A Representation Theorem for Algebras with Involution

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ABSTRACT

Algebras with involution are represented as commutants of two adjoint vector-space endomorphisms.

Let K be a field with an involution $\bar{\ }$. We shall consider finite-dimensional K-algebras A with an involution $*$ extending $\bar{\ }$ (for short: algebras with involution). So $\alpha \mapsto \alpha^*$ will be an antiautomorphism of the ring A satisfying $\alpha^* = \bar{\alpha}$ for $\alpha \in K$ and $\alpha^{**} = \alpha$ for all $\alpha \in A$. Everyone knows that if (V, h) is a nondegenerate ε-hermitian space over K (where $\varepsilon \in K$ satisfies $\varepsilon\bar{\varepsilon} = 1$), then $\mathrm{End}_K(V)$ carries the "adjoint" involution $f \mapsto f^h$ given by

$$h(fv, w) = h(v, f^h w), \qquad v, w \in V.$$

If a subalgebra B of $\mathrm{End}_K(V)$ is stable under this involution, the same is clearly true for its centralizer $C = \mathrm{End}_B(V)$. For example, we may pick one $f \in \mathrm{End}_K(V)$, and let B be generated over K by f and f^h; hence

$$C = \left\{ g \in \mathrm{End}_K(V) \middle| gf = fg,\ gf^h = f^h g \right\} =: \mathrm{End}_K(V; f, f^h).$$

This special construction of an algebra with involution turns out to be already the most general case.

THEOREM 1. *Any algebra with involution is isomorphic to an algebra* $\mathrm{End}_K(V; f, f^h)$ *with the adjoint involution, for some triple* (V, h, f) *as above* (ε *fixed*).

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Proof. Given $A, *$, we first find a space (V_0, h_0) such that $(A, *)$ is isomorphic to $(\mathrm{End}_B(V_0), {}^0)$ for some subalgebra $B = B^0$ of $\mathrm{End}_K(V_0)$, where ^{0} denotes the adjoint involution with respect to h_0. This is done by a "hyperbolic" construction: Set $V_0 = A \oplus \mathrm{Hom}(A, K)$, a left A-module by $\alpha(\beta, \lambda) = (\alpha\beta, \alpha\lambda)$ for $\alpha, \beta \in A$, $\lambda \in \mathrm{Hom}(A, K)$, where $\alpha\lambda$ is defined as the functional $\gamma \mapsto \lambda(\alpha^* \gamma)$ on A. Furthermore, set

$$h_0((\beta, \lambda), (\gamma, \mu)) = \overline{\lambda(\gamma)} + \varepsilon\mu(\beta),$$

an A-invariant nondegenerate ε-hermitian form. In this way $(A, *)$ becomes a subalgebra (with involution) of $(\mathrm{End}_K(V_0), {}^0)$. Since V_0 contains the "regular" A-module A as a direct summand, it is balanced (cf. [2, p. 451]), i.e., we have $A = \mathrm{End}_B(V_0)$ for $B = \mathrm{End}_A(V_0)$.

The final step of the proof depends on a well-known lemma (which can be proved by direct computation):

LEMMA. *In the ring of $m \times m$ matrices over an arbitrary ring R, let $\Phi = (\varphi_{ij})$ commute with some matrix of type*

$$\begin{pmatrix} 0 & 1 & & & * \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & & 0 \end{pmatrix}.$$

Then Φ is upper triangular and $\varphi_{11} = \cdots = \varphi_{mm}$.

This will be applied to $R = \mathrm{End}_K(V_0)$. Let $\beta_1, \ldots, \beta_n$ be a K-basis of B, and put

$$(V, h) = \underbrace{(V_0, h_0) \perp \cdots \perp (V_0, h_0)}_{n+2 \text{ copies}},$$

$$f = \begin{pmatrix} 0 & 1 & \beta_1 & & & 0 \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \beta_n \\ & & & & \cdot & 1 \\ 0 & & & & & 0 \end{pmatrix}.$$

Clearly, $\mathrm{End}_B(V_0)$ is embedded diagonally into $\mathrm{End}_K(V; f, f^h)$. Now consider an arbitrary $g \in \mathrm{End}_K(V; f, f^h)$. Then both g and g^h commute with f. So, by the lemma, g must be both upper and lower triangular, i.e., $g = \varphi \perp \cdots \perp \varphi$, $\varphi \in \mathrm{End}_K(V_0)$. Furthermore, $gf = fg$ implies $\varphi\beta_i = \beta_i\varphi$ for all i, which means $\varphi \in \mathrm{End}_B(V_0)$. This proves that $(\mathrm{End}_B(V_0), {}^0)$ is isomorphic to the algebra with involution associated to (V, h, f). ■

The main conclusion is that one cannot expect to classify the triples (V, h, f). In the second part of the proof it was merely used that (V_0, h_0) defined an ε-hermitian representation of B, i.e., a representation $B \to \mathrm{End}_K(V_0)$ compatible with the involutions; what we have proved then is:

THEOREM 2. *For any algebra with involution there is a full embedding of the category of its ε-hermitian representations into the category of triples (V, h, f) as above.*

This result has been modeled after analogous ones for linear representations (cf. [1]). It was contained in my thesis (unpublished) and recently dug out [3].

REFERENCES

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