Coloring the vertices of a graph with measurable sets in a probability space

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Abstract

We introduce a class of “chromatic” graph parameters that includes the chromatic number, the circular chromatic number, the fractional chromatic number, and an uncountable horde of others. We prove some basic results about this class and pose some problems.

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1. Introduction

Here is a well known interpretation of the chromatic number in terms of scheduling. Suppose that the vertices of a graph $G$ are committees, with distinct committees adjacent if and only if they have members in common. Committee meetings are properly scheduled if adjacent committees are given non-overlapping meeting times. Then $\chi(G)$ is the smallest number of hours that the full schedule of committee meetings will require, with clever but proper scheduling, if each meeting is to last exactly one hour, and is to start at the “top” of an hour—at 9 A.M., for instance. We ignore problems of passing from one meeting room to another.

As noted in [5], the fractional chromatic number of $G$, $\chi_f(G)$, can be similarly characterized. It is the smallest number of hours the full schedule of committee meetings will require, with clever but proper scheduling, if each meeting is to last exactly one hour, but the scheduler is permitted to break each committee’s hour into a finite number of disjoint intervals. (There is no requirement on the starting times of the meetings, or of
the submeetings.) Also, the circular chromatic number of $G$, $\chi_c(G)$, which, like $\chi_f(G)$, has a discouragingly large number of equivalent definitions, counts among them this one (extractable from [7]): $\chi_c(G)$ is the smallest number of hours the full schedule of committee meetings will require, with clever but proper scheduling, if each meeting is to last exactly one hour, and is to take place either during one uninterrupted hour interval, or during two intervals, one at the beginning of the sessions and the other at the very end.

If we invert our view of time and declare that the length of time to be devoted to full schedule of meetings is to be fixed, at one unit, then the job of the clever scheduler changes from minimizing the total amount of time required for all the meetings to maximizing the amount of time to be allotted for each meeting within the session (assuming for the moment that different committees will meet for the same amount of time). Each of the chromatic numbers $\chi(G)$, $\chi_f(G)$, and $\chi_c(G)$ is seen to be the reciprocal of the largest amount of time that the scheduler can arrange for each meeting’s duration, under the different strictures on the kind of subset of the unit (time) interval that is allowed for a meeting. These considerations lead us to what follows.

Let $(X, \mu)$ be a probability measure space, and $S$ a family of $\mu$-measurable subsets of $X$ of positive measure satisfying: for each positive integer $n$, there exist $n$ pairwise disjoint members of $S$.

Suppose $r > 0$. A proper $(r, S)$-coloring of a simple graph $G$ is a function $\psi: V(G) \to S$ such that, for each $u, v \in V(G)$, (i) $\mu(\psi(u)) \geq r$, and (ii) if $u$ and $v$ are adjacent in $G$, then $\psi(u)$ and $\psi(v)$ are disjoint. (For those watching closely: in (ii), disjointness could be replaced by $\mu(\psi(u) \cap \psi(v)) = 0$, but there is really no point to this relaxation, and it complicates some things later.) We define a “chromatic number” $\chi(G; X, \mu, S) = \inf \{1/r; \text{there is a proper } (r, S)\text{-coloring of } G\}$.

As the preceding examples indicate, the choice of $S$ is crucial: for fixed $(X, \mu)$, you can get different parameters for different choices of $S$. Taking $X = (0, 1)$ and $\mu = \text{Lebesgue measure}$, we get $\chi_f$ by taking $S = \{\text{unions of open intervals}\}$, $\chi_c$ by taking $S = \{\text{open intervals}\} \cup \{(0, a) \cup (b, 1); 0 < a < b < 1\}$, and $\chi$ by taking $S = \{(\frac{1}{k}, \frac{1}{r}); k, r \text{ are positive integers and } k \leq r\}$. $\chi_c$ is also obtainable by letting $X$ be a circle of unit circumference, with Lebesgue measure, and letting $S$ be the collection of all open arcs on the circle (see [7]).

In the next section we give what we know of these chromatic numbers. Proofs are postponed until Section 4. In Section 3 we pose a number of problems.

2. Some results

Throughout, $(X, \mu)$ will be a probability measure space, and $S, S_1, S_2$, etc. will be collections of measurable subsets of $X$ of positive $\mu$-measure satisfying the admissibility criterion, that for each positive integer $n$ there can be found $n$ pairwise disjoint sets in the collection. Lebesgue measure will be denoted by $\lambda$, and, as above, $(0, 1)$ will denote the open unit interval.

The first two lemmas are straightforward from the definitions.

**Lemma 1.** $S_1 \subseteq S_2$ implies that $\chi(G; X, \mu, S_1) \geq \chi(G; X, \mu, S_2)$ for each graph $G$.

**Lemma 2.** If $H$ is a subgraph of $G$, then $\chi(H; X, \mu, S) \leq \chi(G; X, \mu, S).$
Corollary 1. For any admissible $S \subseteq \mathcal{M}, \chi_f \leq \chi(\cdot; (0,1), \lambda, S)$.

That is, the chromatic parameters you get by taking $(X, \mu) = ((0,1), \lambda)$ have a minimal element, $\chi_f$. We will see below, in Corollary 2, that they do not have a maximal element.

The next result appears as Theorem 2 in [7], where it is claimed that its proof is straightforward from the definition of the usual chromatic number. We do not think so, but leave it to the reader to judge. Observe that the result does contain a practical message for the clever scheduler, whether the scheduler is trying to minimize the total length of the meeting session, or to maximize the duration of each meeting within a prescribed time period for all the meetings to take place. The message is: if each meeting is to take place in an uninterrupted time interval, then you gain nothing by staggering the meeting starting times. That is, you may as well declare that the meeting starting times will be equally spaced points starting at the beginning.

Proposition 4. $\chi = \chi(\cdot; (0,1), \lambda, \{\text{open subintervals of } (0,1)\})$.

Proposition 3. Suppose that $a_1 \geq a_2 \geq \cdots > 0$ is a sequence satisfying $\sum_{i=1}^{\infty} a_i = 1$. Let $\mathbb{P} = \{1, 2, \ldots\}$ be equipped with the “weighted counting” measure $\mu$ defined by $\mu(S) = \sum_{i \in S} a_i$ for each $S \subseteq \mathbb{P}$. Let $S = \{i; i = 1, 2, \ldots\}$. Then $\chi(G; \mathbb{P}, \mu, S) = a_i^{-1}$ for each (finite simple) graph $G$.

Corollary 2. There are an uncountable infinity of different chromatic parameters $\chi(\cdot; X, \mu, S)$, and no maximal parameter, even among those where $(X, \mu) = ((0,1), \lambda)$.

Although the proofs are mostly postponed until Section 4, we must divulge to the reader that the proof of the last assertion of Corollary 2 arises from the observation that if we take $\hat{S} = \left\{ \left( \sum_{i < j} a_i, \sum_{i \leq j} a_i \right); j \in \mathbb{P} \right\}$, then $\chi(\cdot; \mathbb{P}, \mu, \hat{S}) = \chi(\cdot; (0,1), \lambda, \hat{S})$.

With the $a_i$ as in Proposition 3, let $\chi_{(a_i)} = \chi(\cdot; \mathbb{P}, \mu, 2^{\mathbb{P}})$, a somewhat more interesting parameter than the parameter in Proposition 3. It is straightforward to see that $\chi_{(a_i)} = \lambda(\cdot; (0,1), \lambda, S(a_i))$, where $S(a_i)$ is the collection of all possible unions of the intervals $\left( \sum_{i < j} a_i, \sum_{i \leq j} a_i \right), j \in \mathbb{P}$. This realization of $\chi_{(a_i)}$ can be used to demonstrate the following.

Proposition 4. The parameters $\chi(\cdot; (0,1), \lambda, S)$ are not linearly ordered value-wise. That is, there exist admissible $S_1, S_2$ and graphs $G_1, G_2$ such that $\chi(G_1; (0,1), \lambda, S_1) < \chi(G_1; (0,1), \lambda, S_2)$ and $\chi(G_2; (0,1), \lambda, S_1) > \chi(G_2; (0,1), \lambda, S_2)$.

Let $\alpha(G)$ denote the vertex independence number of $G$. We define

$$s_f(G) = \max \left[ \frac{|V(H)|}{\alpha(H)}; H \text{ is a subgraph of } G \right].$$

The reason for this choice of notation may be found in [2].

Proposition 5. For any $X, \mu$, and $\mathcal{S}$, $\chi(G; X, \mu, \mathcal{S}) \geq s_f(G)$, for all graphs $G$.

3. Problems and questions

PQ 1. Is $\chi_f$ minimal among all the parameters $\chi(\cdot; X, \mu, S)$?
PQ 2. Is every parameter \( \chi(\cdot;X, \mu, S) \) definable in \((0, 1), \lambda)\)? That is, given \( X, \mu, \) and \( S \), does there necessarily exist \( \hat{S} \subseteq \mathcal{M} \) (see Proposition 1) such that \( \chi(G; X, \mu, \hat{S}) = \chi(G; (0, 1), \lambda, \hat{S}) \) for all \( G \)?

By Corollary 1, the answer “yes” to PQ2 implies the answer “yes” to PQ1. Also, “yes” to PQ2 would imply “yes” to PQ3, below, because PQ3 is the same question as PQ2 confined to a certain class of probability measure spaces.

PQ 3. Suppose that \( \mu \) is a probability measure on \((0, 1)\) absolutely continuous with respect to \( \lambda \).

(That is, for some non-negative Lebesgue measurable function \( h \) on \((0, 1)\) satisfying \( \int_0^1 h d\lambda = 1, \mu \) is defined by \( \mu(A) = \int_A h d\lambda \) for every Lebesgue measurable subset \( A \) of \((0, 1)\).) Is every parameter \( \chi(\cdot; (0, 1), \mu, S) \) definable in \((0, 1), \lambda)\)?

PQ 4. Do there exist \( X, \mu, \) and \( S \) such that \( s_f = \chi(\cdot; X, \mu, S) \)?

A “yes” to PQ1 implies “no” to PQ4, because there exist graphs \( G \) such that \( s_f(G) < \chi_f(G) \) [1].

PQ 5. Is the “inf” in the definition of \( \chi(\cdot; X, \mu, S) \) always really a “min”?

The obvious motivation for PQ5 is that it is known that the answer is “yes” for \( \chi, \chi_f, \) and \( \chi_c \). Also, it is straightforward that the answer is “yes” for each of the parameters \( \chi_{(a_i)} \) defined after Proposition 3.

PQ 6. Is there a reasonable axiomatization of the theory of the parameters \( \chi(\cdot; X, \mu, S) \) in the ordinary language of graph theory?

“Reasonable” does not necessarily mean “finite”, although a finite axiomatization is certainly reasonable. What is called for is a list of requirements of a graph theoretic nature, with no mention of probability measure spaces, such that a graph parameter \( P \) satisfies these requirements if and only if \( P = \chi(\cdot; X, \mu, S) \) for some choice of \( X, \mu, \) and \( S \).

For the next problem, let, for each positive integer \( k \), \( \mathcal{U}_k \) denote the collection of unions of \( k \) or fewer open subintervals of \((0, 1)\), and \( \chi_k = \chi(\cdot; (0, 1), \lambda, \mathcal{U}_k) \). Thus, we have that \( \chi_1 = \chi, \chi_2 \leq \chi_c, \chi_{i+1} \leq \chi_i, i = 1, 2, \ldots \), and, for each \( G \), \( \chi_k(G) = \chi_f(G) \) for all \( k \) sufficiently large.

PQ 7. Suppose that \( k > 1 \) is the smallest positive integer such that \( \chi_k(G) = \chi_f(G); \) is it necessarily the case that \( \chi_i(G) > \chi_{i+1}(G), i = 1, \ldots, k - 1 \)?

The parameters \( \chi_k, k = 1, 2, \ldots, \) appear to be new, and of interest, because they “run between” \( \chi \) and \( \chi_f \), and we could ask a great many questions about them. But we content ourselves with PQ7 for now.

PQ 8. Suppose that \((a_i)\) is a sequence satisfying the hypothesis of Proposition 3, and \( \chi_{(a_i)} \) is as defined after Corollary 2. It is necessarily the case that \( \chi(G) \leq \chi_{(a_i)}(G) \) for all \( G \) with sufficiently large clique number (where “sufficiently large” depends on the sequence \((a_i)\))? What about for all \( G \) with \( \chi(G) \) sufficiently large?

4. Proofs

Proof of Proposition 1. In the course of proving Proposition 1, we will verify what was baldly asserted in Section 1, that \( \chi_f = \chi(\cdot; (0, 1), \lambda, \mathcal{U}) \), where \( \mathcal{U} \) is the collection of all unions of open subintervals of \((0, 1)\). Let us start from the first definition in history
of $\chi_f$ (see [3]). For integers $a$ and $b$, $1 \leq a \leq b$, a proper $(a, b)$-coloring of a graph $G$ is an assignment to the vertices of $G$ of $a$-subsets of $\{1, \ldots, b\}$ such that adjacent vertices are assigned disjoint sets. Then $\chi_f(G) = \inf \{b/a; \text{there is a proper } (a, b)\text{-coloring of } G\}$.

For any proper $(a, b)$-coloring $\psi$ of $G$, color the vertices of $G$ with members of $\mathcal{U}$ by assigning $\bigcup_{v \in \psi(v)}(\frac{a - 1}{b}, \frac{a}{b})$ to $v \in V(G)$. This gives a proper $(a/b, \mathcal{U})$-coloring of $G$. Thus $\chi(h; (0, 1), \lambda, \mathcal{M}) \leq \chi$. By Lemma 1, it suffices now to show that $\chi_f \leq \chi(h; (0, 1), \lambda, \mathcal{M})$.

Let $\psi$ be a proper $(r, \mathcal{M})$-coloring of a graph $G$, for some $r > 0$. Let $\delta$ be a (small) positive number. It is an elementary lemma in the theory of Lebesgue measure (see [4]) that for each measurable set of finite measure on the real line there is a finite union of intervals such that the measure of the symmetric difference of the union of intervals and the original set is less than $\delta$. Therefore, for each $v \in V(G)$ there is a set $\psi_1(v) \subseteq \mathcal{U}$ such that $\lambda(\psi(v) \setminus \psi_1(v)) \cup (\psi_1(v) \setminus \psi(v)) < \delta$. Since $\lambda(\psi(v)) > r$ for each $v \in V(G)$, it follows that $\lambda(\psi_1(v)) > r - \delta$ for each $v \in V(G)$.

If $u, v \in V(G)$ are adjacent, then $\psi_1(u) \cap \psi_1(v) \subseteq (\psi_1(u) \setminus \psi(u)) \cup (\psi_1(v) \setminus \psi(v))$, because $\psi(u), \psi(v)$ are disjoint, so $\lambda(\psi_1(u) \cap \psi_1(v)) < 2\delta$. Define $\psi_2(v) = \psi_1(v) \bigcup_{u \in V(G): u \text{ and } v \text{ are adjacent}} \psi(u)$ for each $v \in V(G)$, with $\psi$ denoting closure. Noting that $\mathcal{U}$ is closed under finite intersections, we see that $\psi_2(v) \in \mathcal{U}$ for each $v \in V(G)$. Furthermore, if $u, v \in V(G)$ are adjacent, then $\psi_2(u) \cap \psi_2(v) = \emptyset$. Finally, for $v \in V(G)$, $\lambda(\psi_2(v)) \geq \lambda(\psi_1(v)) - \sum_{u \in V(G): u \text{ and } v \text{ are adjacent}} \lambda(\psi_1(u) \cap \psi_1(v)) > r - 2n\delta = r - (2n + 1)\delta$, where $n = |V(G)|$.

Now, because the sets $\psi_2(v), v \in V(G)$, are finite unions of open intervals, we can find a positive integer $b$ such that for each $v \in V(G)$, there is a set $\psi_3(v)$, a union of intervals of the form $(\frac{a - 1}{b}, \frac{a}{b})$, $a$ a positive integer, such that $\psi_3(v) \subseteq \psi_2(v)$, and $\lambda(\psi_2(v) \setminus \psi_3(v)) < \delta$. Then $\psi_3$ assigns disjoint sets to adjacent vertices, and $\lambda(\psi_3(v)) > \lambda(\psi_2(v)) - \delta > r - (2n + 2)\delta$ for each $v \in V(G)$. By the definition of $\delta$, $\delta$ is small enough that $r - (2n + 2)\delta > 0$.

So if $\hat{\psi}(v) = \{i \in \{1, \ldots, b\}; (\frac{a - 1}{b}, \frac{a}{b}) \subseteq \psi_3(v)\}$, then $|\hat{\psi}(v)|/b = \lambda(\psi_3(v)) > r - (2n + 2)\delta$. Let $a = \lfloor br - (2n + 2)\delta \rfloor$ and let $\psi(v)$, for $v \in V(G)$, be some $a$-subset of $\hat{\psi}(v)$. Then $\psi$ is a proper $(a, b)$-coloring of $G$, so $\chi_f(G) \leq \frac{b}{a} \leq (r - (2n + 2)\delta)^{-1}$.

Since $\delta$ was arbitrary, and $r$ was an arbitrary positive number such that $G$ has a proper $(r, \mathcal{M})$-coloring, it follows that $\chi_f(G) \leq \chi(h; (0, 1), \lambda, \mathcal{M})$. □

**Proof of Proposition 2.** Following the notation introduced before PQ7, the claim is that $\chi_1 = \chi$. Clearly $\chi_1 \leq \chi$. Suppose that $\psi$ is a proper $(r, \mathcal{U}_1)$-coloring of a graph $G$, for some $r > 0$. For $v \in V(G)$, $\psi(v) = (a(v), b(v))$ is an open subinterval of $(0, 1)$ of length $\geq r$.

If $0 \notin \{a(v); v \in V(G)\}$ then we can shift all the intervals $\psi(v), v \in V(G)$ down to obtain a new proper $(r, \mathcal{U}_1)$-coloring of $G$ such that 0 does occur as a left hand end-point of at least one interval assigned to a vertex of $G$. So assume that $0 \notin \{a(v); v \in V(G)\}$.

Also, we can assume that each interval $\psi(v)$ is of length exactly $r$, since replacing $\psi(v)$ with a subinterval of $\psi(v)$ preserves the properness of the coloring.

So $(0, r)$ is one of the intervals $\psi(v), v \in V(G)$. All the intervals $(a, a + r), 0 \leq a < r$, intersect each other, and so are assigned by $\psi$ to non-adjacent vertices. Modify $\psi$ by replacing each of these by $(0, r)$; the new $\psi$ is still a proper $(r, \mathcal{U}_1)$-coloring of $G$. Also, we have for each $v \in V(G)$ that if $\psi(v) \neq (0, r)$ then $a(v) \geq r$. 


If \( r \) does not occur as a left hand end-point of any interval \( \psi(v) \), but there are intervals \( \psi(v) = (a, a + r) \) with \( a > r \), modify \( \psi \) by shifting the ensemble of such intervals down so at least one of them becomes \( (r, 2r) \), but no left hand end-point is shifted past \( r \). Now replace each interval \( \psi(v) = (a, a + r), r < a < 2r \), by \( (r, 2r) \). The new \( \psi \) is still a proper \((r, \mathcal{U}_t)\)-coloring of \( G \).

Continuing in this fashion, we eventually replace the original by a proper \((r, \mathcal{U}_t)\)-coloring of \( G \) which assigns only intervals of the form \(((j - 1)r, jr), j = 1, \ldots, k \), for some integer \( k \leq \frac{1}{r} \). These intervals have the property that they are disjoint iff they are not identical. Thus \( \chi(G) \leq k \leq \frac{1}{r} \). Since \( r \) was an arbitrary positive number such that there is a proper \((r, \mathcal{U}_t)\)-coloring of \( G \), it follows that \( \chi(G) \leq \chi_1(G) \). \( \square \)

**Proof of Proposition 3.** The sets in \( S \) also have the property that they are disjoint iff they are different. Thus, to properly color \( G \) with these sets, you will need at least \( \chi(G) \) of them, and because the \( a_i \) are non-increasing, the \( \chi(G) \) members of \( S \) with the greatest minimum \( \mu \)-measure are \( \{1\}, \ldots, \{\chi(G)\} \). \( \square \)

**Proof of Proposition 4.** Let \( a_i = 1/(5 + \epsilon) \), \( i = 1, \ldots, 5 \), for some small positive \( \epsilon \) (actually, \( \epsilon < 1 \) is small enough for our purpose here), and let \( a_6 \geq a_7 \geq \cdots \geq 0 \) satisfy \( \sum_{i=6}^{\infty} a_i = \epsilon/(5 + \epsilon) \), so that \( \sum_{i=1}^{\infty} a_i = 1 \). Let \( \chi_{(a)} \) be as defined before the proposition; it will be convenient to apply the incarnation of \( \chi_{(a)} \) as \( \chi(\cdot; \mathbb{P}, \mu, 2^E) \), rather than as \( \chi(\cdot; (0, 1), \lambda, S(a_i)) \).

We will see that \( \chi_{(a)}(C_5) \leq \frac{5\epsilon}{\epsilon + 2} < \chi(G) \), and that \( \chi_{(a)}(K_6) \geq (5 + \epsilon)/\epsilon > 6 = \chi(K_6) \). To see that \( \chi_{(a)}(C_5) \leq \frac{5\epsilon}{\epsilon + 2} \), assign to the vertices of \( C_5 \), going around the cycle, the sets \( \{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\} \). To see that \( \chi_{(a)}(K_6) \geq (5 + \epsilon)/\epsilon \), observe that in any proper \((r, 2^E)\)-coloring of \( K_6 \), not all six of the vertices can be assigned sets containing one of the integers \( 1, \ldots, 5 \). That is, one of the assigned sets must be a subset of \( \{6, 7, \ldots\} \). Consequently, \( r \leq \mu(\{6, 7, \ldots\}) = \epsilon/(5 + \epsilon) \), so \( 1/r \geq (5 + \epsilon)/\epsilon \). Since \( r \) is arbitrary, \( \chi_{(a)}(K_6) \geq (5 + \epsilon)/\epsilon \). \( \square \)

**Proof of Proposition 5.** By Lemma 2, it suffices to show that \( \chi(G; X, \mu, S) \geq |V(G)|/\alpha(G) \) for each \( G \). Let \( \psi \) be a proper \((r, S)\)-coloring of \( G \). Let \( \text{char}_{\psi(v)} \) denote the characteristic function of \( \psi(v) \), for each \( v \in V(G) \). Because \( \psi \) is proper, each \( x \in X \) is in at most \( \alpha(G) \) of the sets \( \psi(v), v \in V(G) \); therefore, \( \sum_{v \in V(G)} \text{char}_{\psi(v)}(x) \leq \alpha(G) \). Consequently,

\[
\alpha(G) = \alpha(G)\mu(X) \geq \int_X \left( \sum_{v \in V(G)} \text{char}_{\psi(v)} \right) d\mu
\]

\[
= \sum_{v \in V(G)} \int_X \text{char}_{\psi(v)} \ d\mu
\]

\[
= \sum_{v \in V(G)} \mu(\psi(v)) \geq r|V(G)|.
\]

Therefore, \( 1/r \geq |V(G)|/\alpha(G) \). Since \( r \) was arbitrary, \( \chi(G; X, \mu, S) \geq |V(G)|/\alpha(G) \). \( \square \)
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