},$$

for $X := (t, x)$ and $Y := (s, y) \in \mathbb{R}^{1+n}$. Denote by $Q_r(X)$ the ball of radius $r > 0$ and center $X := (t, x) \in \mathbb{R}^{1+n}$ with $x \in \mathbb{R}^n$:

$$Q_r(X) := \left\{Y \in \mathbb{R}^{1+d}\colon \delta(X, Y) < r\right\} = \left(t - r^2, t + r^2\right) \times B_r(x),$$
$$B_r(x) := \left\{y \in \mathbb{R}^n\colon |y - x| < r\right\},$$

and by $|Q_r(X)|$ the volume.

Denote by $\partial\Pi$ the boundary of domain $\Pi \subset \mathbb{R}^n$. Throughout this paper, we assume that $\mathcal{O}$ is a bounded domain of $\mathbb{R}^n$ with $\partial\mathcal{O} \in C^1$. The set $S_T := [0, T] \times \partial\mathcal{O}$ is called the lateral boundary of Q and the set $\partial_{\mathrm{p}} Q := S_T \cup (\{T\} \times \mathcal{O})$ is called the parabolic boundary of Q.

For domain $\Pi \subset \mathbb{R}^n$, we denote by $C_c^\infty(\Pi)$ the totality of infinitely differentiable functions of compact supports in Π, and the spaces like $L^\infty(\Pi)$, $L^p(\Pi)$ and $W^{k,p}(\Pi)$ are defined as usual for integer k and real number $p \in [1, \infty)$. We denote by $\langle\langle\cdot,\cdot\rangle\rangle_\Pi$ the inner product of $L^2(\Pi)$ and the subscript Π will be omitted for $\Pi = \mathcal{O}$. Set $\Pi_t := [t, T] \times \Pi$ for $t \in [0, T)$. For each integer k and real number $p \in [1, \infty)$, we denote by $W^{k,p}_{\mathscr{F}}(\Pi_t)$ the totality of the $W^{k,p}(\Pi)$-valued predictable processes u on $[t, T]$ such that

$$\|u\|_{W^{k,p}_{\mathscr{F}}(\Pi_t)} := \left(E\left[\int_t^T \|u(s,\cdot)\|^p_{W^{k,p}(\Pi)}\,ds\right]\right)^{1/p} < \infty.$$

Then $(W^{k,p}_{\mathscr{F}}(\Pi_t), \|\cdot\|_{W^{k,p}_{\mathscr{F}}(\Pi_t)})$ is a Banach space.

Definition 2.1. For $(p, t, k) \in [1, \infty) \times [0, T) \times \mathbb{Z}$, define $\mathcal{M}^{k,p}(\Pi_t)$ as the totality of $u \in W^{k,p}_{\mathscr{F}}(\Pi_t)$ such that

$$\|u\|_{k,p;\Pi_t} := \left(\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{s\in[t,T]} E\left[\int_s^T \|u(\omega,\tau,\cdot)\|^p_{W^{k,p}(\Pi)}\,d\tau \,\Big|\mathscr{F}_s\right]\right)^{1/p} < \infty.$$

For $u \in W^{k,p}_{\mathscr{F}}(\Pi_t)$, we deduce from [3, Theorem 6.3] that the process

$$\left\{1_{[t,T]}(s)E\left[\int_s^T \|u(\omega,\tau,\cdot)\|^p_{W^{k,p}(\Pi)}\,d\tau\,\Big|\mathscr{F}_s\right],\ s\in[0,T]\right\} \in S^\beta(\mathbb{R}) \quad \text{for any } \beta \in (0,1).$$

This shows that the norm $\|\cdot\|_{k,p;\Pi_t}$ in the preceding definition makes a sense. Moreover, $(\mathcal{M}^{k,p}(\Pi_t), \|\cdot\|_{k,p;\Pi_t})$ is a Banach space.

To simplify notations, $k = 0$ appearing in either superscript or subscript of spaces or norms will be omitted and therefore the notations $W^{0,p}_{\mathscr{F}}(\Pi_t)$, $\|\cdot\|_{W^{0,p}_{\mathscr{F}}(\Pi_t)}$, $\mathcal{M}^{0,p}(\Pi_t)$ and $\|\cdot\|_{0,p;\Pi_t}$ will be abbreviated as $W^p_{\mathscr{F}}(\Pi_t)$, $\|\cdot\|_{W^p_{\mathscr{F}}(\Pi_t)}$, $\mathcal{M}^p(\Pi_t)$ and $\|\cdot\|_{p;\Pi_t}$. Note that $W^{0,p}(\Pi) \equiv L^p(\Pi)$.

Moreover, we introduce the following spaces of random fields. $\mathcal{L}^\infty(\Pi_t)$ is the totality of $u \in W^p_{\mathscr{F}}(\Pi_t)$ such that

$$\|u\|_{\infty;\Pi_t} := \operatorname*{ess\,sup}_{(\omega,s,x)\in\Omega\times\Pi_t} \big|u(\omega,s,x)\big| < \infty.$$

$\mathcal{L}^{\infty,p}(\Pi_t)$ is the totality of $u \in W^p_{\mathscr{F}}(\Pi_t)$ such that

$$\|u\|_{\infty,p;\Pi_t} := \operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[t,T]} \big\|u(\omega,s,\cdot)\big\|_{L^p(\Pi)} < \infty.$$

$\mathcal{V}_2(\Pi_t)$ is the totality of $u \in W^{1,2}_{\mathscr{F}}(\Pi_t)$ such that

$$\|u\|_{\mathcal{V}_2(\Pi_t)} := \left(\|u\|^2_{\infty,2;\Pi_t} + \|\nabla u\|^2_{2;\Pi_t}\right)^{1/2} < \infty. \tag{2.1}$$

$\mathcal{V}_{2,0}(\Pi_t)$, equipped with the norm (2.1), is the totality of $u \in \mathcal{V}_2(\Pi_t)$ such that

$$\lim_{r\to 0}\big\|u(s+r,\cdot)-u(s,\cdot)\big\|_{L^2(\Pi)}=0, \quad \text{for all } s, s+r\in[t,T]$$

holds almost surely.

We denote by $\dot{\mathcal{V}}_2(Q)$ ($\dot{\mathcal{V}}_{2,0}(\Pi_t)$, $\dot{W}^{1,p}_{\mathscr{F}}(\Pi_t)$ and $\dot{\mathcal{M}}^{1,p}(\Pi_t)$, respectively) all the random fields $u\in\mathcal{V}_2(Q)$ ($\mathcal{V}_{2,0}(\Pi_t)$, $W^{1,p}_{\mathscr{F}}(\Pi_t)$ and $\mathcal{M}^{1,p}(\Pi_t)$, respectively), satisfying

$$u(\omega,s,\cdot)|_{\partial\Pi}=0, \quad \text{a.e. } (\omega,s)\in\Omega\times[t,T].$$

By convention, we treat elements of spaces defined above like $W^{k,p}(\Pi)$ and $\mathcal{M}^{k,p}(\Pi_t)$ as functions rather than distributions or classes of equivalent functions, and if we know that a function of this class has a modification with better properties, then we always consider this modification. For example, if $u\in W^{1,p}(\Pi)$ with $p>n$, then u has a modification lying in $C^\alpha(\Pi)$ for $\alpha\in(0,\frac{p-n}{p})$, and we always adopt the modification $u\in W^{1,p}(\Pi)\cap C^\alpha(\Pi)$. By saying a finite dimensional vector-valued function $v:=(v_i)_{i\in\mathcal{I}}$ belongs to a space like $W^{k,p}(\Pi)$, we mean that each component v_i belongs to the space and the norm is defined by

$$\|v\|_{W^{k,p}(\Pi)}=\left(\sum_{i\in\mathcal{I}}\|v_i\|^p_{W^{k,p}(\Pi)}\right)^{1/p}.$$

For a real number $M\in(0,\infty)$, denote by $\mathcal{C}^2_M$ the totality of $\phi\in C(\mathbb{R}^{n+2})$ such that $\phi:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ is twice differentiable, $\phi'(t,x,0)=0$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}^n$ and

$$\begin{aligned}\operatorname*{ess\,sup}_{(t,x)\in\mathbb{R}^{n+1},s\in\mathbb{R}\setminus\{0\}}&\Bigg\{\big|\phi''(t,x,s)\big|+\frac{1}{|s|}\sum_{i=1}^n\big|\partial_{x_i}\phi'(t,x,s)\big|\\&+\frac{1}{s^2}\big|\partial_t\phi(t,x,s)-\partial_t\phi(t,x,0)\big|\Bigg\}<M,\end{aligned}$$

where $\phi'(t,x,s):=\partial_s\phi(t,x,s)$ and $\phi''(t,x,s):=\partial_{ss}\phi(t,x,s)$. Moreover we denote by $\mathcal{C}^{2,0}_M$ the totality of $u\in\mathcal{C}^2_M$ with the second-order derivatives being continuous.

Consider quasi-linear BSPDE (1.1). We define the following assumptions.

($\mathcal{A}1$) *The pair of random functions*

$$f(\cdot,\cdot,\cdot,\vartheta,y,z):\Omega\times[0,T]\times\mathcal{O}\to\mathbb{R}^n \quad \textit{and} \quad g(\cdot,\cdot,\cdot,\vartheta,y,z):\Omega\times[0,T]\times\mathcal{O}\to\mathbb{R}$$

are $\mathscr{P}\otimes\mathcal{B}(\mathcal{O})$-measurable for any $(\vartheta,y,z)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m$. There exist positive constants L,κ and β such that for all $(\vartheta_1,y_1,z_1),(\vartheta_2,y_2,z_2)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n\times m}$ and $(\omega,t,x)\in\Omega\times[0,T]\times\mathcal{O}$

$$\big|f(\omega,t,x,\vartheta_1,y_1,z_1)-f(\omega,t,x,\vartheta_2,y_2,z_2)\big|\leqslant L|\vartheta_1-\vartheta_2|+\frac{\kappa}{2}|y_1-y_2|+\beta^{1/2}|z_1-z_2|,$$

$$\big|g(\omega,t,x,\vartheta_1,y_1,z_1)-g(\omega,t,x,\vartheta_2,y_2,z_2)\big|\leqslant L\big(|\vartheta_1-\vartheta_2|+|y_1-y_2|+|z_1-z_2|\big).$$

($\mathcal{A}2$) *The pair functions a and σ are $\mathscr{P}\otimes\mathcal{B}(\mathcal{O})$-measurable. There exist positive constants $\varrho>1$, λ and Λ such that the following hold for all $\xi\in\mathbb{R}^n$ and $(\omega,t,x)\in\Omega\times[0,T]\times\mathcal{O}$*

$$\lambda|\xi|^2 \leqslant \big(2a^{ij}(\omega,t,x) - \varrho\sigma^{ir}\sigma^{jr}(\omega,t,x)\big)\xi^i\xi^j \leqslant \Lambda|\xi|^2;$$
$$\big|a(\omega,t,x)\big| + \big|\sigma(\omega,t,x)\big| \leqslant \Lambda;$$
$$\textit{and}\quad \lambda-\kappa-\varrho'\beta>0 \quad \textit{with}\quad \varrho' := \frac{\varrho}{\varrho-1}.$$

($\mathcal{A}3$) *$G \in L^\infty(\Omega, \mathscr{F}_T, L^2(\mathcal{O}))$. There exist two real numbers $p > n+2$ and $q > (n+2)/2$ such that*

$$f_0 := f(\cdot,\cdot,\cdot,0,0,0) \in \mathcal{M}^p(Q), \qquad g_0 := g(\cdot,\cdot,\cdot,0,0,0) \in \mathcal{M}^{\frac{p(n+2)}{p+n+2}}(Q),$$

and $(b^i)^2$, $(\varsigma^r)^2$, $c \in \mathcal{M}^q(Q)$, $i = 1,\dots,n$; $r = 1,\dots,m$. Define

$$\Lambda_0 := B_q(b,c,\varsigma) := \big\||b|^2\big\|_{q;Q} + \|c\|_{q;Q} + \big\||\varsigma|^2\big\|_{q;Q}. \tag{2.2}$$

($\mathcal{A}3$)$_0$ *$G \in L^\infty(\Omega, \mathscr{F}_T, L^2(\mathcal{O}))$, $f_0 \in \mathcal{M}^2(Q)$, $g_0 \in \mathcal{M}^2(Q)$ and $b, \varsigma, c \in \mathcal{L}^\infty(Q)$.*

($\mathcal{A}4$) *There exists a nonnegative constant L_0 such that $c \leqslant L_0$.*

For $p \in [2,\infty)$, define the functional A_p:

$$A_p(u,v) := \|u\|_{p;Q} + \|v\|_{\frac{p(n+2)}{p+n+2};Q}, \quad (u,v) \in \mathcal{M}^p(Q) \times \mathcal{M}^{\frac{p(n+2)}{p+n+2}}(Q),$$

and the functional H_p:

$$H_p(u,v) := \|u\|_{p;Q} + \|v\|_{p;Q}, \quad (u,v) \in \mathcal{M}^p(Q) \times \mathcal{M}^p(Q).$$

Definition 2.2. A pair of processes $(u,v) \in W^{1,2}_{\mathscr{F}}(Q) \times W^2_{\mathscr{F}}(Q)$ is called a weak solution to BSPDE (1.1) if it holds in the weak sense, i.e. for any $\varphi \in C_c^\infty(\mathcal{O})$ there holds almost surely

$$\begin{aligned}
\langle\!\langle \varphi, u(t)\rangle\!\rangle &= \langle\!\langle \varphi, G\rangle\!\rangle - \int_t^T \langle\!\langle \varphi, v^r(s)\rangle\!\rangle\, dW_s^r + \int_t^T \big\langle\!\big\langle \varphi, g\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\big\rangle\!\big\rangle\, ds \\
&\quad - \int_t^T \big\langle\!\big\langle \partial_{x_j}\varphi, a^{ij}\partial_{x_i}u(s) + \sigma^{jr}v^r(s) + f^j\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\big\rangle\!\big\rangle\, ds \\
&\quad + \int_t^T \langle\!\langle \varphi, b^i\partial_{x_i}u(s) + cu(s) + \varsigma^r v^r(s)\rangle\!\rangle\, ds, \quad \forall t \in [0,T].
\end{aligned} \tag{2.3}$$

Denote by $\mathscr{U} \times \mathscr{V}(G,f,g)$ the set of all the weak solutions $(u,v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^2(Q)$ of BSPDE (1.1).

Remark 2.1. Let $(u, v) \in W^{1,2}_{\mathscr{F}}(Q) \times W^2_{\mathscr{F}}(Q)$ be a weak solution to BSPDE (1.1). For each $\zeta(t,x) = \psi(t)\varphi(x)$ with $\varphi \in C_c^\infty(\mathcal{O})$ and $\psi \in C_c^\infty(\mathbb{R})$, in view of (2.3), we have almost surely

$$\begin{aligned}
&\langle\!\langle \zeta(s''), u(s'')\rangle\!\rangle - \langle\!\langle \zeta(s'), u(s')\rangle\!\rangle \\
&\quad= \langle\!\langle \zeta(s'') - \zeta(s'), u(s'')\rangle\!\rangle + \langle\!\langle \zeta(s'), u(s'') - u(s')\rangle\!\rangle \\
&\quad= \left[\psi(s'') - \psi(s')\right]\langle\!\langle \varphi, u(s'')\rangle\!\rangle + \psi(s')\left(\langle\!\langle \varphi, u(s'')\rangle\!\rangle - \langle\!\langle \varphi, u(s')\rangle\!\rangle\right) \\
&\quad= \left[\psi(s'') - \psi(s')\right]\langle\!\langle \varphi, u(s'')\rangle\!\rangle \\
&\qquad - \psi(s')\Bigg(\int_{s'}^{s''} \langle\!\langle \varphi, g(s,\cdot,u(s),\nabla u(s), v(s))\rangle\!\rangle\,ds - \int_{s'}^{s''} \langle\!\langle \varphi, v^r(s)\rangle\!\rangle\,dW_s^r \\
&\qquad - \int_{s'}^{s''} \langle\!\langle \partial_{x_j}\varphi, a^{ij}\partial_{x_i}u(s) + \sigma^{jr}v^r(s) + f^j(s,\cdot,u(s),\nabla u(s), v(s))\rangle\!\rangle\,ds \\
&\qquad + \int_{s'}^{s''} \langle\!\langle \varphi, b^i\partial_{x_i}u(s) + cu(s) + \varsigma^r v^r(s)\rangle\!\rangle\,ds\Bigg)
\end{aligned}$$

for $s'' = t_{i+1}$ and $s' = t_i$, where $t = t_0 < t_1 < t_2 < \cdots < t_N = T$, $2 < N \in \mathbb{N}$ and $t_{i+1} - t_i = T/N$, $i = 1, 2, \ldots, N$. Summing up both sides of these equations and passing to the limit, we have almost surely

$$\begin{aligned}
\langle\!\langle \zeta(t), u(t)\rangle\!\rangle &= \langle\!\langle \zeta(T), G\rangle\!\rangle - \int_t^T \langle\!\langle \partial_s\zeta(s), u(s)\rangle\!\rangle\,ds - \int_t^T \langle\!\langle \zeta(s), v^r(s)\rangle\!\rangle\,dW_s^r \\
&\quad - \int_t^T \langle\!\langle \partial_{x_j}\zeta(s), a^{ij}\partial_{x_i}u(s) + \sigma^{jr}v^r(s) + f^j(s,\cdot,u(s),\nabla u(s), v(s))\rangle\!\rangle\,ds \\
&\quad + \int_t^T \langle\!\langle \zeta(s), b^i\partial_{x_i}u(s) + cu(s) + \varsigma^r v^r(s)\rangle\!\rangle\,ds \\
&\quad + \int_t^T \langle\!\langle \zeta(s), g(s,\cdot,u(s),\nabla u(s), v(s))\rangle\!\rangle\,ds, \quad \forall t \in [0,T]. \qquad (2.4)
\end{aligned}$$

Since the linear space

$$\left\{\sum_{i=1}^N \psi_i(t)\varphi_i(x),\ (t,x) \in \mathbb{R}\times\mathcal{O}\colon\ N \in \mathbb{N},\ (\varphi_i, \psi_i) \in C_c^\infty(\mathcal{O}) \times C_c^\infty(\mathbb{R}),\ i = 1, 2, \ldots, N\right\}$$

is dense in $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathcal{O})$, under certain assumptions on the structure terms of BSPDE (1.1) such as $a, b, c, \sigma, \varsigma, f$ and g, (2.4) holds for any test function $\zeta \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathcal{O})$.

Under assumptions $(\mathcal{A}1)$, $(\mathcal{A}2)$ and $(\mathcal{A}3)_0$, we deduce from [21, Theorem 2.1] that there exists a unique weak solution $(u,v)\in(\dot{W}^{1,2}_{\mathscr{F}}(Q)\cap S^2(L^2(\mathcal{O})))\times W^2_{\mathscr{F}}(Q)$, which admits $L^2(\mathcal{O})$-valued continuous trajectories for u, and which is also said to satisfy the null Dirichlet condition on the lateral boundary since u vanishes in a generalized sense on the boundary $\partial\mathcal{O}$. Denote by $\dot{\mathscr{U}}\times\dot{\mathscr{V}}(G,f,g)$ all the random fields lying in $\mathscr{U}\times\mathscr{V}(G,f,g)$ which satisfy the null Dirichlet boundary condition.

3. Auxiliary results

In what follows, $C>0$ is a constant which may vary from line to line and $C(a_1,a_2,\ldots)$ is a constant to depend on the parameters $a_1,a_2,\ldots$.

First, we give the following embedding lemma.

Lemma 3.1. *For $u\in\dot{\mathcal{V}}_2(\Pi_t)$ with $t\in[0,T)$, we have $u\in\mathcal{M}^{\frac{2(n+2)}{n}}(\Pi_t)$ and*

$$\|u\|_{\frac{2(n+2)}{n};\Pi_t}\leqslant C(n)\|\nabla u\|_{2;\Pi_t}^{n/(n+2)}\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[t,T]}\big\|u(\omega,s,\cdot)\big\|_{L^2(\Pi)}^{2/(n+2)}\leqslant C(n)\|u\|_{\mathcal{V}_2(\Pi_t)}.$$

Proof. By the well-known Gagliard–Nirenberg inequality (cf. [12,15] or [17]), we have

$$\big\|u(\omega,s,\cdot)\big\|_{L^q(\Pi)}^q\leqslant C\big\|\nabla u(\omega,s,\cdot)\big\|_{L^2(\Pi)}^{\alpha q}\big\|u(\omega,s,\cdot)\big\|_{L^2(\Pi)}^{q(1-\alpha)},\quad \text{a.e. }(\omega,s)\in\Omega\times[t,T],$$

where $\alpha=n/(n+2)$ and $q=2(n+2)/n$. Integrating on $[\tau,T]$ for $\tau\in[t,T)$ and taking conditional expectations on both sides, we obtain almost surely

$$\begin{aligned}E\bigg[\int_{\Pi_\tau}\big|u(s,x)\big|^q\,dx\,ds\big|\mathscr{F}_\tau\bigg]&\leqslant C\|\nabla u\|_{2;\Pi_t}^2\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[t,T]}\big\|u(\omega,s,\cdot)\big\|_{L^2(\Pi)}^{(1-\alpha)q}\\&\leqslant C\|u\|_{\mathcal{V}_2(\Pi_t)}^q.\end{aligned}$$

Therefore, $u\in\mathcal{M}^{\frac{2(n+2)}{n}}(\Pi_t)$ and

$$\|u\|_{\frac{2(n+2)}{n};\Pi_t}\leqslant C\|\nabla u\|_{2;\Pi_t}^{n/(n+2)}\operatorname*{ess\,sup}_{(\omega,s)\in\Omega\times[t,T]}\big\|u(\omega,s,\cdot)\big\|_{L^2(\Pi)}^{2/(n+2)}\leqslant C\|u\|_{\mathcal{V}_2(\Pi_t)}$$

with C only depending on n. □

Lemma 3.2. *For any $r\in\mathbb{R}$ and $u\in\mathcal{V}_{2,0}(\Pi_t)$ with $t\in[0,T)$ we have*

$$(u-r)^+:=(u-r)\vee 0\in\mathcal{V}_{2,0}(\Pi_t).$$

Moreover, if $\{u_k,k\in\mathbb{N}\}$ is a Cauchy sequence in $\mathcal{V}_{2,0}(\Pi_t)$ with limit $u\in\mathcal{V}_{2,0}(\Pi_t)$, then

$$\lim_{k\to\infty}\big\|(u_k-r)^+-(u-r)^+\big\|_{\mathcal{V}_2(\Pi_t)}=0.$$

Proof. It can be checked that $(u-r)^+ \in \mathcal{V}_2(\Pi_t)$. Since

$$\left|(u-r)^+ - (v-r)^+\right| \leqslant |u-v|,$$

then we have

$$\left\|(u-r)^+(s+h) - (u-r)^+(s)\right\|_{L^2(\Pi)} \leqslant \left\|u(s+h) - u(s)\right\|_{L^2(\Pi)}, \quad \forall s,\ s+h \in [t,T].$$

Hence, the continuity of u implies that of $(u-r)^+$. The other assertions follow in a similar way. We complete our proof. □

In contrast to Lebesgue's integral, the integrand of Itô's stochastic integral is required to be adapted, and the technique of Steklov time average (see [16, p. 100]) finds difficulty in our stochastic situation. We directly establish some Itô formula to get around the difficulty.

Lemma 3.3. *Let $\phi \in \mathcal{C}_M^{2,0}$ and assume that the equation*

$$u(t,x) = u(T,x) + \int_t^T \left(h^0(s,x) + \partial_{x_i} h^i(s,x)\right) ds - \int_t^T z^r(s,x)\, dW_s^r, \quad t \in [0,T], \tag{3.1}$$

holds in the weak sense of Definition 2.2, where $u(T) \in L^2(\Omega, \mathscr{F}_T, L^2(\mathcal{O}))$; $h^i \in W^2_{\mathscr{F}}(Q)$, $i = 0, 1, \ldots, n$; and $z \in W^2_{\mathscr{F}}(Q)$. If $u \in \dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))$, we have almost surely

$$\begin{aligned}
\int_{\mathcal{O}} \phi\big(t,x,u(t,x)\big)\, dx &= \int_{\mathcal{O}} \phi\big(T,x,u(T,x)\big)\, dx - \int_t^T \int_{\mathcal{O}} \partial_s \phi\big(s,x,u(s,x)\big)\, dx\, ds \\
&\quad - \int_t^T \left\langle\!\left\langle \phi'\big(s,\cdot,u(s)\big), z^r(s) \right\rangle\!\right\rangle dW_s^r + \int_t^T \left\langle\!\left\langle \phi'\big(s,\cdot,u(s)\big), h^0(s) \right\rangle\!\right\rangle ds \\
&\quad - \int_t^T \left\langle\!\left\langle \phi''\big(s,\cdot,u(s)\big)\partial_{x_i} u(s) + \partial_{x_i}\phi'\big(s,\cdot,u(s)\big), h^i(s) \right\rangle\!\right\rangle ds \\
&\quad - \frac{1}{2} \int_t^T \left\langle\!\left\langle \phi''\big(s,\cdot,u(s)\big), \left|z(s)\right|^2 \right\rangle\!\right\rangle ds, \quad \forall t \in [0,T].
\end{aligned} \tag{3.2}$$

Remark 3.1. Lemma 3.3 is more general than both Itô formulas of [7] and [20] since our test function ϕ is allowed to depend on both time and space variables. The extension is motivated by the subsequent study of the local maximum principle where Itô formula for truncated solutions of BSPDEs is required.

Proof of Lemma 3.3. All the integrals in (3.2) are well defined. In particular, the stochastic integral

$$I(t) := \int_0^t \langle\!\langle \phi'(s, \cdot, u(s)), z^r(s) \rangle\!\rangle \, dW_s^r, \quad t \in [0, T]$$

is a martingale since

$$E\Big[\sup_{t\in[0,T]} |I(t)|\Big] \leqslant CE\Bigg[\bigg(\int_0^T \big|\langle\!\langle |\phi'(s, \cdot, u(s))|, |z(s)| \rangle\!\rangle\big|^2 \, ds\bigg)^{1/2}\Bigg]$$
$$\leqslant CM\|u\|_{S^2(L^2(\mathcal{O}))}\|z\|_{W^2_{\mathscr{F}}(Q)}.$$

We extend the random fields $u, h^0, h^1, \ldots, h^n$ and z from their domain $\Omega \times [0, T] \times \mathcal{O}$ to $\Omega \times [0, T] \times \mathbb{R}^n$ by setting them all to be zero outside $\mathcal{O}$, and we still use themselves to denote their respective extensions. Since u satisfies the null Dirichlet condition on the lateral boundary and $\partial\mathcal{O} \in C^1$, we have $u \in W^{1,2}_{\mathscr{F}}([0, T] \times \mathbb{R}^n)$. It is obvious that all the extensions $h^0, h^1, \ldots, h^n$ and z lie in $W^2_{\mathscr{F}}([0, T] \times \mathbb{R}^n)$.

Step 1. Consider $h^i \in \dot{W}^{1,2}_{\mathscr{F}}(\mathcal{O})$, $i = 1, 2, \ldots, n$. Choose a sufficiently large positive integer N_0 so that $\{x \in \mathcal{O}\colon dis(x, \partial\mathcal{O}) > 1/N_0\}$ is a nonempty sub-domain of $\mathcal{O}$. For integer $N > N_0$, define

$$\mathcal{O}^N := \big\{x \in \mathcal{O}\colon dis(x, \partial\mathcal{O}) > 1/N\big\}.$$

Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\operatorname{supp}(\rho) \subset B_1(0) \quad \text{and} \quad \int_{\mathbb{R}^n} \rho(x)\, dx = 1.$$

Define for each positive integer k,

$$\rho_k(x) := (2Nk)^n \rho(2Nkx), \qquad u_k(s, x) := u(s) * \rho_k(x) := \int_{\mathbb{R}^n} \rho_k(x - y) u(s, y)\, dy.$$

In a similar way, we write

$$z_k(s, x) := z(s) * \rho_k(x) \quad \text{and} \quad h_k^i(s, x) := h^i(s) * \rho_k(x), \quad i = 1, 2, \ldots, n.$$

Then for each $x \in \mathcal{O}^N$, we have almost surely

$$u_k(t, x) = u_k(T, x) + \int_t^T \big(\partial_{x_i} h_k^i(s, x) + h_k^0(s, x)\big)\, ds - \int_t^T z_k^r(s, x)\, dW_s^r, \quad \forall t \in [0, T].$$

By using Itô formula for each $x \in \mathcal{O}^N$ and then integrating over $\mathcal{O}^N$ with respect to x, we obtain

$$\int_{\mathcal{O}^N} \phi\big(t,x,u_k(t,x)\big)\,dx = \int_{\mathcal{O}^N} \phi\big(T,x,u_k(T,x)\big)\,dx - \int_t^T \int_{\mathcal{O}^N} \partial_s \phi\big(s,x,u_k(s,x)\big)\,dx\,ds$$
$$+ \int_t^T \langle\!\langle \phi'\big(s,\cdot,u_k(s)\big), h^0(s) \rangle\!\rangle_{\mathcal{O}^N}\,ds$$
$$+ \int_t^T \langle\!\langle \phi'\big(s,\cdot,u_k(s)\big), \partial_{x_i} h_k^i(s) \rangle\!\rangle_{\mathcal{O}^N}\,ds$$
$$- \frac{1}{2}\int_t^T \langle\!\langle \phi''\big(s,\cdot,u_k(s)\big), \big|z_k(s)\big|^2 \rangle\!\rangle_{\mathcal{O}^N}\,ds$$
$$- \int_t^T \langle\!\langle \phi'\big(s,\cdot,u_k(s)\big), z_k^r(s) \rangle\!\rangle_{\mathcal{O}^N}\,dW_s^r. \tag{3.3}$$

For the sake of convenience, we define

$$\delta\phi_k(t,x) := \phi\big(t,x,u(t,x)\big) - \phi\big(t,x,u_k(t,x)\big),$$
$$\delta u_k(t,x) := u(t,x) - u_k(t,x)$$

and in a similar way, we define $\delta\phi_k'$, $\delta\phi_k''$, δh_k^i and δz_k^r, $i = 0,1,\dots,n$; $r = 1,\dots,m$.
Since for almost all $(\omega,s) \in \Omega \times [0,T]$

$$\big\|u_k(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)} \leqslant \big\|u(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)}, \qquad \lim_{k\to\infty} \big\|\delta u_k(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)} \to 0;$$
$$\big\|h_k^0(\omega,s)\big\|_{L^2(\mathbb{R}^n)} \leqslant \big\|h^0(\omega,s)\big\|_{L^2(\mathbb{R}^n)}, \qquad \lim_{k\to\infty} \big\|\delta h_k^0(\omega,s)\big\|_{L^2(\mathbb{R}^n)} \to 0;$$
$$\big\|h_k^i(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)} \leqslant \big\|h^i(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)}, \qquad \lim_{k\to\infty} \big\|\delta h_k^i(\omega,s)\big\|_{W^{1,2}(\mathbb{R}^n)} \to 0, \quad i = 1,2,\dots;$$
$$\big\|z_k(\omega,s)\big\|_{L^2(\mathbb{R}^n)} \leqslant \big\|z(\omega,s)\big\|_{L^2(\mathbb{R}^n)}, \qquad \lim_{k\to\infty} \big\|\delta z_k(\omega,s)\big\|_{L^2(\mathbb{R}^n)} \to 0,$$

by Lebesgue's dominated convergence theorem, we have as $k \to \infty$

$$\sum_{i=1}^n \big\|\delta h_k^i(s)\big\|^2_{W^{1,2}_{\mathscr{F}}([0,T]\times\mathbb{R}^n)} + \big\|\delta h_k^0(s)\big\|^2_{W^2_{\mathscr{F}}([0,T]\times\mathbb{R}^n)} + \big\|\delta z_k(s)\big\|^2_{W^2_{\mathscr{F}}([0,T]\times\mathbb{R}^n)}$$
$$+ \big\|\delta u_k(s)\big\|^2_{W^{1,2}_{\mathscr{F}}([0,T]\times\mathbb{R}^n)} \to 0,$$

$$E\left[\int_0^T \int_{\mathcal{O}} \big|\delta\phi_k(t,x)\big|\,dx\,dt\right] \leqslant E\left[\int_0^T M \langle\!\langle \big|u_k(t)\big| + \big|u(t)\big|, \big|\delta u_k(t)\big| \rangle\!\rangle\,dt\right] \to 0,$$

$$E\left[\int_0^T\int_{\mathbb{R}^n}\left|\phi'\big(s,x,u_k(s,x)\big)\partial_{x_i}h_k^i(s,x)-\phi'\big(s,x,u(s,x)\big)\partial_{x_i}h^i(s,x)\right|dx\,ds\right]$$
$$\leqslant E\left[\int_0^T\int_{\mathbb{R}^n}\big(M\left|\delta u_k(s,x)\partial_{x_i}h_k^i(s,x)\right|+M\left|u(s,x)\right|\left|\partial_{x_i}\big(\delta h_k^i\big)(s,x)\right|\big)\,dx\,ds\right]\to 0,$$
$$i=1,\dots,n$$

and

$$E\left[\int_0^T\int_{\mathbb{R}^n}\left|\phi'\big(s,x,u_k(s,x)\big)h_k^0(s,x)-\phi'\big(s,x,u(s,x)\big)h^0(s,x)\right|dx\,ds\right]$$
$$\leqslant E\left[\int_0^T\int_{\mathbb{R}^n}\big(M\left|\delta u_k(s,x)h_k^0(s,x)\right|+M\left|u(s,x)\right|\left|\delta h_k^0(s,x)\right|\big)\,dx\,ds\right]\to 0.$$

Since the convergence

$$\lim_{k\to\infty}\|\delta u_k\|_{W^{1,2}_{\mathscr{F}}([0,T]\times\mathbb{R}^n)}=0$$

implies that $u_k(\omega,t,x)$ converges to $u(\omega,t,x)$ in measure $dP\otimes dt\otimes dx$, from the dominated convergence theorem we conclude that

$$\lim_{k\to\infty}E\left[\int_0^T\int_{\mathcal{O}}\left|\partial_s\phi\big(s,x,u_k(s,x)\big)\;-\partial_s\phi\big(s,x,u(s,x)\big)\right|dx\,ds\right]=0.$$

In a similar way, we obtain

$$E\left[\int_0^T\int_{\mathcal{O}}\left|\phi''\big(s,x,u(s,x)\big)\left|z(s,x)\right|^2-\phi''\big(s,x,u_k(s,x)\big)\left|z_k(s,x)\right|^2\right|dx\,ds\right]\to 0$$

and

$$E\left[\sup_{t\in[0,T]}\left|\sum_{r=1}^m\int_t^T\int_{\mathbb{R}^n}\big(\phi'\big(s,x,u_k(s,x)\big)z_k^r(s,x)-\phi'\big(s,x,u(s,x)\big)z^r(s,x)\big)\,dx\,dW_s^r\right|\right]$$
$$\leqslant CE\left[\left(\int_0^T\left|\big\langle\!\big\langle\phi'\big(s,\cdot,u_k(s)\big),\ z_k(s)\big\rangle\!\big\rangle_{\mathbb{R}^n}-\big\langle\!\big\langle\phi'\big(s,\cdot,u(s)\big),\ z(s)\big\rangle\!\big\rangle_{\mathbb{R}^n}\right|^2ds\right)^{1/2}\right]$$

$$\leqslant CE\left[\left(\int_0^T \left(\left\|\delta u_k(s)\right\|^2_{L^2(\mathbb{R}^n)}\left\|z(s)\right\|^2_{L^2(\mathbb{R}^n)} + \left\|u_k(s)\right\|^2_{L^2(\mathbb{R}^n)}\left\|\delta z_k(s)\right\|^2_{L^2(\mathbb{R}^n)}\right)ds\right)^{1/2}\right]$$
$$\to 0 \quad \text{as } k\to\infty.$$

Hence taking limits in $L^1(\Omega\times[0,T],\mathscr{P})$ as $k\to\infty$ on both sides of (3.3) and noting the path-wise continuity of u, we have almost surely

$$\begin{aligned}\int_{\mathcal{O}^N}\phi\big(t,x,u(t,x)\big)\,dx &= \int_{\mathcal{O}^N}\phi\big(T,x,u(T,x)\big)\,dx - \int_t^T\int_{\mathcal{O}^N}\partial_s\phi\big(s,x,u(s,x)\big)\,dx\,ds \\ &\quad + \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u(s)\big), h^0(s)\big\rangle\!\big\rangle_{\mathcal{O}^N}\,ds \\ &\quad + \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u(s)\big), \partial_{x_i}h^i(s)\big\rangle\!\big\rangle_{\mathcal{O}^N}\,ds \\ &\quad - \frac{1}{2}\int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u(s)\big), \big|z(s)\big|^2\big\rangle\!\big\rangle_{\mathcal{O}^N}\,ds \\ &\quad - \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u(s)\big), z^r(s)\big\rangle\!\big\rangle_{\mathcal{O}^N}\,dW_s^r, \quad \forall t\in[0,T]. \end{aligned} \tag{3.4}$$

Passing to the limit in $L^1(\Omega\times[0,T],\mathscr{P})$ by letting $N\to\infty$ on both sides of (3.4), in view of the path-wise continuity of u and the integration-by-parts formula, we conclude (3.2).

Step 2. For the general $h^i\in W^2_{\mathscr{F}}(Q)$, we choose sequences $\{h^i_k\}$, $\{z^r_k\}$ and $\{u_k\}$ from $S^2(\mathbb{R})\otimes C_c^\infty(\mathcal{O})$ such that

$$\lim_{k\to\infty}\left\{\sum_{i=0}^n\left\|\delta h^i_k\right\|_{W^2_{\mathscr{F}}(Q)} + \|\delta z_k\|_{W^2_{\mathscr{F}}(Q)} + \|\delta u_k\|_{W^{1,2}_{\mathscr{F}}(Q)} + \left\|\delta u_k(0)\right\|_{L^2(\mathcal{O})}\right\} = 0.$$

Consider

$$\begin{aligned}\bar u(t,x) &= u(0,x) + \int_0^t\big(\Delta\bar u(s,x) + \partial_{x_i}\tilde h^i(s,x) - h^0(s,x)\big)\,ds \\ &\quad + \int_0^t z^r(s,x)\,dW_s^r, \quad t\in[0,T]\end{aligned} \tag{3.5}$$

with

$$\tilde h^i(s,x) := -\partial_{x_i}u(s,x) - h^i(s,x).$$

From Remark 2.1 and [5, Theorem 2.1], there are unique weak solutions $u \in \dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))$ to SPDE (3.5) in the sense of [5, Definition 1] or equivalently [7, Definition 4]), and $u^k \in \dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))$ to SPDE (3.5) with $u(0,x)$, $z(s,x)$ and $\tilde{h}^i(s,x)$ being replaced by $u_k(0,x)$, $z_k(s,x)$ and

$$\tilde{h}^i_k(s,x) := -\partial_{x_i} u_k(s,x) - h^i_k(s,x), \quad k = 1, 2, \ldots.$$

Then we deduce from [5, Propositions 6 and 7, and Theorem 9] that $u^k \in W^{2,2}_{\mathscr{F}}(Q) \cap \dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))$ and

$$\begin{aligned}
&\lim_{k\to\infty} \big\{ \|u^k - u\|_{W^{1,2}_{\mathscr{F}}(Q)} + \|u^k - u\|_{S^2(L^2(\mathcal{O}))} \big\} \\
&\quad \leqslant C \lim_{k\to\infty} \left\{ \|\delta u_k\|_{W^2_{\mathscr{F}}(Q)} + \|\delta z_k\|_{W^2_{\mathscr{F}}(Q)} + \|\delta u_k(0)\|_{L^2(\mathcal{O})} + \sum_{i=0}^n \|\delta h^i_k\|_{W^2_{\mathscr{F}}(Q)} \right\} = 0 \qquad (3.6)
\end{aligned}$$

with the constant C being independent of k. For each k, by Step 1 we have

$$\begin{aligned}
&\int_{\mathcal{O}} \phi\big(t,x,u^k(t,x)\big)\,dx \\
&\quad = \int_{\mathcal{O}} \phi\big(T,x,u^k(T,x)\big)\,dx - \int_t^T \int_{\mathcal{O}} \partial_s \phi\big(s,x,u^k(s,x)\big)\,dx\,ds \\
&\qquad + \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^k(s)\big), h^0_k(s) \big\rangle\!\big\rangle\,ds \\
&\qquad + \int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^k(s)\big)\partial_{x_i} u^k(s) + \partial_{x_i}\phi'\big(s,\cdot,u^k(s)\big), \partial_{x_i} u^k(s) \big\rangle\!\big\rangle\,ds \\
&\qquad + \int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^k(s)\big)\partial_{x_i} u^k(s) + \partial_{x_i}\phi'\big(s,\cdot,u^k(s)\big), \tilde{h}^i_k(s) \big\rangle\!\big\rangle\,ds \\
&\qquad - \frac{1}{2}\int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^k(s)\big), \big|z_k(s)\big|^2 \big\rangle\!\big\rangle\,ds - \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^k(s)\big), z^r_k(s) \big\rangle\!\big\rangle\,dW^r_s,
\end{aligned}$$

for all $t \in [0,T]$, P-a.s. By taking limits as $k \to \infty$, we complete our proof. □

Rewritting (3.1) into

$$u(t,x) = u(0,x) + \int_0^t \big(\Delta u(s,x) + \partial_{x_i}\tilde{h}^i(s,x) - h^0(s,x)\big)\,ds + \int_0^t z^r(s,x)\,dW^r_s$$

with

$$\tilde{h}^i(s,x) := -\partial_{x_i}u(s,x) - h^i(s,x),$$

we obtain

Lemma 3.4. *Let $\phi \in \mathcal{C}_M^{2,0}$ with $\phi'(s,x,r) \leqslant M$ for any $(s,x,r) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ and* (3.1) *hold in the weak sense of Definition* 2.2 *with $u(T) \in L^2(\Omega, \mathscr{F}_T, L^2(\mathcal{O}))$, $z \in W^2_{\mathscr{F}}(Q)$, $h^i \in W^2_{\mathscr{F}}(Q)$, $i = 1,\dots,n$ and $h^0 \in W^1_{\mathscr{F}}(Q)$. If $u \in \dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))$, then* (3.2) *holds almost surely.*

The proof is very similar to that of [6, Proposition 2] and is omitted here. The only difference lies in the fact that Lemma 3.3 instead of [7, Lemma 7] is used.

Through a standard procedure we obtain by Lemma 3.3 the following

Lemma 3.5. *Let $\phi \in \mathcal{C}_M^{2,0}$. If the function u in* (3.1) *belongs to $W^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(Q))$ with $u^+ \in \dot{W}^{1,2}_{\mathscr{F}}(Q)$, we have almost surely*

$$\begin{aligned}
&\int_{\mathcal{O}} \phi\big(t,x,u^+(t,x)\big)\,dx + \frac{1}{2}\int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^+(s)\big), \big|z^u(s)\big|^2 \big\rangle\!\big\rangle\,ds \\
&\quad= \int_{\mathcal{O}} \phi\big(T,x,u^+(T,x)\big)\,dx - \int_t^T\int_{\mathcal{O}} \partial_s\phi\big(s,x,u^+(s,x)\big)\,dx\,ds \\
&\qquad+ \int_t^T \big\langle\!\big\langle \phi\big(s,\cdot,u^+(s)\big), h^{0,u}(s)\big\rangle\!\big\rangle\,ds \\
&\qquad- \int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^+(s)\big)\partial_{x_i}u^+(s) + \partial_{x_i}\phi'\big(s,\cdot,u^+(s)\big), h^{i,u}(s)\big\rangle\!\big\rangle\,ds \\
&\qquad- \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^+(s)\big),\ z^{r,u}(s)\big\rangle\!\big\rangle\,dW_s^r, \quad t \in [0,T]
\end{aligned} \tag{3.7}$$

with

$$h^{i,u} := 1_{\{u>0\}}h^i, \quad i = 0,1,\dots,n$$

and

$$z^{r,u} = 1_{\{u>0\}}z^r, \quad r = 1,\dots,m; \qquad z^u := \big(z^{1,u},\dots,z^{m,u}\big).$$

Remark 3.2. Note that the assumption $u^+ \in \dot{W}^{1,2}_{\mathscr{F}}(Q)$ does not imply that u vanishes in a generalized sense on the boundary $\partial\mathcal{O}$ and therefore Lemma 3.3 cannot be applied directly to get the corresponding equation (3.1) for u^+.

Sketch of the proof. *Step* 1. For $k \in \mathbb{N}$, define

$$\psi(s) = \psi_k(s) := \begin{cases} 0, & s \in (-\infty, \frac{1}{k}]; \\ \frac{8k^2}{9}(s - \frac{1}{k})^3, & s \in (\frac{1}{k}, \frac{5}{4k}]; \\ \frac{2k}{3}(s - \frac{5}{4k})^2 + \frac{1}{6}(s - \frac{5}{4k}) + \frac{1}{72k}, & s \in (\frac{5}{4k}, \frac{7}{4k}]; \\ -\frac{8k^2}{9}(s - \frac{2}{k})^3 + s - \frac{3}{2k}, & s \in (\frac{7}{4k}, \frac{2}{k}]; \\ s - \frac{3}{2k}, & s \in (\frac{2}{k}, +\infty). \end{cases} \tag{3.8}$$

Then the assumptions on u^+ imply that $\psi(u) \in \dot{W}^{1,2}_{\mathscr{F}}(Q)$.

Take $\varphi \in C_c^\infty(\mathcal{O})$ and set $\mathscr{V} := \varphi u$. Then $\mathscr{V} \in \dot{W}^{1,2}_{\mathscr{F}}(Q)$. Since (3.1) holds in the weak sense of Definition 2.2, we have almost surely for any $\xi \in C_c^\infty(\mathcal{O})$

$$\langle\langle \xi,\ \varphi u(t) \rangle\rangle = \langle\langle \xi,\ \varphi u(T) \rangle\rangle + \int_t^T \langle\langle \xi, \varphi h^0(s) - \partial_{x_i}\varphi h^i(s) \rangle\rangle\, ds$$
$$- \int_t^T \langle\langle \partial_{x_i}\xi, \varphi h^i(s) \rangle\rangle\, ds - \int_t^T \langle\langle \xi, \varphi z^r(s) \rangle\rangle\, dW_s^r, \quad \forall t \in [0, T].$$

Hence, there holds

$$\mathscr{V}(t,x) = \mathscr{V}(T,x) + \int_t^T \bigl[\varphi(x)h^0(s,x) - \partial_{x_i}\varphi(x)h^i(s,x) + \partial_{x_i}\bigl(\varphi(x)h^i(s,x)\bigr)\bigr]\, ds$$
$$- \int_t^T \varphi(x)z^r(s,x)\, dW_s^r, \quad t \in [0, T]$$

in the weak sense of Definition 2.2.

For $\tilde{\varphi} \in C_c^\infty(O)$, by Lemma 3.3 we have almost surely

$$\langle\langle \psi\bigl(\mathscr{V}(t)\bigr), \tilde{\varphi} \rangle\rangle + \frac{1}{2}\int_t^T \langle\langle \psi''\bigl(\mathscr{V}(s)\bigr)\tilde{\varphi}, \bigl|\varphi z(s)\bigr|^2 \rangle\rangle\, ds$$
$$= \langle\langle \psi\bigl(\mathscr{V}(T)\bigr), \tilde{\varphi} \rangle\rangle + \int_t^T \langle\langle \psi'\bigl(\mathscr{V}(s)\bigr)\tilde{\varphi}, \varphi h^0(s) \rangle\rangle\, ds - \int_t^T \langle\langle \partial_{x_i}\bigl(\tilde{\varphi}\psi'\bigl(\mathscr{V}(s)\bigr)\varphi\bigr), h^i(s) \rangle\rangle\, ds$$
$$- \int_t^T \langle\langle \psi'\bigl(\mathscr{V}(s)\bigr)\tilde{\varphi}, \varphi z^r(s) \rangle\rangle\, dW_s^r, \quad \forall t \in [0, T]. \tag{3.9}$$

Choosing φ such that $\varphi \equiv 1$ in an open subset $\mathcal{O}' \Subset \mathcal{O}$ (i.e., $\overline{\mathcal{O}'} \subset \mathcal{O}$) and $\operatorname{supp}(\tilde{\varphi}) \subset \mathcal{O}'$, we have almost surely

$$\begin{aligned}
&\langle\!\langle \tilde{\varphi}, \psi\big(u(t)\big)\rangle\!\rangle + \frac{1}{2}\int_t^T \langle\!\langle \tilde{\varphi}, \psi''\big(u(s)\big)\big|z(s)\big|^2\rangle\!\rangle\, ds \\
&\quad = \langle\!\langle \tilde{\varphi}, \psi\big(u(T)\big)\rangle\!\rangle + \int_t^T \langle\!\langle \tilde{\varphi}, \psi'\big(u(s)\big)h^0(s)\rangle\!\rangle\, ds \\
&\qquad - \int_t^T \langle\!\langle \partial_{x_i}\big(\tilde{\varphi}\psi'\big(u(s)\big)\big), h^i(s)\rangle\!\rangle\, ds - \int_t^T \langle\!\langle \tilde{\varphi}, \psi'\big(u(s)\big)z^r(s)\rangle\!\rangle\, dW_s^r \\
&\quad = \langle\!\langle \tilde{\varphi}, \psi\big(u(T)\big)\rangle\!\rangle + \int_t^T \langle\!\langle \tilde{\varphi}, \psi'\big(u(s)\big)h^0(s)\rangle\!\rangle\, ds \\
&\qquad - \int_t^T \langle\!\langle \tilde{\varphi}, \psi'\big(u(s)\big)z^r(s)\rangle\!\rangle\, dW_s^r - \int_t^T \langle\!\langle \partial_{x_i}\tilde{\varphi}, \psi'\big(u(s)\big)h^i(s)\rangle\!\rangle\, ds \\
&\qquad - \int_t^T \langle\!\langle \tilde{\varphi}, \psi''\big(u(s)\big)\partial_{x_i}u(s)h^i(s)\rangle\!\rangle\, ds, \quad \forall t \in [0, T].
\end{aligned}$$

Since $\tilde{\varphi}$ is arbitrary, we have

$$\begin{aligned}
\psi\big(u(t,x)\big) &= \psi\big(u(T,x)\big) + \int_t^T \psi'\big(u(s,x)\big)h^0(s,x)\, ds - \frac{1}{2}\int_t^T \psi''\big(u(s,x)\big)\big|z(s,x)\big|^2\, ds \\
&\quad - \int_t^T \psi'\big(u(s,x)\big)z^r(s,x)\, dW_s^r - \int_t^T \psi''\big(u(s,x)\big)\partial_{x_i}u(s,x)h^i(s,x)\, ds \\
&\quad + \int_t^T \partial_{x_i}\big(\psi'\big(u(s,x)\big)h^i(s,x)\big)\, ds \qquad (3.10)
\end{aligned}$$

holds in the weak sense of Definition 2.2.

Step 2. It is sufficient to prove (3.7) holds for test functions ϕ of bounded first and second derivatives. Since (3.10) holds for $\psi = \psi_k$, $k = 1, 2, \ldots$, in view of Lemma 3.4 we obtain

$$\begin{aligned}
&\int_{\mathcal{O}} \phi\big(t, x, \psi_k\big(u(t,x)\big)\big)\, dx + \frac{1}{2}\int_t^T \langle\!\langle \phi''\big(s, \cdot, \psi_k\big(u(s)\big)\big), \big|\psi_k'\big(u(s)\big)z^u(s)\big|^2\rangle\!\rangle\, ds \\
&\quad = \int_{\mathcal{O}} \phi\big(T, x, \psi_k\big(u(T,x)\big)\big)\, dx + \int_t^T \langle\!\langle \phi'\big(s, \cdot, \psi_k\big(u(s)\big)\big), \psi_k'\big(u(s)\big)h^{0,u}(s)\rangle\!\rangle\, ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_t^T \int_{\mathcal{O}} \partial_s \phi\big(s, x, \psi_k\big(u(s,x)\big)\big)\, dx\, ds - \frac{1}{2} \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, \psi_k\big(u(s)\big)\big), \psi_k''\big(u(s)\big)\big|z(s)\big|^2 \big\rangle\!\big\rangle ds \\
&- \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, \psi_k\big(u(s)\big)\big), \psi_k''\big(u(s)\big)\partial_{x_i} u(s) h^i(s) \big\rangle\!\big\rangle ds \\
&- \int_t^T \big\langle\!\big\langle \phi''\big(s, \cdot, \psi_k\big(u(s)\big)\big)\psi_k'\big(u(s)\big)\partial_{x_i} u(s) + \partial_{x_i}\phi'\big(s, \cdot, \psi_k\big(u(s)\big)\big), \psi_k'\big(u(s)\big)h^{i,u}(s) \big\rangle\!\big\rangle ds \\
&- \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, \psi_k\big(u(s)\big)\big), \psi_k'\big(u(s)\big)z^{r,u}(s) \big\rangle\!\big\rangle dW_s^r
\end{aligned}
\tag{3.11}
$$

holds almost surely for all $t \in [0, T]$. From properties of ϕ, we have $\phi'(t, x, r) \leqslant M|r|$ for any $(t, x, r) \in [0, T] \times \mathcal{O} \times \mathbb{R}$. It follows that for any $(s, x) \in [0, T] \times \mathcal{O}$,

$$
\begin{aligned}
\big|\phi'\big(s, x, \psi_k\big(u(s,x)\big)\big)\psi_k''\big(u(s,x)\big)\big| &\leqslant M\big|\psi_k\big(u(s,x)\big)\big|\big|\psi_k''\big(u(s)\big)\big| \\
&\leqslant \frac{M}{2k} \times \frac{4k}{3} 1_{[\frac{1}{k}, \frac{2}{k}]}\big(u(s,x)\big) \\
&\leqslant M 1_{[\frac{1}{k}, \frac{2}{k}]}\big(u(s,x)\big).
\end{aligned}
\tag{3.12}
$$

On the other hand, we check that $\lim_{k\to\infty} \|\psi_k(u) - u^+\|_{W^{1,2}_{\mathscr{F}}(Q)} = 0$. Therefore, by the dominated convergence theorem and taking limits in $L^1([0, T] \times \Omega, \mathscr{P}, \mathbb{R})$ on both sides of (3.11), we prove our assertion. □

4. Solvability of Eq. (1.1)

Theorem 4.1. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *be satisfied and* $\{h^i,\ i = 0, 1, \ldots, n\} \subset \mathcal{M}^2(Q)$. *Then* $\dot{\mathscr{U}} \times \dot{\mathscr{V}}(G, f + h, g + h^0)$ *(with* $h := (h^1, \ldots, h^n)$*) admits one and only one element* (u, v) *which satisfies the following estimate*

$$
\|u\|_{\mathcal{V}_2(Q)} + \|v\|_{2;Q} \leqslant C\big\{\|G\|_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))} + A_p(f_0, g_0) + H_2\big(h, h^0\big)\big\},
\tag{4.1}
$$

where C *is a constant depending on* $n, p, q, \kappa, \lambda, \beta, \varrho, \Lambda_0, T, |\mathcal{O}|$ *and* L.

Proof. *Step* 1. Let $(\mathcal{A}3)_0$ be satisfied. From [21, Theorem 2.1], there is a unique weak solution $(u, v) \in (\dot{W}^{1,2}_{\mathscr{F}}(Q) \cap S^2(L^2(\mathcal{O}))) \times W^2_{\mathscr{F}}(Q)$ to Eq. (1.1).

We have

$$
\textit{Claim } (*)\text{:} \quad (u, v) \in \dot{\mathscr{U}} \times \dot{\mathscr{V}}\big(G, f + h, g + h^0\big)
$$

which will be proved in Step 2.

By Lemma 3.3, we have almost surely

$$\begin{aligned}
&\|u(t)\|^2_{L^2(\mathcal{O})}+\int_t^T\|v(s)\|^2_{L^2(\mathcal{O})}\,ds\\
&\quad=\|G\|^2_{L^2(\mathcal{O})}+2\int_t^T\langle\!\langle u(s),b^i\partial_{x_i}u(s)+cu(s)+\varsigma^r v^r(s)+h^0(s)\rangle\!\rangle\,ds\\
&\qquad-2\int_t^T\langle\!\langle\partial_{x_j}u(s),a^{ij}\partial_{x_i}u(s)+\sigma^{jr}v^r(s)+f^j\big(s,\cdot,u(s),\nabla u(s),v(s)\big)+h^j(s)\rangle\!\rangle\,ds\\
&\qquad-2\int_t^T\langle\!\langle u(s),v^r(s)\rangle\!\rangle\,dW^r_s+2\int_t^T\langle\!\langle u(s),g\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\rangle\!\rangle\,ds
\end{aligned}\tag{4.2}$$

for all $t\in[0,T]$. Therefore, we obtain that almost surely

$$\begin{aligned}
&E\bigg[\|u(t)\|^2_{L^2(\mathcal{O})}+\int_t^T\|v(s)\|^2_{L^2(\mathcal{O})}\,ds|\mathscr{F}_t\bigg]\\
&\quad=E\big[\|G\|^2_{L^2(\mathcal{O})}|\mathscr{F}_t\big]+2E\bigg[\int_t^T\langle\!\langle u(s),g\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\bigg]\\
&\qquad+2E\bigg[\int_t^T\langle\!\langle u(s),b^i\partial_{x_i}u(s)+cu(s)+\varsigma^r v^r(s)+h^0(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\bigg]\\
&\qquad-2E\bigg[\int_t^T\langle\!\langle\partial_{x_j}u(s),a^{ij}\partial_{x_i}u(s)+\sigma^{jr}v^r(s)\\
&\qquad+f^j\big(s,\cdot,u(s),\nabla u(s),v(s)\big)+h^j(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\bigg],\quad\forall t\in[0,T].
\end{aligned}\tag{4.3}$$

Using the Lipschitz condition and Hölder inequality, we have the three estimates:

$$\begin{aligned}
&2E\bigg[\int_t^T\big(\langle\!\langle u(s),h^0(s)\rangle\!\rangle-\langle\!\langle\partial_{x_j}u(s),h^j(s)\rangle\!\rangle\big)\,ds\Big|\mathscr{F}_t\bigg]\\
&\quad\leqslant E\bigg[\int_t^T\big(\|u(s)\|^2_{L^2(\mathcal{O})}+\|h^0(s)\|^2_{L^2(\mathcal{O})}+\varepsilon^{-1}\|h(s)\|^2_{L^2(\mathcal{O})}+\varepsilon\|\nabla u(s)\|^2_{L^2(\mathcal{O})}\big)\,ds\Big|\mathscr{F}_t\bigg],
\end{aligned}\tag{4.4}$$

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle u(s),\ g\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\quad\leqslant \operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle u(s),\ g_0(s)+L\big(|u(s)|+|\nabla u(s)|+|v(s)|\big)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+C(\varepsilon,\varepsilon_1,L)\|u\|_{2;\mathcal{O}_t}^2\\
&\qquad+\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle |u(s)|,\ |g_0(s)|\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+C(\varepsilon,\varepsilon_1,L)\|u\|_{2;\mathcal{O}_t}^2+2|\mathcal{O}_t|^{\frac12-\frac1p}\|g_0\|_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}\|u\|_{\frac{2(n+2)}{n};\mathcal{O}_t}\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+C(\varepsilon,\varepsilon_1,L)\|u\|_{2;\mathcal{O}_t}^2+c(n)|\mathcal{O}_t|^{\frac12-\frac1p}\|g_0\|_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}\|u\|_{\mathcal{V}_2(\mathcal{O}_t)}\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+C(\varepsilon,\varepsilon_1,L)\|u\|_{2;\mathcal{O}_t}^2+\delta\|u\|_{\mathcal{V}_2(\mathcal{O}_t)}^2\\
&\qquad+C\big(\delta,n,p,|Q|\big)\|g_0\|_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}^2
\end{aligned}\tag{4.5}
$$

and

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle u(s), b^i(s)\partial_{x_i}u(s)+c(s)\,u(s)+\varsigma^r(s)v^r(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\quad\leqslant \big(\varepsilon^{-1}+\varepsilon_1^{-1}\big)\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} E\left[\int_\tau^T \langle\!\langle |b(s)|^2+|c(s)|+|\varsigma(s)|^2, u^2(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\qquad+\varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+\big(\varepsilon^{-1}+\varepsilon_1^{-1}\big)B_q(b,c,\varsigma)\|u\|_{\frac{2q}{q-1};\mathcal{O}_t}^2\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+\big(\varepsilon^{-1}+\varepsilon_1^{-1}\big)B_q(b,c,\varsigma)\|u\|_{\frac{2(n+2)}{n};\mathcal{O}_t}^{2\alpha}\|u\|_{2;\mathcal{O}_t}^{2(1-\alpha)}\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2\\
&\qquad+\big(\varepsilon^{-1}+\varepsilon_1^{-1}\big)B_q(b,c,\varsigma)\big(C(n)\|u\|_{\mathcal{V}_2(\mathcal{O}_t)}\big)^{2\alpha}\|u\|_{2;\mathcal{O}_t}^{2(1-\alpha)}\quad\text{(by Lemma 3.1)}\\
&\quad\leqslant \varepsilon\|\nabla u\|_{2;\mathcal{O}_t}^2+\varepsilon_1\|v\|_{2;\mathcal{O}_t}^2+\delta\|u\|_{\mathcal{V}_2(\mathcal{O}_t)}^2\\
&\qquad+C(\delta,n,q)\big|\big(\varepsilon^{-1}+\varepsilon_1^{-1}\big)B_q(b,c,\varsigma)\big|^{\frac{1}{1-\alpha}}\|u\|_{2;\mathcal{O}_t}^2
\end{aligned}\tag{4.6}
$$

with $\alpha := \frac{n+2}{2q} \in (0,1)$ and the three positive small parameters ε, ε_1 and δ waiting to be determined later. Also, there exists a constant $\theta > \varrho' = \frac{\varrho}{\varrho-1}$ such that $\lambda-\kappa-\beta\theta > 0$ and

$$
\begin{aligned}
&-E\left[\int_t^T 2\langle\!\langle \partial_{x_j}u(s), a^{ij}\partial_{x_i}u(s)+\sigma^{jr}v^r(s)+f^j\big(s,u(s),\nabla u(s),v(s)\big)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right]\\
&\quad\leqslant -E\left[\int_t^T \langle\!\langle \partial_{x_j}u(s),\big(2a^{ij}(s)-\varrho\sigma^{jr}(s)\sigma^{ir}(s)\big)\partial_{x_i}u(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right]\\
&\qquad+\frac{1}{\varrho}E\left[\int_t^T \big\|v(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\qquad+2E\left[\int_t^T \langle\!\langle \big|\nabla u(s)\big|, L\big|u(s)\big|+\frac{\kappa}{2}\big|\nabla u(s)\big|+\beta^{\frac12}\big|v(s)\big|+\big|f_0(s)\big|\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right]\\
&\quad\leqslant -(\lambda-\kappa-\beta\theta-\varepsilon)E\left[\int_t^T \big\|\nabla u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]+C(\varepsilon)\|f_0\|^2_{2;\mathcal{O}_t}\\
&\qquad+\left(\frac{1}{\varrho}+\frac{1}{\theta}\right)E\left[\int_t^T \big\|v(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]+C(\varepsilon,L)E\left[\int_t^T \big\|u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad\leqslant -(\lambda-\kappa-\beta\theta-\varepsilon)E\left[\int_t^T \big\|\nabla u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\qquad+C\big(\varepsilon,|Q|,p,L\big)\left\{E\left[\int_t^T \big\|u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]+\|f_0\|^2_{p;\mathcal{O}_t}\right\}\\
&\qquad+\left(\frac{1}{\varrho}+\frac{1}{\theta}\right)E\left[\int_t^T \big\|v(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right],\quad \forall t\in[0,T]\ \text{a.s.}
\end{aligned}
\tag{4.7}
$$

Choosing ε and ε_1 to be small enough, we get

$$
\begin{aligned}
&\|u\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\|v\|^2_{2;\mathcal{O}_t}\\
&\quad\leqslant 3\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]}\left\{\big\|u(\tau)\big\|^2_{L^2(\mathcal{O})}+E\left[\int_\tau^T\big(\big\|\nabla u(s)\big\|^2_{L^2(\mathcal{O})}+\big\|v(s)\big\|^2_{L^2(\mathcal{O})}\big)\,ds\Big|\mathscr{F}_\tau\right]\right\}\\
&\quad\leqslant C_1\bigg\{\|G\|^2_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))}+\|f_0\|^2_{p;\mathcal{O}_t}+\big|H_2\big(h,h^0\big)\big|^2\\
&\qquad+\delta\|u\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+C(\delta,n,q,\Lambda_0)\int_t^T\|u\|^2_{\mathcal{V}_2(\mathcal{O}_s)}\,ds+C\big(\delta,n,p,|Q|\big)\|g_0\|^2_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}\bigg\}
\end{aligned}
$$

with the constant C_1 being independent of δ. Then by choosing δ to be so small that $C_1\delta < 1/2$, we obtain

$$\|u\|^2_{\mathcal{V}_2(\mathcal{O}_t)} + \|v\|^2_{2;\mathcal{O}_t} \leqslant C\left\{\|G\|^2_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))} + \int_t^T \|u\|^2_{\mathcal{V}_2(\mathcal{O}_s)}\,ds + \left|A_p(f_0,g_0)\right|^2 + \left|H_2\big(h,h^0\big)\right|^2\right\}. \tag{4.8}$$

Thus, it follows from Gronwall inequality that

$$\|u\|^2_{\mathcal{V}_2(\mathcal{O}_t)} + \|v\|^2_{2;\mathcal{O}_t} \leqslant C\big\{\|G\|^2_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))} + \left|A_p(f_0,g_0)\right|^2 + \left|H_2\big(h,h^0\big)\right|^2\big\} \tag{4.9}$$

with the constant C depending on $T, L, \Lambda_0, \lambda, \beta, \kappa, \varrho, n, p, q$ and $|Q|$.

Step 2. We prove *Claim* $(*)$. It is sufficient to prove $(u,v) \in \dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$. Like (4.4) and (4.7), we have

$$\begin{aligned}
&\big\|u(t)\big\|^2_{L^2(\mathcal{O})} + E\left[\int_t^T \big\|v(s)\big\|^2_{L^2(\mathcal{O})}\,ds|\mathscr{F}_t\right] \\
&= E\big[\|G\|^2_{L^2(\mathcal{O})}|\mathscr{F}_t\big] + 2E\left[\int_t^T \big\langle\!\big\langle u(s), g\big(s,\cdot,u(s),\nabla u(s),v(s)\big)\big\rangle\!\big\rangle\,ds\big|\mathscr{F}_t\right] \\
&\quad + 2E\left[\int_t^T \big\langle\!\big\langle u(s), b^i\partial_{x_i}u(s) + cu(s) + \varsigma^r v^r(s) + h^0(s)\big\rangle\!\big\rangle\,ds\big|\mathscr{F}_t\right] \\
&\quad - 2E\left[\int_t^T \big\langle\!\big\langle \partial_{x_j}u(s), a^{ij}\partial_{x_i}u(s) + \sigma^{jr}v^r(s)\right. \\
&\quad \left. + f^j\big(s,\cdot,u(s),\nabla u(s),v(s)\big) + h^j(s)\big\rangle\!\big\rangle\,ds\big|\mathscr{F}_t\right] \\
&\leqslant -(\lambda-\kappa-\beta\theta-\varepsilon)E\left[\int_t^T \big\|\nabla u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\big|\mathscr{F}_t\right] \\
&\quad + \left(\frac{1}{\varrho}+\frac{1}{\theta}+\varepsilon\right)E\left[\int_t^T \big\|v(s)\big\|^2_{L^2(\mathcal{O})}\,ds\big|\mathscr{F}_t\right] \\
&\quad + E\big[\|G\|^2_{L^2(\mathcal{O})}|\mathscr{F}_t\big] + C(\varepsilon)\big(\left|H_2(f_0,g_0)\right|^2 + \left|H_2\big(h,h^0\big)\right|^2\big) \\
&\quad + C\big(\varepsilon,\lambda,\beta,\kappa,\varrho,L,|\!|\!|b|\!|\!|_{\mathcal{L}^\infty(Q)},\|c\|_{\mathcal{L}^\infty(Q)},|\!|\!|\varsigma|\!|\!|_{\mathcal{L}^\infty(Q)}\big)E\left[\int_t^T \big\|u(s)\big\|^2_{L^2(\mathcal{O})}\,ds\big|\mathscr{F}_t\right]
\end{aligned} \tag{4.10}$$

with the positive constant ε waiting to be determined later. Letting ε be small enough, we have almost surely

$$\|u(t)\|_{L^2(\mathcal{O})}^2 + E\left[\int_t^T \left(\|\nabla u(s)\|_{L^2(\mathcal{O})}^2 + \|v(s)\|_{L^2(\mathcal{O})}^2\right) ds \Big| \mathscr{F}_t\right]$$

$$\leqslant C\left\{\|G\|_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))}^2 + \left|H_2(f_0, g_0)\right|^2 + \left|H_2\left(h, h^0\right)\right|^2 + E\left[\int_t^T \|u(s)\|_{L^2(\mathcal{O})}^2 ds \Big| \mathscr{F}_t\right]\right\}$$

for all $t \in [0, T]$. Then, by Gronwall inequality we obtain

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[0,T]} \left\{\|u(t)\|_{L^2(\mathcal{O})}^2 + E\left[\int_t^T \left(\|\nabla u(s)\|_{L^2(\mathcal{O})}^2 + \|v(s)\|_{L^2(\mathcal{O})}^2\right) ds \Big| \mathscr{F}_t\right]\right\}$$

$$\leqslant C\left\{\|G\|_{L^\infty(\Omega,\mathscr{F}_T,L^2(\mathcal{O}))}^2 + \left|H_2(f_0, g_0)\right|^2 + \left|H_2\left(h, h^0\right)\right|^2\right\}$$

with the constant C depending on $\lambda, \beta, \kappa, \varrho, L, T, |||b|||_{\mathcal{L}^\infty(Q)}, \|c\|_{\mathcal{L}^\infty(Q)}, |||\varsigma|||_{\mathcal{L}^\infty(Q)}$. Hence, $(u, v) \in \dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$. We complete the proof of *Claim* $(*)$.

Step 3. Now we consider the general case of assumption $(\mathcal{A}3)$. The existence of the solution can be shown by approximation. As $p > n + 2$ and $\mathcal{M}^p(Q) \subset \mathcal{M}^2(Q)$, $f_0 \in \mathcal{M}^2(Q)$. We approximate the functions b, c, ς and g by

$$b_k := b1_{\{|b|\leqslant k\}}, \quad c_k := c1_{\{|c|\leqslant k\}}, \quad \varsigma_k := \varsigma 1_{\{|\varsigma|\leqslant k\}} \quad \text{and} \quad g^k := g - g_0 + g_0^k, \tag{4.11}$$

with $g_0^k = g_0 1_{\{|g_0|\leqslant k\}}$. Then we have

$$\lim_{k\to\infty} B_q(b - b_k, c - c_k, \varsigma - \varsigma_k) + A_p\left(0, g_0 - g_0^k\right) = 0.$$

Let $(u_k, v_k) \in \dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$ be the unique weak solution to (1.1) with (b, c, ς, f, g) being replaced by $(b_k, c_k, \varsigma_k, f + h, g^k + h^0)$. Then by estimate (4.9), there exists a positive constant C_0 such that

$$\sup_{k\in\mathbb{N}} \left\{\|u_k\|_{\mathcal{V}_2(Q)}^2 + \|v_k\|_{2;Q}^2\right\} < C_0.$$

For $k, l \in \mathbb{N}$, the pair of random fields $(u_{kl}, v_{kl}) := (u_k - u_l, v_k - v_l) \in \dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$ is the weak solution to the following BSPDE:

$$(k,l)\quad \begin{cases} -du_{kl}(t,x) = \big[\partial_{x_j}\big(a^{ij}(t,x)\partial_{x_i}u_{kl}(t,x) + \sigma^{jr}(t,x)v^r_{kl}(t,x)\big) + b^j_k(t,x)\partial_{x_j}u_{kl}(t,x) \\ \qquad + c_k(t,x)u_{kl}(t,x) + \varsigma^r_k(t,x)v^r_{kl}(t,x) \\ \qquad + b^j_{kl}(t,x)\partial_{x_j}u_l(t,x) + c_{kl}(t,x)u_l(t,x) + \varsigma^r_{kl}(t,x)v^r_l(t,x) \\ \qquad + \bar{g}_{kl}\big(t,x,u_{kl}(t,x),\nabla u_{kl}(t,x),v_{kl}(t,x)\big) \\ \qquad + \partial_{x_j}\bar{f}^j_{kl}\big(t,x,u_{kl}(t,x),\nabla u_{kl}(t,x),v_{kl}(t,x)\big)\big]\,dt \\ \qquad - v^r_{kl}(t,x)\,dW^r_t, \quad (t,x)\in Q := [0,T]\times\mathcal{O}; \\ u_{kl}(T,x) = 0, \quad x\in\mathcal{O} \end{cases}$$

with

$$\begin{aligned} \bar{f}_{kl}(t,x,R,Y,Z) &:= f\big(t,x,R+u_l(t,x),Y+\nabla u_l(t,x),Z+v_l(t,x)\big) \\ &\quad - f\big(t,x,u_l(t,x),\nabla u_l(t,x),v_l(t,x)\big), \\ \bar{g}_{kl}(t,x,R,Y,Z) &:= g^k\big(t,x,R+u_l(t,x),Y+\nabla u_l(t,x),Z+v_l(t,x)\big) \\ &\quad - g^l\big(t,x,u_l(t,x),\nabla u_l(t,x),v_l(t,x)\big), \\ (b_{kl},c_{kl},\varsigma_{kl})(t,x) &:= (b_k-b_l,c_k-c_l,\varsigma_k-\varsigma_l)(t,x). \end{aligned}$$

Since

$$\begin{aligned} &\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \big\langle\!\big\langle u_{kl}(s), b^i_{kl}\partial_{x_i}u_l(s) + c_{kl}\,u_l(s) + \varsigma^r_{kl}v^r_l(s)\big\rangle\!\big\rangle\,ds\,\big|\mathscr{F}_\tau\right] \\ &\quad\leqslant 2\bar{\varepsilon}^{-1}\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} E\left[\int_\tau^T \big\langle\!\big\langle \big|b_{kl}(s)\big|^2 + \big|c_{kl}(s)\big| + \big|\varsigma_{kl}(s)\big|^2, u^2_{kl}(s)\big\rangle\!\big\rangle\,ds\,\big|\mathscr{F}_\tau\right] \\ &\qquad + \bar{\varepsilon}\big(\|\nabla u_l\|^2_{2;\mathcal{O}_t} + \|v_l\|^2_{2;\mathcal{O}_t}\big) \\ &\quad\leqslant \bar{\varepsilon}\big(\|u_l\|^2_{\mathcal{V}_2(Q)} + \|v_l\|^2_{2;Q}\big) + 2\bar{\varepsilon}^{-1}B_q(b_{kl},c_{kl},\varsigma_{kl})\|u_{kl}\|^2_{\frac{2q}{q-1};\mathcal{O}_t} \\ &\quad\leqslant \bar{\varepsilon}C_0 + \delta\|u_{kl}\|^2_{\mathcal{V}_2(\mathcal{O}_t)} + C(\delta,n,q)\big|\bar{\varepsilon}^{-1}B_q(b_{kl},c_{kl},\varsigma_{kl})\big|^{\frac{2q}{2q-n-2}}\|u_{kl}\|^2_{2;\mathcal{O}_t} \quad \text{(by Lemma 3.1)}, \end{aligned}$$

in a similar way to the derivation of (4.8), we obtain

$$\begin{aligned} &\|u_{kl}\|^2_{\mathcal{V}_2(\mathcal{O}_t)} + \|v_{kl}\|^2_{2;\mathcal{O}_t} \\ &\quad\leqslant C\left\{\bar{\varepsilon} + \big|A_p\big(0,g^k_0-g^l_0\big)\big|^2 + \big(1+\big|\bar{\varepsilon}^{-1}B_q(b_{kl},c_{kl},\varsigma_{kl})\big|^{\frac{2q}{2q-n-2}}\big)\int_t^T \|u_{kl}\|^2_{\mathcal{V}_2(\mathcal{O}_s)}\,ds\right\} \end{aligned}$$

which, by Gronwall inequality, implies

$$\begin{aligned} &\|u_{kl}\|^2_{\mathcal{V}_2(Q)} + \|v_{kl}\|^2_{2;Q} \\ &\quad\leqslant C\big(\bar{\varepsilon} + \big|A_p\big(0,g^k_0-g^l_0\big)\big|^2\big)\exp\big[T\big(1+\big|\bar{\varepsilon}^{-1}B_q(b_{kl},c_{kl},\varsigma_{kl})\big|^{\frac{2q}{2q-n-2}}\big)\big] \end{aligned} \tag{4.12}$$

with the constant C being independent of k, l and $\bar{\varepsilon}$. By choosing $\bar{\varepsilon}$ to be small and then k and l to be sufficiently large, we conclude that (u_k, v_k) is a Cauchy sequence in $\dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$. Passing to the limit, we check that the limit $(u, v) \in \dot{\mathscr{U}} \times \dot{\mathscr{V}}(G, f + h, g + h^0)$. In view of estimate (4.9) we prove estimate (4.1).

Step 4. It remains to prove the uniqueness. Assume that (u', v') and (u, v) are two weak solutions in $\dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$. Then their difference $(\bar{u}, \bar{v}) := (u - u', v - v') \in \dot{\mathscr{U}} \times \dot{\mathscr{V}}(0, \bar{f}, \bar{g})$ with

$$\begin{aligned}
\bar{f}(t, x, R, Y, Z) &:= f\big(t, x, R + u'(t, x), Y + \nabla u'(t, x), Z + v'(t, x)\big) \\
&\quad - f\big(t, x, u'(t, x), \nabla u'(t, x), v'(t, x)\big), \\
\bar{g}(t, x, R, Y, Z) &:= g\big(t, x, R + u'(t, x), Y + \nabla u'(t, x), Z + v'(t, x)\big) \\
&\quad - g\big(t, x, u'(t, x), \nabla u'(t, x), v'(t, x)\big).
\end{aligned}$$

Since $\bar{f}_0 = 0$, $\bar{g}_0 = 0$ and $\bar{u}(T) = 0$, we deduce from (4.9) that $\bar{u} = 0$ and $\bar{v} = 0$. The proof is complete. □

Remark 4.1. On the basis of the monotone operator theory, Qiu and Tang [21] established a theory of solvability for quasi-linear BSPDEs in an abstract framework. However even for the linear case $(f, g) \equiv (f_0, g_0)$, assumptions $(\mathcal{A}1)$–$(\mathcal{A}3)$ go beyond the framework of Qiu and Tang [21] since our b, c, and ς may be unbounded.

Corollary 4.2. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *be true,* $\{h^i,\ i = 0, 1, \ldots, n\} \subset \mathcal{M}^2(Q)$ *and* $(u, v) \in \dot{\mathscr{U}} \times \dot{\mathscr{V}}(G, f + h, g + h^0)$ *with* $h := (h^1, \ldots, h^n)$. *Let* $\phi \in \mathcal{C}^{2,0}_M$. *Then we have almost surely*

$$\begin{aligned}
&\int_{\mathcal{O}} \phi\big(t, x, u(t, x)\big)\, dx + \frac{1}{2} \int_t^T \big\langle\!\big\langle \phi''\big(s, \cdot, u(s)\big), \big|v(s)\big|^2 \big\rangle\!\big\rangle\, ds \\
&\quad = \int_{\mathcal{O}} \phi\big(T, x, G(x)\big)\, dx - \int_t^T \int_{\mathcal{O}} \partial_s \phi\big(s, x, u(s, x)\big)\, dx\, ds \\
&\qquad + \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u(s)\big), b^i \partial_{x_i} u(s) + cu(s) + \varsigma^r v^r(s) + h^0(s) \big\rangle\!\big\rangle\, ds \\
&\qquad + \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u(s)\big), g\big(s, \cdot, u(s), \nabla u(s), v(s)\big) \big\rangle\!\big\rangle\, ds \\
&\qquad - \int_t^T \big\langle\!\big\langle \phi''\big(s, \cdot, u(s)\big) \partial_{x_i} u(s) + \partial_{x_i} \phi'\big(s, \cdot, u(s)\big), a^{ji} \partial_{x_j} u(s) + \sigma^{ri} v^r(s) \\
&\qquad + f^i\big(s, \cdot, u(s), \nabla u(s), v(s)\big) + h^i(s) \big\rangle\!\big\rangle\, ds \\
&\qquad - \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u(s)\big), v^r(s) \big\rangle\!\big\rangle\, dW^r_s, \quad \forall t \in [0, T]. \qquad (4.13)
\end{aligned}$$

The proof of Corollary 4.2 is rather standard and is sketched below.

Sketch of the proof. First, one can check that all the terms involved in our assertion is well defined. Similar to the proof of Theorem 4.1, we still approximate (b, c, ς, g) by $(b_k, c_k, \varsigma_k, g^k)$ which is defined in (4.11). By Theorem 4.1, there is a unique weak solution $(u_k, v_k) \in \dot{\mathcal{V}}_{2,0}(Q) \times \mathcal{M}^2(Q)$ to BSPDE (1.1) with (b, c, ς, f, g) being replaced by $(b_k, c_k, \varsigma_k, f + h, g^k + h^0)$. Then by Lemma 3.3, we have for each $k \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathcal{O}} \phi\big(t, x, u_k(t, x)\big)\, dx + \frac{1}{2}\int_t^T \big\langle\!\big\langle \phi''\big(s, \cdot, u_k(s)\big), \big|v_k(s)\big|^2 \big\rangle\!\big\rangle\, ds \\
&= \int_{\mathcal{O}} \phi\big(T, x, G(x)\big)\, dx - \int_t^T \int_{\mathcal{O}} \partial_s \phi\big(s, x, u_k(s, x)\big)\, dx\, ds \\
&\quad + \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u_k(s)\big), b_k^i \partial_{x_i} u_k(s) + c_k\, u_k(s) + \varsigma_k^r v_k^r(s) + h^0(s) \big\rangle\!\big\rangle\, ds \\
&\quad + \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u_k(s)\big), g^k\big(s, \cdot, u_k(s), \nabla u_k(s), v_k(s)\big) \big\rangle\!\big\rangle\, ds \\
&\quad - \int_t^T \big\langle\!\big\langle \phi'\big(s, \cdot, u_k(s)\big), v_k^r(s) \big\rangle\!\big\rangle\, dW_s^r \\
&\quad - \int_t^T \big\langle\!\big\langle \phi''\big(s, \cdot, u_k(s)\big)\partial_{x_i} u_k(s) + \partial_{x_i}\phi'\big(s, \cdot, u_k(s)\big), a^{ji}\partial_{x_j} u_k(s) + \sigma^{ri} v_k^r(s) \\
&\quad + f^i\big(s, \cdot, u_k(s), \nabla u_k(s), v_k(s)\big) + h^i(s) \big\rangle\!\big\rangle\, ds
\end{aligned} \tag{4.14}$$

almost surely for all $t \in [0, T]$. On the other hand, from the proof of Theorem 4.1 it follows that

$$\lim_{k\to\infty} \big\{ \|u - u_k\|_{\mathcal{V}_2(Q)} + \|v - v_k\|_{2;Q} \big\} = 0.$$

Hence passing to the limit in $L^1(\Omega, \mathscr{F})$ and taking into account the path-wise continuity of u, we prove our assertion. □

We have

Proposition 4.3. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *be satisfied,* $\{h^i,\ i = 0, 1, \ldots, n\} \subset \mathcal{M}^2(Q)$ *and* $(u, v) \in \mathscr{U} \times \mathscr{V}(G, f + h, g + h^0)$ *with* $h := (h^1, \ldots, h^n)$ *and* $u^+ \in \dot{\mathcal{V}}_{2,0}(Q)$. *Let* $\phi \in \mathcal{C}_M^{2,0}$. *Then, with probability* 1, *the following relation*

$$\int_{\mathcal{O}} \phi\big(t,x,u^+(t,x)\big)\,dx + \frac{1}{2}\int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^+(s)\big), \big|v^u(s)\big|^2 \big\rangle\!\big\rangle\,ds$$

$$= \int_{\mathcal{O}} \phi\big(T,x,G^+(x)\big)\,dx - \int_t^T\int_{\mathcal{O}} \partial_s\phi\big(s,x,u^+(s,x)\big)\,dx\,ds$$

$$- \int_t^T \big\langle\!\big\langle \phi''\big(s,\cdot,u^+(s)\big)\partial_{x_i}u^+(s) + \partial_{x_i}\phi'\big(s,\cdot,u^+(s)\big), a^{ji}(s)\partial_{x_j}u^+(s)$$

$$+ \sigma^{ri}(s)v^{r,u}(s) + f^{i,u}(s)\big\rangle\!\big\rangle\,ds$$

$$+ \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^+(s)\big), b^i(s)\partial_{x_i}u^+(s) + c(s)u^+(s) + \varsigma^r(s)v^{r,u}(s)\big\rangle\!\big\rangle\,ds$$

$$+ \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^+(s)\big), g^u(s)\big\rangle\!\big\rangle\,ds - \int_t^T \big\langle\!\big\langle \phi'\big(s,\cdot,u^+(s)\big), v^{r,u}(s)\big\rangle\!\big\rangle\,dW_s^r$$

holds almost surely for all $t \in [0,T]$ *where*

$$g^u(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)\big(h^0(s,x) + g\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big)\big);$$
$$f^{i,u}(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)\big(h^i(s,x) + f^i\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big)\big),$$
$$i = 0,1,\ldots,n;$$

and

$$v^u := \big(v^{1,u},\ldots,v^{m,u}\big), \qquad v^{r,u}(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)v^r(s,x), \quad r = 1,\ldots,m.$$

The proof is very similar to that of Lemma 3.5 and is omitted here. The main difference lies in Step 1 where we use Corollary 4.2 instead of Lemma 3.3.

5. The maximum principles

5.1. The global case

Theorem 5.1. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}4)$ *hold. Assume that* $(u,v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^2(Q)$ *is a weak solution of* (1.1)*. Then we have*

$$\operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times Q} u^{\pm}(\omega,t,x)$$
$$\leqslant C\Big\{ \operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^{\pm}(\omega,t,x) + A_p\big(f_0, g_0^{\pm}\big) + \big\|u^{\pm}\big\|_{2;Q}\Big\} \qquad (*)^{\pm}$$

where C *is a constant depending on* $n, p, q, \kappa, \lambda, \beta, \varrho, \Lambda_0, L_0, T, |\mathcal{O}|$ *and* L.

Remark 5.1. By the inequality $\operatorname{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_p Q} u^+(\omega,t,x) \leqslant L_1$, we mean that $(u - L_1)^+ \in \dot{\mathcal{V}}_{2,0}(Q)$ and with probability 1, for any $\zeta \in C_c^\infty(\mathcal{O})$, there holds

$$\lim_{t\to T_-} \langle\!\langle \zeta, \big(u(t) - L_1\big)^+ \rangle\!\rangle = 0.$$

Remark 5.2. In Theorem 5.1, assume further that

$$\operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_p Q} \big|u(\omega,t,x)\big| \leqslant L_1 < \infty.$$

We have $u \in \mathcal{L}^\infty(Q)$ and

$$\|u\|_{\infty;Q} \leqslant C\big\{L_1 + A_p(f_0, g_0) + \|u\|_{2;Q}\big\} \tag{5.1}$$

where C is a constant depending on $n, p, q, \kappa, \lambda, \beta, \varrho, \Lambda_0, L_0, T, |\mathcal{O}|$ and L.

We start the proof of Theorem 5.1 with borrowing the following lemma either from [4, Lemma 1.2, Chapter 6] or from [15, Lemma 5.6, Chapter 2].

Lemma 5.2. *Let $\{a_k\colon\ k = 0, 1, 2, \ldots\}$ be a sequence of nonnegative numbers satisfying*

$$a_{k+1} \leqslant C_0 b^k a_k^{1+\delta}, \quad k = 0, 1, 2, \ldots$$

where $b > 1$, $\delta > 0$ and C_0 is a positive constant. Then if

$$a_0 \leqslant \theta_0 := C_0^{-\frac{1}{\delta}} b^{-\frac{1}{\delta^2}},$$

we have $\lim_{k\to\infty} a_k = 0$.

Sketch of the proof. We use the induction principle. It is sufficient to prove the following assertion:

$$a_k \leqslant \frac{\theta_0}{\nu^k}, \quad k = 0, 1, 2, \ldots, \tag{5.2}$$

with the parameter $\nu > 1$ waiting to be determined later. It is obvious for $k = 0$ that (5.2) holds. Assume that (5.2) holds for $k = r$. Then we have

$$a_{r+1} \leqslant C_0 b^r a_r^{1+\delta} \leqslant C_0 b^r \left(\frac{\theta_0}{\nu^r}\right)^{1+\delta} = \frac{\theta_0}{\nu^{r+1}} \cdot \frac{C_0 b^r \theta_0^\delta}{\nu^{r\delta - 1}}.$$

Taking $\nu = b^{\frac{1}{\delta}} > 1$, we obtain

$$a_{r+1} \leqslant \frac{\theta_0}{\nu^{r+1}} \cdot C_0 \nu \theta_0^\delta = \frac{\theta_0}{\nu^{r+1}}. \qquad \square$$

Corollary 5.3. *Let $\phi : [r_0, \infty) \to \mathbb{R}^+$ be a nonnegative and decreasing function. Moreover, there exist constants $C_1 > 0$, $\alpha > 0$ and $\zeta > 1$ such that for any $l > r > r_0$,*

$$\phi(l) \leqslant \frac{C_1}{(l-r)^\alpha} \phi(r)^\zeta.$$

Then for

$$d \geqslant C_1^{\frac{1}{\alpha}} \left|\phi(r_0)\right|^{\frac{\zeta-1}{\alpha}} 2^{\frac{\zeta}{\zeta-1}},$$

we have $\phi(r_0 + d) = 0$.

Sketch of the proof. Define

$$r_k := r_0 + d - \frac{d}{2^k}, \quad k = 0, 1, 2, \ldots.$$

Then

$$\phi(r_{k+1}) \leqslant \frac{C_1 2^{(k+1)\alpha}}{d^\alpha} \phi(r_k)^\zeta = \frac{C_1 2^\alpha}{d^\alpha} 2^{k\alpha} \phi(r_k)^\zeta.$$

In view of our assumption on d, since

$$\phi(r_0) \leqslant \theta_0 = \left(\frac{C_1 2^\alpha}{d^\alpha}\right)^{-\frac{1}{\zeta-1}} 2^{-\frac{\alpha}{(\zeta-1)^2}} = d^{\frac{\alpha}{\zeta-1}} C_1^{-\frac{1}{\zeta-1}} 2^{-\frac{\alpha\zeta}{(\zeta-1)^2}},$$

we deduce from Lemma 5.2 that $\lim_{k\to\infty} \phi(r_{k+1}) = 0$. □

Proof of Theorem 5.1. Assume that $L_0 = 0$, or else we consider $\tilde{u}(t,x) := e^{L_0 t} u(t,x)$ instead of u. First to prove $(*)^+$, it is sufficient to prove our theorem for the case

$$\operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+(\omega,t,x) < \infty.$$

Then for $k \geqslant \operatorname{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+(\omega,t,x)$, we have $(u-k, v) \in \mathscr{U} \times \mathscr{V}(G-k, f^k, g^k)$ with

$$\left(f^k, g^k\right)(\omega,t,x,R,Y,Z) := (f,g)(\omega,t,x,R+k,Y,Z) + \left(0, c(\omega,t,x)k\right)$$

for $(\omega,t,x,R,Y,Z) \in \Omega \times [0,T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. From Proposition 4.3, we have almost surely

$$\int_{\mathcal{O}} \left|\left(u(t,x)-k\right)^+\right|^2 dx + \int_t^T \left\|v_k(s)\right\|^2_{L^2(\mathcal{O})} ds$$
$$= -2 \int_t^T \left\langle\!\left\langle \partial_{x_j}\left(u(s)-k\right)^+, a^{ij}\partial_{x_i} u(s) + \sigma^{jr} v_k^r(s)\right.\right.$$

$$
\begin{aligned}
&+ \left(f^k\right)^j\left(s,\cdot,\left(u(s)-k\right)^+,\nabla u(s),v_k(s)\right)\rangle\!\rangle\,ds \\
&+2\int_t^T \langle\!\langle \left(u(s)-k\right)^+, b^i\partial_{x_i}u(s)+c\left(u(s)-k\right)^+ + \varsigma^r v_k^r(s)\rangle\!\rangle\,ds \\
&+2\int_t^T \langle\!\langle \left(u(s)-k\right)^+, g^k\left(s,\cdot,\left(u(s)-k\right)^+,\ \nabla u(s),v_k(s)\right)\rangle\!\rangle\,ds \\
&-2\int_t^T \langle\!\langle \left(u(s)-k\right)^+, v_k^r(s)\rangle\!\rangle\,dW_s^r, \quad \forall t\in[0,T]
\end{aligned}
$$

with $v_k := v1_{u>k}$. Therefore, we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \left|\left(u(t,x)-k\right)^+\right|^2 dx + E\left[\int_t^T \left\|v_k(s)\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right] \\
&\quad = -2E\left[\int_t^T \langle\!\langle \partial_{x_j}\left(u(s)-k\right)^+, a^{ij}\partial_{x_i}u(s)+\sigma^{jr}v_k^r(s)\right. \\
&\qquad \left. + \left(f^k\right)^j\left(s,\cdot,\left(u(s)-k\right)^+,\nabla u(s),v_k(s)\right)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right] \\
&\qquad +2E\left[\int_t^T \langle\!\langle \left(u(s)-k\right)^+, b^i\partial_{x_i}u(s)+c\left(u(s)-k\right)^+ + \varsigma^r v_k^r(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right] \\
&\qquad +2E\left[\int_t^T \langle\!\langle \left(u(s)-k\right)^+, g^k\left(s,\cdot,\left(u(s)-k\right)^+,\nabla u(s),v_k(s)\right)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right], \quad \text{a.s.} \qquad (5.3)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\Omega}\ \sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle \left(u(s)-k\right)^+, g_0^k(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right] \\
&\quad \leqslant 2\left\|\left(g_0^k\right)^+\right\|_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}\left\|(u-k)^+\right\|_{\frac{2(n+2)}{n};\mathcal{O}_t}\left|\{u>k\}\right|^{\frac12-\frac1p}_{\infty;\mathcal{O}_t} \quad \text{(Hölder inequality)} \\
&\quad \leqslant \delta\left\|(u-k)^+\right\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t} + C(\delta)\left\|\left(g_0^k\right)^+\right\|^2_{\frac{p(n+2)}{n+2+p};\mathcal{O}_t}\left|\{u>k\}\right|^{1-\frac2p}_{\infty;\mathcal{O}_t} \\
&\quad \leqslant \delta\left\|(u-k)^+\right\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t} + C(\delta,p,L)\left(\left|A_p\left(f_0,g_0^+\right)\right|^2+k^2\right)\left|\{u>k\}\right|^{1-\frac2p}_{\infty;\mathcal{O}_t}, \qquad (5.4)
\end{aligned}
$$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle \big(u(s)-k\big)^+, b^i\partial_{x_i}u(s)+c\big(u(s)-k\big)^+ +\varsigma^r v_k^r(s)\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\leqslant C(\varepsilon)\operatorname*{ess\,sup}_{\omega\in\Omega}\sup_{\tau\in[t,T]} E\left[\int_\tau^T \langle\!\langle |b(s)|^2+|c(s)|+|\varsigma(s)|^2, \big|\big(u(s)-k\big)^+\big|^2\rangle\!\rangle\,ds\Big|\mathscr{F}_\tau\right]\\
&\quad+\varepsilon\big(\big\|\nabla(u-k)^+\big\|^2_{2;\mathcal{O}_t}+\|v_k\|^2_{2;\mathcal{O}_t}\big)\\
&\leqslant \varepsilon\big(\big\|\nabla(u-k)^+\big\|^2_{2;\mathcal{O}_t}+\|v_k\|^2_{2;\mathcal{O}_t}\big)+C(\varepsilon)\Lambda_0\big\|(u-k)^+\big\|^2_{\frac{2q}{q-1};\mathcal{O}_t}\\
&\leqslant \varepsilon\big(\big\|\nabla(u-k)^+\big\|^2_{2;\mathcal{O}_t}+\|v_k\|^2_{2;\mathcal{O}_t}\big)+\delta\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t}\\
&\quad+C(\delta,n,q,\varepsilon,\Lambda_0)\big\|(u-k)^+\big\|^2_{2;\mathcal{O}_t},
\end{aligned}\tag{5.5}$$

and

$$\begin{aligned}
&2E\left[\int_t^T \langle\!\langle \big|\nabla\big(u(s)-k\big)^+\big|, \big|f_0^k(s)\big|\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right]\\
&\leqslant \varepsilon E\left[\int_t^T \big\|\nabla\big(u(s)-k\big)^+\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]+C(\varepsilon)E\left[\int_t^T \big\|f_0^k 1_{u>k}(s)\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\leqslant \varepsilon E\left[\int_t^T \big\|\nabla\big(u(s)-k\big)^+\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]+C(\varepsilon)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}\big\|f_0^k\big\|^2_{p;\mathcal{O}_t}\\
&\leqslant \varepsilon E\left[\int_t^T \big\|\nabla\big(u(s)-k\big)^+\big\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+C(\varepsilon,p,L)\big(\big|A_p\big(f_0,g_0^+\big)\big|^2+k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t},\quad \text{a.s.}
\end{aligned}\tag{5.6}$$

where ε and δ are two positive parameters waiting to be determined later and

$$\big|\{u>k\}\big|_{\infty;\mathcal{O}_t}:=\operatorname*{ess\,sup}_{\Omega}\sup_{\tau\in[t,T]} E\big[\big|\mathcal{O}_\tau\cap\{u>k\}\big|\big|\mathscr{F}_\tau\big].$$

In a similar way to (4.5) and (4.7) in the proof of Theorem 4.1, we obtain from (5.4) and (5.6) that with probability 1, for all $t\in[0,T]$

$$\begin{aligned}
&-2E\left[\int_t^T \langle\!\langle \partial_{x_j}\big(u(s)-k\big)^+, a^{ij}\partial_{x_i}u(s)+\sigma^{jr}v_k^r(s)\right.\\
&\qquad\left.+\big(f^k\big)^j\big(s,\cdot,\big(u(s)-k\big)^+,\nabla u(s),v_k(s)\big)\rangle\!\rangle\,ds\Big|\mathscr{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -(\lambda-\kappa-\beta\theta-\varepsilon)E\left[\int_t^T \left\|\nabla\big(u(s)-k\big)^+\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+\left(\frac{1}{\varrho}+\frac{1}{\theta}\right)E\left[\int_t^T \left\|v_k(s)\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+C(\varepsilon,L)E\left[\int_t^T \left\|\big(u(s)-k\big)^+\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+2E\left[\int_t^T \big(\left|\nabla\big(u(s)-k\big)^+\right|,\left|f_0^k(s)\right|\big)\,ds\Big|\mathscr{F}_t\right]\\
&\leqslant -(\lambda-\kappa-\beta\theta-2\varepsilon)E\left[\int_t^T \left\|\nabla\big(u(s)-k\big)^+\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+\left(\frac{1}{\varrho}+\frac{1}{\theta}\right)E\left[\int_t^T \left\|v_k(s)\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+C(\varepsilon,L)E\left[\int_t^T \left\|\big(u(s)-k\big)^+\right\|^2_{L^2(\mathcal{O})}\,ds\Big|\mathscr{F}_t\right]\\
&\quad+C(\varepsilon,p,L)\big(\left|A_p\big(f_0,g_0^+\big)\right|^2+k^2\big)\left|\{u>k\}\right|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{\Omega}\sup_{\tau\in[t,T]}2E\left[\int_\tau^T \left\langle\!\left\langle\big(u(s)-k\big)^+, g^k\big(s,\cdot,\big(u(s)-k\big)^+,\nabla u(s),v_k(s)\big)\right\rangle\!\right\rangle ds\Big|\mathscr{F}_\tau\right]\\
&\leqslant \varepsilon\left\|\nabla(u-k)^+\right\|^2_{2;\mathcal{O}_t}+\varepsilon_1\|v_k\|^2_{2;\mathcal{O}_t}+C(\varepsilon,\varepsilon_1,L)\left\|(u-k)^+\right\|^2_{2;\mathcal{O}_t}\\
&\quad+\operatorname*{ess\,sup}_{\Omega}\sup_{\tau\in[t,T]}2E\left[\int_\tau^T \left\langle\!\left\langle\big(u(s)-k\big)^+, g_0^k(s)\right\rangle\!\right\rangle ds\Big|\mathscr{F}_\tau\right]\\
&\leqslant \varepsilon\left\|\nabla(u-k)^+\right\|^2_{2;\mathcal{O}_t}+\varepsilon_1\|v_k\|^2_{2;\mathcal{O}_t}+C(\varepsilon,\varepsilon_1,L)\left\|(u-k)^+\right\|^2_{2;\mathcal{O}_t}\\
&\quad+\delta\left\|(u-k)^+\right\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t}+C(L,\delta,p)\big(\left|A_p\big(f_0,g_0^+\big)\right|^2+k^2\big)\left|\{u>k\}\right|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}
\end{aligned}
\tag{5.8}
$$

where θ, ε, ε_1 and δ are four positive parameters such that

$$
\theta>\frac{\varrho}{\varrho-1}>1,\qquad \frac{1}{\varrho}+\frac{1}{\theta}+\varepsilon+\varepsilon_1<1\quad\text{and}\quad \lambda-\kappa-\beta\theta-4\varepsilon>0.
$$

Combining (5.3), (5.5), (5.7) and (5.8), we have

$$\begin{aligned}&\big\|(u-k)^+\big\|^2_{\mathcal{V}_2(\mathcal{O}_t)}+\|v_k\|^2_{2;\mathcal{O}_t}\\&\quad\leqslant C\big\{C(\delta)\big\|(u-k)^+\big\|^2_{2;\mathcal{O}_t}+\delta\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t}\\&\qquad+C(\delta)\big(\big|A_p\big(f_0,g_0^+\big)\big|^2+k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}\big\},\end{aligned}\tag{5.9}$$

where C is a constant independent of t and δ.

By Lemma 3.1, $\mathcal{V}_{2,0}(\mathcal{O}_t)$ is continuously embedded into $\mathcal{M}^{\frac{2(n+2)}{n}}(\mathcal{O}_t)$. That is

$$\big\|(u-k)^+\big\|_{\frac{2(n+2)}{n};\mathcal{O}_t}\leqslant C\big\|(u-k)^+\big\|_{\mathcal{V}_2(\mathcal{O}_t)}.$$

Therefore, choosing δ to be small enough, we obtain

$$\begin{aligned}\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t}&\leqslant C\big\|(u-k)^+\big\|^2_{2;\mathcal{O}_t}+C\big(\big|A_p\big(f_0,g_0^+\big)\big|^2+k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}\\&\leqslant C\big(|T-t||\mathcal{O}|\big)^{\frac{2}{n+2}}\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t}\\&\quad+C\big(\big|A_p\big(f_0,g_0^+\big)\big|^2+k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}.\end{aligned}$$

Choosing $t_1\in[0,T)$ such that $C(|T-t_1||\mathcal{O}|)^{\frac{2}{n+2}}\leqslant\frac{1}{2}$, we get

$$\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_{t_1}}\leqslant C\big(\big|A_p\big(f_0,g_0^+\big)\big|^2+k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_{t_1}}$$

where the constant C does not depend on t_1.

Define

$$\psi:\mathbb{R}\to\mathbb{R},\qquad\psi(h)=\big|\{u>h\}\big|_{\infty;\mathcal{O}_{t_1}}.$$

Since for any $h>k$,

$$\big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_{t_1}}\geqslant(h-k)^2\big|\{u>h\}\big|^{\frac{n}{n+2}}_{\infty;\mathcal{O}_{t_1}},$$

taking $k\geqslant A_p(f_0,g_0^+)$ we have

$$\psi(h)^{\frac{n}{n+2}}\leqslant\frac{Ck^2}{(h-k)^2}\psi(k)^{1-\frac{2}{p}}$$

which implies

$$\psi(h)\leqslant\frac{Ck^\alpha}{(h-k)^\alpha}\psi(k)^{1+\bar{\varepsilon}}\tag{5.10}$$

where $\alpha = \frac{2(n+2)}{n}$ and $\bar{\varepsilon} = \frac{2(p-n-2)}{pn} > 0$. Take $k_l = k(2 - 2^{-l})$, $l = 0, 1, 2, \ldots$. Then from

$$\psi(k_{l+1}) \leqslant \frac{Ck_l^\alpha}{(k_{l+1} - k_l)^\alpha} \psi(k_l)^{1+\bar{\varepsilon}},$$

it follows that

$$\psi(k_{l+1}) \leqslant \hat{C} 2^{\alpha(l+1)} \psi(k_l)^{1+\bar{\varepsilon}}.$$

By Lemma 5.2, there exists a constant $\theta_0 = \theta_0(\hat{C}, \bar{\varepsilon}) > 0$, such that if $\psi(k_0) \leqslant \theta_0$, $\lim_{l\to\infty} \psi(k_l) = 0$. Note that $k_0 = k$ and $\psi(k_0) = |\{u > k\}|_{\infty;\mathcal{O}_{t_1}}$.

Taking

$$k = \operatorname*{ess\,sup}_{(\omega,s,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+(\omega, s, x) + A_p\big(f_0, g_0^+\big) + \theta_0^{-\frac{1}{2}} \big\|u^+\big\|_{2;\mathcal{O}_{t_1}},$$

we have

$$k^2 \geqslant \frac{1}{\theta_0}\big\|u^+\big\|^2_{2;\mathcal{O}_{t_1}} \geqslant \frac{1}{\theta_0}k^2\big|\{u > k\}\big|_{\infty;\mathcal{O}_{t_1}}$$

which implies

$$\psi(k_0) = \big|\{u > k\}\big|_{\infty;\mathcal{O}_{t_1}} \leqslant \theta_0.$$

Hence, $\psi(k_\infty) = 0$. Since $k_\infty = 2k$, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\omega,s,x)\in\Omega\times\mathcal{O}_{t_1}} u(\omega, s, x) \leqslant 2k \\
&\quad = 2\Big\{\operatorname*{ess\,sup}_{(\omega,s,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+(\omega, s, x) + A_p\big(f_0, g_0^+\big) + \theta_0^{-\frac{1}{2}}\big\|u^+\big\|_{2;Q}\Big\}.
\end{aligned}$$

As $T - t_1$ only depends on the structure terms like $n, \lambda, \kappa, \beta, \varrho, p, q, L, \Lambda_0, |\mathcal{O}|$ and T, by induction, we get estimate $(*)^+$.

In a similar way, we prove estimate $(*)^-$. We complete our proof. □

Theorem 5.4. *Let assumptions* ($\mathcal{A}$1)–($\mathcal{A}$4) *be satisfied and* $(u, v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^2(Q)$ *be a weak solution of* (1.1). *If* $L_0 = 0$ *and with probability* 1

$$f(t, x, r, 0, 0) \equiv f(t, x, 0, 0) \quad \textit{and} \quad g(t, x, r, 0, 0) \quad \textit{are decreasing in } r \in \mathbb{R} \tag{5.11}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$, *then we assert*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times Q} u^\pm(\omega, t, x) \\
&\quad \leqslant \operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^\pm(\omega, t, x) + CA_p\big(f_0, g_0^\pm\big)|\mathcal{O}|^{\frac{1}{n+2}-\frac{1}{p}}
\end{aligned} \tag{$**$}^\pm$$

with the constant C *only depending on* $n, p, q, \kappa, \lambda, \beta, \varrho, T, \Lambda_0$ *and* L.

Proof. We use De Giorgi iteration and the notations in the proof of Theorem 5.1. Similar to the proof of (5.4) and (5.6), under condition (5.11), we have for each $t \in [0, T]$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\Omega} \sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle (u(s)-k)^+, g_0^k(s)\rangle\!\rangle\, ds \,\Big|\, \mathscr{F}_\tau\right] \\
&\quad\leqslant \operatorname*{ess\,sup}_{\Omega} \sup_{\tau\in[t,T]} 2E\left[\int_\tau^T \langle\!\langle (u(s)-k)^+, g_0(s)\rangle\!\rangle\, ds \,\Big|\, \mathscr{F}_\tau\right] \\
&\quad\leqslant \delta \big\|(u-k)^+\big\|^2_{\frac{2(n+2)}{n};\mathcal{O}_t} + C(\delta)\big|A_p\big(f_0, g_0^+\big)\big|^2 \big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}
\end{aligned} \tag{5.12}$$

and almost surely

$$\begin{aligned}
&2E\left[\int_t^T \langle\!\langle \big|\nabla (u(s)-k)^+\big|, \big|f_0^k(s)\big|\rangle\!\rangle\, ds \,\Big|\, \mathscr{F}_t\right] \\
&\quad= 2E\left[\int_t^T \langle\!\langle \big|\nabla (u(s)-k)^+\big|, \big|f_0(s)\big|\rangle\!\rangle\, ds \,\Big|\, \mathscr{F}_t\right] \\
&\quad\leqslant \varepsilon E\left[\int_t^T \big\|\nabla (u(s)-k)^+\big\|^2_{L^2(\mathcal{O})}\, ds \,\Big|\, \mathscr{F}_t\right] + C(\varepsilon)\big|A_p\big(f_0, g_0^+\big)\big|^2 \big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;\mathcal{O}_t}.
\end{aligned} \tag{5.13}$$

Hence instead of (5.10), we obtain

$$\psi(h) \leqslant \frac{C|A_p(f_0, g_0^+)|^\alpha}{(h-k)^\alpha} \psi(k)^{1+\bar{\varepsilon}}.$$

By Corollary 5.3, for any $\bar{\theta}_0 \geqslant C A_p(f_0, g_0^+)|\mathcal{O}_{t_1}|^{\frac{1}{n+2}-\frac{1}{p}}$, we have

$$\Big|\Big\{u > \operatorname*{ess\,sup}_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+(\omega,t,x) + \bar{\theta}_0\Big\}\Big|_{\infty;\mathcal{O}_{t_1}} = 0, \tag{5.14}$$

which implies

$$\sup_{(\omega,t,x)\in\Omega\times\mathcal{O}_{t_1}} u \leqslant \sup_{(\omega,t,x)\in\Omega\times\partial_{\mathrm{p}}Q} u^+ + C A_p\big(f_0, g_0^+\big)|\mathcal{O}_{t_1}|^{\frac{1}{n+2}-\frac{1}{p}} \tag{5.15}$$

where the constant C depends only on $n, \lambda, p, q, \beta, \kappa, \varrho, \Lambda_0$ and L. As $T - t_1$ only depends on the structure terms, by induction, we get estimate $(**)^+$ where the constant C also depends on T.

In a similar way, we obtain estimate $(**)^-$. We complete the proof. □

Remark 5.3. In Theorem 5.4, we can dispense with the assumptions that $L_0 = 0$ and the function $r \mapsto g(t, x, r, 0, 0)$ decreases in r, by considering the function $\tilde{u}(t, x) := e^{2(L+L_0)t} u(t, x)$ instead of u.

Corollary 5.5. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}4)$ *be satisfied with* $L_0 = 0$. *Let the two pair* (f, g^1) *and* (f, g^2) *satisfy condition* (5.11) *in Theorem* 5.4. *Assume that* G^1 *and* G^2 *are two random variables in* $L^\infty(\Omega, \mathscr{F}_T, L^2(\mathcal{O}))$. *Let* $(u_i, v_i) \in \mathscr{U} \times \mathscr{V}(G^i, f, g^i)$, $i = 1, 2$ *and* $(u_1 - u_2)^+ \in \dot{\mathcal{V}}_{2,0}(Q)$. *Then if* $G^1 \leqslant G^2$ $dP \otimes dx$*-a.e. and* $g^1(\omega, t, x, u_2, \nabla u_2, v_2) \leqslant g^2(\omega, t, x, u_2, \nabla u_2, v_2)$, $dP \otimes dt \otimes dx$*-a.e., we have* $u_1(\omega, t, x) \leqslant u_2(\omega, t, x)$, $dP \otimes dt \otimes dx$*-a.e.*

Proof. $(u_1 - u_2, v_1 - v_2)$ belongs to $\mathscr{U} \times \mathscr{V}(\tilde{G}, \tilde{f}, \tilde{g})$ with

$$\begin{aligned}
\tilde{f}(s, x, R, Y, Z) &:= f\big(s, x, R + u_2(s, x), Y + \nabla u_2(s, x), Z + v_2(s, x)\big) \\
&\quad - f\big(s, x, u_2(s, x), \nabla u_2(s, x), v_2(s, x)\big), \\
\tilde{g}(s, x, R, Y, Z) &:= g^1\big(s, x, R + u_2(s, x), Y + \nabla u_2(s, x), Z + v_2(s, x)\big) \\
&\quad - g^2\big(s, x, u_2(s, x), \nabla u_2(s, x), v_2(s, x)\big)
\end{aligned}$$

and $\tilde{G} := G^1 - G^2$. Since $\tilde{G} \leqslant 0$, $\tilde{g}_0 \leqslant 0$ and $f_0 = 0$, the assertion follows from Theorem 5.4. □

5.2. The local case

This subsection is devoted to the local regularity of weak solutions.

Definition 5.1. For domain $Q' \subset Q$, a function $\zeta(\cdot, \cdot)$ is called a cut-off function on Q' if

(i) $\zeta \in \dot{W}_1^{2,2}(Q')$, i.e. there exists a sequence $\{\zeta^l, l \in \mathbb{N}\} \subset C_c^\infty(Q')$ such that

$$\begin{aligned}
\|\zeta^l - \zeta\|_{W_1^{2,2}} := \bigg\{ \int_{Q'} \big(\big|(\zeta^l - \zeta)(t, x)\big|^2 + \big|\partial_t(\zeta^l - \zeta)(t, x)\big|^2 \\
+ \big|\nabla(\zeta^l - \zeta)(t, x)\big|^2 + \big|\nabla^2(\zeta^l - \zeta)(t, x)\big|^2 \big)\, dx\, dt \bigg\}^{\frac{1}{2}} \qquad (5.16)
\end{aligned}$$

converges to zero as l tends to infinity with $\nabla^2(\zeta^l - \zeta)(t, x)$ being the Hessian matrix of the function $(\zeta^l - \zeta)(t, \cdot)$ at x;

(ii) $0 \leqslant \zeta \leqslant 1$;

(iii) there exists a domain $Q'' \Subset Q'$ and a nonempty domain $Q''' \Subset Q''$ such that

$$\zeta(t, x) = \begin{cases} 1, & (t, x) \in Q''', \\ 0, & (t, x) \in Q' \setminus Q''; \end{cases}$$

(iv) $|\nabla\zeta|, \partial_t\zeta \in L^\infty(Q')$.

For simplicity, we denote

$$\|\nabla\zeta\|_{L^\infty(Q')} := \||\nabla\zeta|\|_{L^\infty(Q')}.$$

First, to study the local behavior of our weak solutions, we shall generalize the deterministic parabolic De Giorgi class (for instance, see [4,15,16,24]) to our stochastic version and introduce the definition of De Giorgi class in the backward stochastic parabolic case.

Definition 5.2. We say that a function $u \in \mathcal{V}_{2,0}(Q)$ belongs to the backward stochastic parabolic De Giorgi class (BSPDG, for short) if for any $k \in \mathbb{R}$, $Q_{\rho,\tau} := [t_0 - \tau, t_0) \times B_\rho(x_0) \subset Q$ (with $\rho, \tau \in (0,1)$) and any cut-off function ζ on $Q_{\rho,\tau}$, we have

$$\begin{aligned}\left\|\zeta(u-k)^{\pm}\right\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} \leqslant \gamma\Big\{&\left\|(u-k)^{\pm}\right\|^2_{2;Q_{\rho,\tau}}\big(1+\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}+\|\partial_t\zeta\|_{L^\infty(Q_{\rho,\tau})}\big)\\&+\big(k^2+a_0^2\big)\big|\big\{(u-k)^{\pm}>0\big\}\big|^{1-\frac{2}{\mu}}_{\infty;Q_{\rho,\tau}}\Big\}\end{aligned} \qquad (\mathfrak{D}^{\pm})$$

for some triplet $(a_0, \mu, \gamma) \in [0,\infty) \times (n+2,\infty) \times [0,\infty)$. We call a_0, μ, and γ the structural parameters of *BSPDG*$^{\pm}$. We mean that $u \in \mathcal{V}_{2,0}(Q)$ satisfies $(\mathfrak{D}^+)$ ($(\mathfrak{D}^-)$, respectively) by the inclusion $u \in BSPDG^+(a_0,\mu,\gamma;Q)$ ($u \in BSPDG^-(a_0,\mu,\gamma;Q)$, respectively). We say $u \in BSPDG(a_0,\mu,\gamma;Q)$ if both inclusions $u \in BSPDG^+(a_0,\mu,\gamma;Q)$ and $u \in BSPDG^-(a_0,\mu,\gamma;Q)$ hold.

Proposition 5.6. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *hold. Assume that* $(u,v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^2(Q)$ *is a weak solution of* (1.1). *Then we assert that* $u \in BSPDG(a_0,\mu,\gamma;Q)$, *with* $a_0 := A_p(f_0,g_0)$, $\mu := \min\{p, 2q\}$, *and some parameter* γ *depending on* $n, p, q, \kappa, \lambda, \beta, \varrho, \Lambda, \Lambda_0$ *and* L.

Remark 5.4. In Proposition 5.6, assumption $(\mathcal{A}4)$ is not imposed.

The proof requires the following lemma.

Lemma 5.7. *Let assumptions* $(\mathcal{A}1)$–$(\mathcal{A}3)$ *hold,* ζ *be a cut-off function on* $Q_{\rho,\tau} := [t_0-\tau,t_0) \times B_\rho(x_0) \subset Q$, *and* $(u,v) \in \mathcal{V}_{2,0}(Q) \times \mathcal{M}^2(Q)$ *be a weak solution of* (1.1). *Then, we have almost surely*

$$\begin{aligned}&\big\langle\!\big\langle\zeta^2(t), \big|u^+(t)\big|^2\big\rangle\!\big\rangle_{B_\rho(x_0)} + \int_t^{t_0}\big\langle\!\big\langle\zeta^2(s), \big|v^u(s)\big|^2\big\rangle\!\big\rangle_{B_\rho(x_0)}\,ds\\&= -\int_t^{t_0} 2\big\langle\!\big\langle\zeta\,\partial_s\zeta(s), \big|u^+(s)\big|^2\big\rangle\!\big\rangle_{B_\rho(x_0)}\,ds\\&\quad+\int_t^{t_0} 2\big\langle\!\big\langle\zeta^2(s)u^+(s), g^u(s)\big\rangle\!\big\rangle_{B_\rho(x_0)}\,ds\\&\quad+\int_t^{t_0} 2\big\langle\!\big\langle\zeta^2(s)u^+(s), b^i(s)\partial_{x_i}u(s)+c(s)\,u^+(s)+\varsigma^r(s)v^{r,u}(s)\big\rangle\!\big\rangle_{B_\rho(x_0)}\,ds\end{aligned}$$

$$- \int_t^{t_0} \langle\!\langle 2\partial_{x_i}\big(\zeta^2(s)u^+(s)\big), a^{ji}(s)\partial_{x_j}u^+(s) + \sigma^{ir}(s)v^{r,u}(s) + f^{i,u}(s)\rangle\!\rangle_{B_\rho(x_0)}\,ds$$

$$- \int_t^{t_0} 2\langle\!\langle \zeta^2(s)u^+(s), v^{r,u}(s)\rangle\!\rangle_{B_\rho(x_0)}\,dW_s^r, \quad \forall t \in [t_0-\tau, t_0] \tag{5.17}$$

where

$$g^u(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)g\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big);$$
$$f^{i,u}(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)f^i\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big), \quad i = 0,1,\ldots,n;$$

and

$$v^u := \big(v^{1,u},\ldots,v^{m,u}\big), \qquad v^{r,u}(s,x) := 1_{\{(s,x):\, u(s,x)>0\}}(s,x)v^r(s,x), \quad r = 1,\ldots,m.$$

The proof is rather standard and is sketched below.

Sketch of the proof. We use approximation. By the definition of a cut-off function, all terms of (5.17) are well defined and there is a nonnegative sequence $\{\zeta^l, l \in \mathbb{N}\} \subset C_c^\infty(Q_{\rho,\tau})$ such that $\lim_{l\to\infty} \|\zeta^l - \zeta\|_{W_1^{2,2}(Q_{\rho,\tau})} = 0$. In view of Definition 2.2 and Remark 2.1, we verify like in Step 1 of the proof of Lemma 3.5 that for each l there holds

$$\zeta^l u(t,x) = \int_t^T \big[\partial_{x_j}\big(a^{ij}\partial_{x_i}\big(\zeta^l u\big)(s,x) + \sigma^{jr}\zeta^l v^r(s,x) + \tilde{f}_l^j(s,x)\big) + b^i\partial_{x_j}\big(\zeta^l u\big)(s,x)$$
$$+ c\zeta^l u(s,x) + \varsigma^r\zeta^l v^r(s,x) + \tilde{g}_l(s,x)\big]\,ds - \int_t^T \zeta^l v^r(s,x)\,dW_s^r, \quad t \in [0,T],$$

in the weak sense of Definition 2.2, where

$$\tilde{g}_l(s,x) := -\partial_s\zeta^l u(s,x) + \zeta^l(s,x)g\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big)$$
$$- b^i\partial_{x_i}\zeta^l u(s,x) - \partial_{x_j}\zeta^l\bar{f}_l^j(s,x),$$

$$\bar{f}_l(s,x) := a^{i\cdot}\partial_{x_i}u(s,x) + \sigma^{\cdot r}v^r(s,x) + f\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big),$$
$$\tilde{f}_l(s,x) := -a^{i\cdot}\partial_{x_i}\zeta^l u(s,x) + \zeta^l(s,x)f\big(s,x,u(s,x),\nabla u(s,x),v(s,x)\big).$$

Thus, $(\zeta^l u, \zeta^l v) \in \dot{\mathscr{U}} \times \dot{\mathscr{V}}(0, \tilde{f}_l, \tilde{g}_l)$. From Proposition 4.3 we conclude that (5.17) holds with ζ being replaced by ζ^l. Passing to the limit in $L^1(\Omega \times Q)$ and taking into account the path-wise continuity of u, we prove our assertion. □

Proof of Proposition 5.6. Consider the cylinder

$$Q_{\rho,\tau}(X) = X + [-\tau, 0) \times B_\rho(0) \subset Q \quad \text{with} \quad X := (t_0, x_0).$$

For simplicity, we denote $Q_{\rho,\tau}(X)$ and $B_\rho(x_0)$ by $Q_{\rho,\tau}$ and B_ρ respectively. Let ζ be a cut-off function on $Q_{\rho,\tau}$. Denote $\bar{u} := (u-k)^+$. From Lemma 5.7, it follows that

$$\begin{aligned}
&E\left[\left\|\zeta(t)\bar{u}(t)\right\|^2_{L^2(B_\rho)} + \int_t^{t_0}\left\|\zeta(s)v_k(s)\right\|^2_{L^2(B_\rho)}\,ds\Big|\mathscr{F}_t\right]\\
&= E\left[\int_t^{t_0} 2\langle\!\langle \zeta^2(s)\bar{u}(s), g^k\big(s,\cdot,\bar{u}(s),\nabla\bar{u}(s),v_k(s)\big)\rangle\!\rangle_{B_\rho}\,ds\Big|\mathscr{F}_t\right]\\
&\quad + E\left[\int_t^{t_0} 2\langle\!\langle \zeta^2(s)\bar{u}(s), b^i(s)\partial_{x_i}u(s) + c(s)\bar{u}(s) + \varsigma^r(s)v_k^r(s)\rangle\!\rangle_{B_\rho}\Big|\mathscr{F}_t\right]\\
&\quad - E\left[\int_t^{t_0} 2\langle\!\langle \zeta(s)\partial_s\zeta(s), \left|\bar{u}(s)\right|^2\rangle\!\rangle_{B_\rho}\,ds\Big|\mathscr{F}_t\right]\\
&\quad - E\left[\int_t^{t_0} \langle\!\langle 2\partial_{x_j}\big(\zeta^2(s)\bar{u}(s)\big), a^{ij}(s)\partial_{x_i}\bar{u}(s) + \sigma^{jr}(s)v_k^r(s)\right.\\
&\quad \left. + \big(f^k\big)^j\big(s,\cdot,\bar{u}(s),\nabla\bar{u}(s),v_k(s)\big)\rangle\!\rangle_{B_\rho}\,ds\Big|\mathscr{F}_t\right]
\end{aligned} \tag{5.18}$$

holds almost surely for all $t \in [t_0-\tau, t_0)$ where $v_k := v1_{u>k}$ and for $(\omega,t,x,R,Y,Z) \in \Omega \times [t_0-\tau,t_0) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$

$$\big(f^k, g^k\big)(\omega,t,x,R,Y,Z) := (f,g)(\omega,t,x,R+k,Y,Z) + \big(0, c(\omega,t,x)k\big).$$

In view of (4.5)–(4.7) and (5.4)–(5.8), we have almost surely for all $t \in [t_0-\tau, t_0)$

$$\begin{aligned}
&E\left[\int_t^{t_0} 2\langle\!\langle \zeta^2\bar{u}(s), g^k\big(s,\cdot,\bar{u}(s),\nabla\bar{u}(s),v_k(s)\big)\rangle\!\rangle_{B_\rho}\,ds\Big|\mathscr{F}_t\right]\\
&\leqslant E\left[\int_t^{t_0} 2\langle\!\langle \zeta^2\bar{u}(s), g_0^k(s) + L\big(\left|\bar{u}(s)\right| + \left|\nabla\bar{u}(s)\right| + \left|v_k(s)\right|\big)\rangle\!\rangle_{B_\rho}\,ds\Big|\mathscr{F}_t\right]\\
&\leqslant \varepsilon_1\|\zeta\bar{u}\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} + \varepsilon_2 E\left[\int_t^{t_0}\left\|\zeta(s)v_k(s)\right\|^2_{L^2(B_\rho)}\,ds\Big|\mathscr{F}_t\right]
\end{aligned}$$

$$+ C(L)\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}\|\bar u\|^2_{2;Q_{\rho,\tau}} + C(\varepsilon_1, L, n, p)\big(\big|A_p(f_0, g_0)\big|^2 + k^2\big)\big|\{u > k\}\big|^{1-\frac{2}{p}}_{\infty;Q_{\rho,\tau}}$$

$$+ C(\varepsilon_1, \varepsilon_2, L)\|\zeta\bar u\|^2_{2;Q_{\rho,\tau}} + E\left[\int_t^{t_0} 2\big\langle\!\big\langle \zeta^2\bar u(s), \big|c(s)k\big|\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right],$$

$$E\left[\int_t^{t_0} 2\big\langle\!\big\langle \zeta^2\bar u(s), \big|c(s)k\big|\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$\leqslant E\left[\int_t^{t_0} \big\langle\!\big\langle \zeta^2\big|\bar u(s)\big|^2, \big|c(s)\big|\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right] + k^2 E\left[\int_t^{t_0} \big|\big\langle\!\big\langle \zeta^2(s), \big|c(s)1_{u>k}\big|\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$\leqslant E\left[\int_t^{t_0} \big\langle\!\big\langle \zeta^2\big|\bar u(s)\big|^2, \big|c(s)\big|\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right] + k^2\Lambda_0\big|\{u > k\}\big|^{1-\frac{1}{q}}_{\infty;Q_{\rho,\tau}},$$

$$E\left[\int_t^{t_0} 2\big\langle\!\big\langle \zeta^2(s)\bar u(s), b^i\partial_{x_i}u(s) + c\bar u(s) + \varsigma^r v_k^r(s)\big\rangle\!\big\rangle_{B_\rho}\Big|\mathscr{F}_t\right]$$

$$\leqslant \varepsilon_1\|\zeta\bar u\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} + \varepsilon_2\|\zeta v_k\|^2_{2;Q_{\rho,\tau}} + \varepsilon_1\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}\|\bar u\|^2_{2;Q_{\rho,\tau}}$$

$$+ C(\varepsilon_1, \varepsilon_2, n)E\left[\int_t^{t_0} \big\langle\!\big\langle \zeta^2\bar u^2(s),\ \big|b(s)\big|^2 + \big|c(s)\big| + \big|\varsigma(s)\big|^2\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$\leqslant \varepsilon_1\|\zeta\bar u\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} + \varepsilon_2\|\zeta v_k\|^2_{2;Q_{\rho,\tau}} + \varepsilon_1\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}\|\bar u\|^2_{2;Q_{\rho,\tau}}$$

$$+ C(\varepsilon_1, \varepsilon_2, n)\Lambda_0\|\zeta\bar u\|^2_{\frac{2q}{q-1};Q_{\rho,\tau}}$$

$$\leqslant 2\varepsilon_1\|\zeta\bar u\|^2_{\mathcal{V}_2(Q_{\rho,\tau})} + \varepsilon_2\|\zeta v_k\|^2_{2;Q_{\rho,\tau}} + \varepsilon_1\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}\|\bar u\|^2_{2;Q_{\rho,\tau}}$$

$$+ C(\varepsilon_1, \varepsilon_2, n, q, \Lambda_0)\|\zeta\bar u\|^2_{2;Q_{\rho,\tau}}$$

and

$$-2E\left[\int_t^{t_0} \big\langle\!\big\langle \partial_{x_j}\big(\zeta^2\bar u(s)\big), a^{ij}\partial_{x_i}\bar u(s) + \sigma^{jr}v_k^r(s) + \big(f^k\big)^j\big(s, \bar u(s), \nabla\bar u(s), v_k(s)\big)\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$= -2E\left[\int_t^{t_0} \big\langle\!\big\langle \zeta^2\partial_{x_j}\bar u(s), a^{ij}\partial_{x_i}\bar u(s) + \sigma^{jr}v_k^r(s) + \big(f^k\big)^j\big(s, \bar u(s), \nabla\bar u(s), v_k(s)\big)\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$- 4E\left[\int_t^{t_0} \big\langle\!\big\langle \bar u\zeta\partial_{x_j}\zeta(s), a^{ij}\partial_{x_i}\bar u(s) + \sigma^{jr}v_k^r(s) + \big(f^k\big)^j\big(s, \bar u(s), \nabla\bar u(s), v_k(s)\big)\big\rangle\!\big\rangle_{B_\rho}\, ds\Big|\mathscr{F}_t\right]$$

$$\leqslant -\lambda_0 E\left[\int_t^{t_0} \|\zeta\nabla\bar{u}(s)\|^2_{L^2(B_\rho)}\,ds\big|\mathscr{F}_t\right] + \alpha_0 E\left[\int_t^{t_0} \|\zeta v_k(s)\|^2_{L^2(B_\rho)}\,ds\big|\mathscr{F}_t\right]$$

$$+ C\big(\|\zeta\bar{u}\|^2_{2;Q_{\rho,\tau}} + \big(\big|A_p(f_0,g_0)\big|^2 + k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;Q_{\rho,\tau}}\big)$$

$$+ CE\left[\int_t^{t_0} \big\langle\!\big\langle \big|\bar{u}\nabla\zeta(s)\big|, \big|f_0^k(s)\big| + \big|\bar{u}(s)\big| + \big|\zeta\nabla\bar{u}(s)\big| + \big|\zeta v_k(s)\big|\big\rangle\!\big\rangle\,ds\big|\mathscr{F}_t\right]$$

$$\leqslant -(\lambda_0-\varepsilon)E\left[\int_t^{t_0} \big\|\nabla\big(\zeta\bar{u}(s)\big)\big\|^2_{L^2(B_\rho)}\,ds\big|\mathscr{F}_t\right] + (\alpha_0+\varepsilon)E\left[\int_t^{t_0} \|\zeta v_k(s)\|^2_{L^2(B_\rho)}\,ds\big|\mathscr{F}_t\right]$$

$$+ C\big\{\|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})}\|\bar{u}\|^2_{2;Q_{\rho,\tau}} + \|\bar{u}\zeta\|^2_{Q_{\rho,\tau}} + \big(\big|A_p(f_0,g_0)\big|^2 + k^2\big)\big|\{u>k\}\big|^{1-\frac{2}{p}}_{\infty;Q_{\rho,\tau}}\big\}$$

with $C := C(\varepsilon, p, \lambda, \beta, \varrho, \kappa, \Lambda, L)$, where $\alpha_0 \in (0,1)$ and λ_0 are two positive constants depending only on structure terms such as $\kappa, p, \lambda, \varrho, \beta, \Lambda$ and L, and the three parameters $\varepsilon, \varepsilon_1, \varepsilon_2$ are waiting to be determined later. On the other hand, it is obvious that almost surely

$$-E\left[\int_t^{t_0} 2\big\langle\!\big\langle\zeta\partial_s\zeta(s), \big|\bar{u}(s)\big|^2\big\rangle\!\big\rangle_{B_\rho}\,ds\big|\mathscr{F}_t\right] \leqslant 2\|\partial_s\zeta\|_{L^\infty(Q_{\rho,\tau})}\|\bar{u}\|^2_{2;Q_{\rho,\tau}}, \quad \forall t\in[t_0-\tau, t_0).$$

Therefore, combining the above estimates and (5.18) and choosing the parameters $\varepsilon, \varepsilon_1$ and ε_2 to be small enough, we obtain

$$\begin{aligned}&\big\|\zeta(u-k)^+\big\|^2_{\infty,2;Q_{\rho,\tau}} + \big\|\nabla\big(\zeta(u-k)^+\big)\big\|^2_{2;Q_{\rho,\tau}}\\ &\quad\leqslant \gamma\big\{\big(1 + \|\nabla\zeta\|^2_{L^\infty(Q_{\rho,\tau})} + \|\partial_t\zeta\|_{L^\infty(Q_{\rho,\tau})}\big)\big\|(u-k)^+\big\|^2_{2;Q_{\rho,\tau}}\\ &\qquad + \big(k^2 + \big|A_p(f_0,g_0)\big|^2\big)\big|\big\{(u-k)^+>0\big\}\big|^{1-\frac{2}{p\wedge(2q)}}_{\infty;Q_{\rho,\tau}}\big\}\end{aligned}$$

where γ is a positive constant depending on the structure terms such as $n, p, q, \kappa, \lambda, \varrho, \beta, L, \Lambda$ and Λ_0. Hence $u \in BSPDG^+(a_0, \mu, \gamma; Q)$.

In a similar way, we show $u \in BSPDG^-(a_0, \mu, \gamma; Q)$. The proof is complete. □

Theorem 5.8. *If $u \in BSPDG^\pm(a_0, \mu, \gamma; Q)$, we assert that for any*

$$Q_\rho = \big[t_0, t_0+\rho^2\big) \times B_\rho(x_0) \subset Q, \quad \rho \in (0,1),$$

there holds

$$\operatorname*{ess\,sup}_{\Omega\times Q_{\frac{\rho}{2}}} u^\pm \leqslant C\big\{\rho^{-\frac{n+2}{2}}\big\|u^\pm\big\|_{2;Q_\rho} + a_0\rho^{1-\frac{n+2}{\mu}}\big\}, \tag{5.19}$$

where C is a constant depending only on a_0, μ, γ and n.

Proof. Consider $u \in BSPDG^+(a_0, \mu, \gamma; Q)$. Take

$$R_l = \frac{\rho}{2} + \frac{\rho}{2^{l+1}}, \qquad k_l = k\left(2 - \frac{1}{2^l}\right), \quad l = 0, 1, 2, \ldots$$

where k is a parameter waiting to be determined later. Denote $Q^l := Q_{R_l} = [t_0, t_0 + R_l^2) \times B_{R_l}(x_0)$. Choose ζ_l to be a cut-off function on Q^l such that

$$\zeta_l(t,x) = \begin{cases} 1, & (t,x) \in Q^{l+1}; \\ 0, & (t,x) \in Q^l \setminus Q_{\frac{R_l + R_{l+1}}{2}} \end{cases}$$

and

$$\|\nabla \zeta_l\|^2_{L^\infty(Q_\rho)} + \|\partial_t \zeta_l\|_{L^\infty(Q_\rho)} \leqslant \frac{C(n)}{(R_l - R_{l+1})^2}.$$

From $(\mathfrak{D}^+)$, it follows that

$$\begin{aligned} &\big\|\zeta_l(u - k_{l+1})^+\big\|^2_{\mathcal{V}_2(Q^l)} \\ &\quad \leqslant C2^{2l}\rho^{-2}\big\|(u - k_{l+1})^+\big\|^2_{2;Q^l} + C\big(k^2 + a_0^2\big)\big|\{u > k_{l+1}\}\big|^{1-\frac{2}{\mu}}_{\infty;Q^l}. \end{aligned}$$

For $k \geqslant a_0 \rho^{1-\frac{n+2}{\mu}}$, we obtain from Lemma 3.1 that

$$\begin{aligned} \big\|\zeta_l(u - k_{l+1})^+\big\|^2_{\frac{2(n+2)}{n};Q^l} &\leqslant C\big\|\zeta_l(u - k_{l+1})^+\big\|^2_{\mathcal{V}_2(Q^l)} \\ &\leqslant C2^{2l}\rho^{-2}\big\|(u - k_{l+1})^+\big\|^2_{2;Q^l} + Ck^2\rho^{-2(1-\frac{n+2}{\mu})}\big|\{u > k_{l+1}\}\big|^{1-\frac{2}{\mu}}_{\infty;Q^l}. \end{aligned}$$

Setting

$$\phi_l := \big\|(u - k_l)^+\big\|^2_{2;Q^l},$$

we have

$$\begin{aligned} \phi_{l+1} &\leqslant \big\|\zeta_l(u - k_{l+1})^+\big\|^2_{2;Q^l} \\ &\leqslant \big|\{u > k_{l+1}\}\big|^{\frac{2}{n+2}}_{\infty;Q^l}\big\|\zeta_l(u - k_{l+1})^+\big\|^2_{\frac{2(n+2)}{n};Q^l} \quad \text{(Hölder inequality)} \\ &\leqslant C2^{2l}\rho^{-2}\phi_l\big|\{u > k_{l+1}\}\big|^{\frac{2}{n+2}}_{\infty;Q^l} + Ck^2\rho^{-2(1-\frac{n+2}{\mu})}\big|\{u > k_{l+1}\}\big|^{1-\frac{2}{\mu}+\frac{2}{n+2}}_{\infty;Q^l}. \end{aligned}$$

Note that

$$\phi_l = \big\|(u - k_l)^+\big\|^2_{2;Q^l} \geqslant (k_{l+1} - k_l)^2\big|\{u > k_{l+1}\}\big|_{\infty;Q^l} = k^2 2^{-(2l+2)}\big|\{u > k_{l+1}\}\big|_{\infty;Q^l}.$$

Hence,

$$\phi_{l+1} \leqslant C2^{2l(1+\frac{2}{n+2})}\left[\rho^{-2}k^{-\frac{4}{n+2}}\phi_l^{1+\frac{2}{n+2}} + \rho^{-2(1-\frac{n+2}{\mu})}k^{\frac{4}{\mu}-\frac{4}{n+2}}\phi_l^{1-\frac{2}{\mu}+\frac{2}{n+2}}\right]$$
$$= C2^{2l(1+\frac{2}{n+2})}\rho^{-2(1-\frac{n+2}{\mu})}k^{\frac{4}{\mu}-\frac{4}{n+2}}\phi_l^{1-\frac{2}{\mu}+\frac{2}{n+2}}\left[\left(k^{-2}\rho^{-n-2}\phi_l\right)^{\frac{2}{\mu}} + 1\right].$$

For $k \geqslant a_0\rho^{1-\frac{n+2}{\mu}} + \rho^{-\frac{n+2}{2}}\|u^+\|_{2;Q_\rho}$, we have $k^{-2}\rho^{-n-2}\phi_l \leqslant 1$ and therefore

$$\phi_{l+1} \leqslant C2^{2l(1+\frac{2}{n+2})}\rho^{-2(1-\frac{n+2}{\mu})}k^{\frac{4}{\mu}-\frac{4}{n+2}}\phi_l^{1-\frac{2}{\mu}+\frac{2}{n+2}}.$$

Setting

$$\alpha_l := \rho^{-n-2}k^{-2}\phi_l,$$

we have

$$\alpha_{l+1} \leqslant C_1 2^{2l(1+\frac{2}{n+2})}\alpha_l^{1-\frac{2}{\mu}+\frac{2}{n+2}}.$$

From Lemma 5.2, we see that the following

$$\alpha_0 = k^{-2}\rho^{-n-2}\left\|(u-k)^+\right\|_{2;Q_\rho}^2 \leqslant k^{-2}\rho^{-n-2}\left\|u^+\right\|_{2;Q_\rho}^2 \leqslant \theta_0 := C_1^{-\frac{1}{\alpha}}2^{-\frac{2}{\alpha^2}(1+\frac{2}{n+2})}$$

with $\alpha := \frac{2}{n+2} - \frac{2}{\mu}$, implies

$$\lim_{l\to\infty}\alpha_l = 0 \quad \text{and thus} \quad \lim_{l\to\infty}\phi_l = 0.$$

In conclusion, the two inequalities

$$k^2 \geqslant \theta_0^{-1}\rho^{-n-2}\left\|u^+\right\|_{2;Q_\rho}^2 \quad \text{and} \quad k \geqslant a_0\rho^{1-\frac{n+2}{\mu}} + \rho^{-\frac{n+2}{2}}\left\|u^+\right\|_{2;Q_\rho}$$

imply the following one:

$$\|u\|_{\infty;Q_{\frac{\rho}{2}}} \leqslant 2k. \tag{5.20}$$

Hence, (5.20) holds for the following choice

$$k := a_0\rho^{1-\frac{n+2}{\mu}} + \left(1+\theta_0^{-\frac{1}{2}}\right)\rho^{-\frac{n+2}{2}}\left\|u^+\right\|_{2;Q_\rho},$$

which implies our desired estimate.

For $u \in BSPDG^-(a_0, \mu, \gamma; Q)$, the desired assertion follows in a similar way. We complete our proof. □

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