Uniform tag systems for paperfolding sequences

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Abstract

One folds lengthwise a strip of paper in \( p \geq 2 \) equal parts several times on itself (all folds being parallel). After unfolding, one obtains a sequence of “valley” and “ridge” folds. If one codes these folds over a two-letter alphabet, one obtains a \( p \)-paperfolding word associated with the sequence of \( p \)-folding instructions. A \( p \)-paperfolding sequence is an infinite \( p \)-paperfolding word. This paper is devoted to the effective construction of \( p \)-uniform tag systems which generate every \( p \)-paperfolding sequence associated with ultimately periodic sequence of \( p \)-unfolding instructions.

Keywords: Automata; Automatic sequence

1. Introduction

One folds lengthwise a strip of paper in \( p \geq 2 \) equal parts several times on itself (all folds being parallel). After unfolding, one obtains a sequence of “valley” and “ridge” folds. In particular, if one represents the “valley” folds (resp. the “ridge” folds) by moving left (resp. right), one obtains the classic dragon curves (see [14–16]). If one codes these folds over a two-letter alphabet, one obtains a \( p \)-paperfolding word associated with the sequence of \( p \)-folding instructions. A \( p \)-paperfolding sequence is an infinite \( p \)-paperfolding word and, as in [15], we say that it is associated with a sequence of \( p \)-unfolding instructions.

For \( p = 2 \), these sequences had been widely studied (see for instance [2, 4–7, 15, 18–20, 22] and [1] for an extensive bibliography on this subject). The case \( p \geq 3 \) has been less studied. Davis and Knuth [14] define the ternary foldings and their associated ternary dragon curves. Dekking et al. [15] generate dragon curves derived from the iterated \( p \)-foldings (where one always executes the same sequence of \( p \)-folding instructions). Razafy Andriamampianina [23–25] shows that these sequences as well...
as polynomial subsequences are \( p \)-automatic. Koskas [17] defines the \( p \)-paperfolding sequences as Toeplitz sequences and computes their complexity functions.

It is well-known that ultimately periodic sequences can be generated by (finite) automata. Cobham in [13] characterizes the set of sequences than can be generated by uniform tag system (note that this set strictly contains the set of non-ultimately periodic sequences). Roughly speaking, a (uniform) tag system is the application of a coding to an infinite word generated by infinite iteration of a morphism (uniform if the images of the letters have all the same length).

Up to now, only the 2-paperfolding sequences have been generated by 2-uniform tag system (see [1, 15] for the iterated sequences and [7] for the general case). This paper gives a universal construction for all \( p \)-paperfolding sequences. For a fixed integer \( p \geq 2 \), we introduce a unique infinite word (called the \( p \)-Toeplitz word) over an infinite set which, though it is not ultimately periodic, is generated by iterated morphism. More precisely, the \( p \)-Toeplitz word is generated by an infinite iteration of a \( \kappa \)-uniform morphism if and only if \( \kappa = p^v, v > 0 \) (this is our Theorem 1). A specific coding of the \( p \)-Toeplitz word provides all \( p \)-paperfolding sequences. But this process of generation is not a tag system because the \( p \)-Toeplitz word is defined over an infinite set. In fact, as in the case where \( p = 2 \) (see [1, 7, 20]), only a particular class of \( p \)-paperfolding sequences are \( p \)-automatic (i.e., generated by a \( p \)-uniform tag system). Namely, it is easy to deduce from the very definition of \( p \)-paperfolding (this can also be read in [20]) that: "a \( p \)-paperfolding sequence is \( p \)-automatic if and only if the associated sequence of \( p \)-unfolding instructions is ultimately periodic". So, we characterize the \( p \)-paperfolding sequences associated with an ultimately periodic sequence of \( p \)-unfolding instructions and we explain how to construct effectively a family of \( p \)-uniform tag systems which generate all automatic \( p \)-paperfolding sequences (this is our Theorem 2).

2. Definitions and notations

Let \( A \) be a finite set called an alphabet. The elements of \( A \) are letters and finite strings over \( A \) are words. \( A^* \) is the semi-group generated by \( A \) with the concatenation (the identity element is the empty word \( \varepsilon \)) and \( A^+ = A^* \setminus \{ \varepsilon \} \). For a word \( w \) over \( A \), \( |w| \) denotes its length, i.e., the number of letters of this word (in particular, \( |\varepsilon| = 0 \)).

Every function \( x : \mathbb{N} \setminus \{ 0 \} \to A \) is called an infinite word over \( A \) and denoted by \( x = (x(n))_{n \geq 1} \) where \( x(n) \) is the letter of index \( n \) in \( x \). \( A^\omega \) is the set of infinite words over \( A \) and \( A^+ = A^* \cup A^\omega \). Sometimes, for convenience, we consider an infinite word \( x \) as an application \( x : \mathbb{N} \to A \). In this case \( x = (x(n))_{n \geq 0} \).

For \( x \in A^\omega \) and \( d \in \mathbb{N} \setminus \{ 0 \} \), the prefix of length \( d \) of \( x \) will be denoted by \( x[d] = x(1)x(2) \ldots x(d) \) if \( x = (x(n))_{n \geq 1} \).

A word \( u \in A^* \) is a proper prefix of \( w \in A^* \) if there exists \( v \in A^+ \) such that \( w = uv \). Let \((w_n)_{n \geq 1}\) be a sequence of words over \( A \) such that, for all \( n \geq 1 \), \( w_n \) is a proper prefix
of \( w_{n+1} \). Then for the topology on \( A^* \) of which \( A^\infty = A^* \cup A^\omega \) is the completion, one can define an infinite word by \( w[n] = w_n \) for all \( n \geq 1 \). In this case, we will write \( w = \lim_{n \to \infty} w_n \). In fact, only this topology makes \( A^\infty \) a topological monoid (see [10] for a survey on topologies over \( A^\infty \)).

A sequence \( x = (x(r))_{r \geq 1} \) is said to be **ultimately periodic** if there exist two words \( u \in A^* \) and \( v \in A^+ \) such that \( x = uv^\omega \) (\( v^\omega \) denote the infinite repetition of word \( v \)).

Let \( A \) be a finite or infinite set of letters. A morphism \( \theta : A^* \to A^* \) (i.e., a map such that \( \forall u, v \in A^*, \theta(uv) = \theta(u)\theta(v) \)) is expansive on \( a \in A \) if \( \exists w \in A^+ \) such that \( \theta(a) = aw \).

A morphism \( \theta : A^* \to A^* \) is nonerasing if \( \forall a \in A, \theta(a) \neq \epsilon \). A morphism \( \theta : A^* \to A^* \) is p-uniform \( (p > 0) \) if \( \forall a \in A, |\theta(a)| = p \). If \( p = 1 \), \( \theta \) is called a **literal morphism**. Note that a p-uniform morphism is nonerasing.

Now we put, for all integer \( n \geq 0 \),

\[
\theta^n = \begin{cases} 
\text{the identity} & \text{if } n = 0 \\
\theta^{n-1} \circ \theta & \text{if } n > 0.
\end{cases}
\]

If \( \theta \) is a morphism nonerasing and expansive on \( a \in A \) then, for all integer \( n \geq 0 \), \( \theta^n(a) \) is a proper prefix of \( \theta^{n+1}(a) \). In this case, there exists \( x \in A^\omega \) defined, for \( n \in \mathbb{N} \), by \( x[d] = \theta^n(a) \) for \( d = |\theta^n(a)| \) and \( x = \lim_{n \to \infty} \theta^n(a) = \theta^\omega(a) \)" and we say that \( x \) is **morphic** of base \( \theta \) (resp. **p-morphic** of base \( \theta \) if \( \theta \) is p-uniform, with \( p \geq 2 \) using the definition of expansive morphism). It is obvious that \( x \) is also \( p^* \)-morphic of base \( \theta^v \) with \( v > 0 \) (see "speed up" in [21, 26]). Note that the term "morphic" is commonly used in more general acception (see, e.g., [9]).

Of course, it is not always possible to generate an infinite word by iterating a morphism (see [8] for an example of such a word). But for the study of infinite words, it is often useful to be able to generate them automatically. Such an automatic process to generate infinite words was developed by Cobham in [13]: let \( A, B \) be alphabets, \( \theta : A^* \to A^* \) a morphism (resp. a p-uniform morphism) expansive on \( a \in A \) and \( \eta : A^* \to B^* \) a literal morphism. Then \( \mathcal{Z} = \langle A, a, \theta, B, \eta \rangle \) is called a **tag system** (resp. **p-uniform tag system**). If \( \mathcal{Z} \) is a p-uniform tag system, then the infinite word \( \theta^\omega(a) \) is p-morphic of base \( \theta \) and the infinite word \( \eta[\theta^\omega(a)] \) is said to be p-automatic.

Now, we introduce some notation. With every integer \( r > 0 \), we associate two letters denoted by \( r \) and \( \bar{r} \) (by abuse of notation). Note that \( r \neq \bar{r} \). This allows us to build an infinite ordered set \( \Sigma \cup \bar{\Sigma} \) where \( \Sigma = \{r, r \in \mathbb{N} \setminus \{0\}\} \) and \( \bar{\Sigma} = \{\bar{r}, r \in \mathbb{N} \setminus \{0\}\} \).

Again by abuse of notation, we shall denote for all \( r \in \Sigma \cup \bar{\Sigma} \)

\[
r + 1 = \begin{cases} 
i + 1 \in \Sigma & \text{if } r = i \in \Sigma, \\
\bar{i} + 1 \in \bar{\Sigma} & \text{if } r = \bar{i} \in \bar{\Sigma}.
\end{cases}
\]

In other words, \( r + 1 \) is the successor of \( r \) in the ordered set \( \Sigma \) if \( r \in \Sigma \) (resp. in the ordered set \( \bar{\Sigma} \) if \( r \in \bar{\Sigma} \)).
For all \( r \in \Sigma \cup \tilde{\Sigma} \cup \{\varepsilon\} \), we define \( \bar{r} \) the video of \( r \) by

\[
\bar{\varepsilon} = \varepsilon, \\
\bar{r} = \begin{cases} 
\bar{i} \in \tilde{\Sigma} & \text{if } r = i \in \Sigma, \\
\bar{i} \in \Sigma & \text{if } r = \bar{i} \in \tilde{\Sigma}.
\end{cases}
\]

For all word \( w \in (\Sigma \cup \tilde{\Sigma})^* \), we define \( w^R \) the reversal word of \( w \) by

\[
\begin{cases} 
\varepsilon^R = \varepsilon, \\
(wr)^R = rw^R & \text{for all } r \in \Sigma \cup \tilde{\Sigma}.
\end{cases}
\]

For all \( w \in (\Sigma \cup \tilde{\Sigma})^* \), \( \bar{w}^R \) is called the video-reversal word of \( w \).

**Fact.** For all words \( v, w \in (\Sigma \cup \tilde{\Sigma})^* \),

\[
\bar{w}^R = \bar{w} \text{ if } |w| = 1, (w^R)^R = w \text{ and } (\bar{w}^R)^R = \bar{w}^R v^R.
\]

### 3. \( p \)-paperfolding sequences and \( p \)-Toeplitz word

#### 3.1. The \( p \)-paperfolding sequences

Given a strip of paper.

- **First step:** We fold it lengthwise in \( p \geq 2 \) equal parts, putting each \((i + 1)\)th part \((1 \leq i < p)\) either over or under the \(i\)th part.
- **Second step:** We again fold the folded strip as above, the new folds being parallel to the first ones. For this second folding there are also \( 2^{p-1} \) possible ways of folding.
- And so on ...

Unfolding this folded strip of paper such that the folds form an angle, we obtain a sequence of "valley" and "ridge" folds. If we code these folds over the two-letter alphabet \( \{a, \bar{a}\} \), we obtain a word called \( p \)-paperfolding word. A \( p \)-paperfolding sequence is an infinite \( p \)-paperfolding word.

A recursive definition of \( p \)-paperfolding sequences (see for example [5, 7] if \( p = 2 \)) can be obtained from the following remarks:

(i) All integers \( r > 0 \) have a unique expansion in base \( p \geq 2 \) (also called base-\( p \)-representation) as follows: there exists a unique \( 3 \)-uplet of integers \((s, j, k)\) where \( s \geq 0, j \geq 0 \) and \( 1 \leq k < p \), such that \( r = p^s(j + k) \) (note that the integer \( s \geq 0 \) is called in [6] valuation \( p \)-adic of positive integer \( r \)).

(ii) The folds (coming from the same step of folding) of two consecutive parts of strip folded as above, are associated in twos, the second being "valley" if the first is
"ridge" (and vice versa) in the following way:

If \( w(r) \) is the fold of index \( r \), then, writing \( r = p^{s}(pj + k) \) for all integers \( s, j \geq 0 \) and \( 1 \leq k < p \), we have: \( w(p^{s}(pj + k)) = \overline{w(p^{s}(p(j + 1) + p - k))} \).

**Definition 1.** Let \( p \geq 2 \). The sequence \( w = (w(r))_{r \geq 1} \) over \( \{a, \bar{a}\} \) is a \( p \)-paperfolding sequence if for all \( s, j \in \mathbb{N} \) and \( 1 \leq k < p \), we have:

\[
\begin{align*}
\text{if } j \text{ is even,} & & \quad w(p^{s}(pj + k)) = \overline{w(p^{s}(p^2j + 2k))}, \\
\text{otherwise.} & & \quad w(p^{s}(pj + k)) = w(p^{s}(p^2j + 2k)).
\end{align*}
\]

**Example.** \( w = (w(r))_{r \geq 1} \) is a 3-paperfolding sequence if

\[
\begin{align*}
\text{for } j \text{ even} & & \quad w(3j + 1) = w(1), & & \quad w(3j + 2) = w(2), \\
& & \quad w(3(3j + 1)) = w(3), & & \quad w(3(3j + 2)) = w(6), \\
& & \quad w(3^2(3j + 1)) = w(9), & & \quad w(3^2(3j + 2)) = w(18), \\
& & \text{and so on}
\end{align*}
\]

\[
\begin{align*}
\text{for } j \text{ odd} & & \quad w(3j + 1) = \overline{w(2)}, & & \quad w(3j + 2) = \overline{w(1)}, \\
& & \quad w(3(3j + 1)) = \overline{w(6)}, & & \quad w(3(3j + 2)) = \overline{w(3)}, \\
& & \quad w(3^2(3j + 1)) = \overline{w(18)}, & & \quad w(3^2(3j + 2)) = \overline{w(9)}, \\
& & \text{and so on.}
\end{align*}
\]

Definition 1 shows that any \( p \)-paperfolding sequence \( w = (w(r))_{r \geq 1} \) is entirely determined by the subsequences: \( (w(k))_{1 \leq k < p}, (w(pk))_{1 \leq k < p}, \ldots, (w(p^sk))_{1 \leq k < p}, \ldots \). Also, in the sequel, we use the following notations.

For \( p \geq 2 \) fixed, let \( (f_{p,s})_{s \geq 0} \) be a sequence where each \( f_{p,s} \) is any word of length \( (p - 1) \) over the alphabet \( \{a, \bar{a}\} \). Obviously, there is an infinite number of such sequences.
In what follows, we consider one of these sequences and we denote it $f_p$. Thus, for any integer $1 \leq k < p$, $f_{p,s}(k)$ denotes the letter of index $k + s(p - 1)$ of $f_p$. In other words, if $p \geq 2$ fixed, every integer $r > 0$ has a unique decomposition: $r = k + s(p - 1)$ where $s \in \mathbb{N}$ and $1 \leq k < p$. This leads to define the morphism $\lambda_p^s: (\Sigma \cup \Sigma)^* \rightarrow (\Sigma \cup \Sigma)^*$ by $\lambda_p^s(r) = r + s(p - 1)$; hence $\lambda_p^s(r) = r + s(p - 1)$ for $s \geq 0$. With these notations, of course, $f_{p,s}(k) = f_p(\lambda_p^s(k))$.

We then state the following obvious characterization:

**Lemma 1.** Let $p \geq 2$ and let $f_p = (f_{p,s})_{s \geq 0}$ be a sequence of words of length $(p - 1)$ over $\{a, \bar{a}\}$. The sequence $w = (w(r))_{r \geq 1}$ defined for all $s, j \geq 0$ and $1 \leq k < p$, by

$$
w(p^s(pj + k)) = \begin{cases} f_{p,s}(k) & \text{if } j \text{ is even}, \\ f_{p,s}(p - k) & \text{otherwise} \end{cases}
$$

is a $p$-paperfolding sequence.

The words $f_{p,s}$ are called $p$-unfolding instructions and the sequence $w$ as defined is called $p$-paperfolding sequence associated with sequence of $p$-unfolding instructions $f_p$.

**Remark.** (1) If all the instructions $f_{p,s}$ are identical, the associated $p$-paperfolding sequence is said iterated and as in [23], it is defined for $1 \leq k < p$ and $j \in \mathbb{N}$ by

$$
w(p(j + k)) = \begin{cases} w(k) & \text{if } j \text{ is even}, \\ w(p - k) & \text{otherwise}, \end{cases}
$$

$$
w(pr) = w(r) \quad \text{for } r > 0.
$$

Consequently, every iterated $p$-paperfolding sequence is entirely defined by the finite sequence: $(w(1), w(2), \ldots, w(p - 1))$.

(2) The other definitions of $p$-paperfolding sequences are available (see e.g., [17, 25]). But the effective construction of our tag systems requires the relation of Lemma 1 between the letters of $w$ and these of the $s$th instruction $f_{p,s}$ (see 4.2 below).

Hence, we define any $p$-paperfolding sequence by an infinity of words of length $p - 1$: $f_{p,0} = w(1)w(2) \ldots w(p - 1)$, $f_{p,1} = w(p)w(2p) \ldots w((p - 1)p)$, $f_{p,s} = w(p^s)w(2p^s) \ldots w((p - 1)p^s)$, $\ldots$ and not as in [25], by the $p - 1$ (infinite) subsequences: $(w(p^s))_{s \geq 0} = w(1)w(p) \ldots w(p^s) \ldots$, $(w(2p^s))_{s \geq 0} = w(2)w(2p) \ldots w(2p^s) \ldots$,

$(w((p - 1)p^s))_{s \geq 0} = w(p - 1)w((p - 1)p) \ldots w((p - 1)p^s) \ldots$

(3) The $p$-paperfolding sequence associated with $f_p$ can be considered as an infinite word over the infinite set: $\{(f_{p,s}(k))_{1 \leq s < p, s \geq 0}, (w(f_{p,s}(k)))_{1 \leq s < p, s \geq 0}\}$.

But in order to simplify, we will associate each letter $f_{p,s}(k)$ to an element of $\Sigma = \{1, 2, 3, \ldots\}$ and each letter $w(f_{p,s}(k))$ to an element of $\Sigma = \{\bar{1}, \bar{2}, \bar{3}, \ldots\}$ as follows:

**Definition 2.** Let $p \geq 2$ and let $f_p = (f_{p,s})_{s \geq 0}$ be a sequence of words of length $(p - 1)$ over $\{a, \bar{a}\}$. The morphism $\sigma_f: (\Sigma \cup \Sigma)^* \rightarrow \{a, \bar{a}\}^*$ defined by

$$
\sigma_f(r) = f_p(r),
$$

$$
\sigma_f(\bar{r}) = \overline{f_p(r)}
$$
is called the morphism associated with \( f_p \). In particular, we will use the relation:
\[
\sigma_j(\lambda_p^s(k)) = f_p.s(k) \quad \text{for } s \geq 0 \text{ and } 1 \leq k < p.
\]

### 3.2. The p-Toeplitz word \( T_p \)

Now we prove that there exists an infinite word over \( \Sigma \cup \Sigma \) (called the p-Toeplitz word) whose image by \( \sigma_f \) is the p-paperfolding sequence associated with \( f_p \).

**Definition 3.** Let \( p \geq 2 \). The word \( T_p = (T(r))_{r \geq 1} \) over \( \Sigma \cup \Sigma \) is called the p-Toeplitz word if for all integers \( s, j \geq 0 \) and \( 1 \leq k < p \), we have
\[
T(p^j(pj + k)) = \begin{cases} 
\lambda_p^s(k) & \text{if } j \text{ is even}, \\
\lambda_p^s(p - k) & \text{otherwise}.
\end{cases}
\]

This definition shows that the p-Toeplitz word is just the paperfolding sequence over the alphabet \( \Sigma \cup \Sigma \) associated to the p-unfolding instructions \( \lambda_p^s \). So, we could define p-paperfolding sequences over the alphabet \( \{(f_p.s(k))_{1 \leq k < p} \cup ((f_p.s(k))_{1 \leq k < p})_{s \geq 0} \} \), but we chose to use the more general notion of Toeplitz sequence (see [4, 25]).

An immediate consequence of Definitions 2 and 3 is the following characterization.

**Proposition 1.** Let \( p \geq 2 \) and let \( f_p = (f_p.s)_{s \geq 0} \) be a sequence of words of length \( (p - 1) \) over \( \{a, a\} \). Then the following assertions are equivalent:

(i) the sequence \( w \) is the p-paperfolding sequence associated with \( f_p \);

(ii) there exists a morphism \( \sigma_f \) such that \( w = \sigma_f(T_p) \).

D. Razafy Andriamampianina in [25] shows that any p-paperfolding sequence is a Toeplitz sequence but we here prefer to distinguish the properties of the word defined over the infinite set \( \Sigma \cup \Sigma \) from its coding over the two-letter alphabet \( \{a, a\} \).

The characterization of the p-Toeplitz word as limit requires the following definition.

**Definition 4.** Let \( (t_{p,n})_{n \geq 1} \) be the sequence of words over \( \Sigma \cup \Sigma \) such that
\[
\begin{align*}
& \begin{cases} 
& t_{p,1} = 12 \ldots (p - 1) = (\lambda_p^0(k))_{1 \leq k < p}, \\
& \text{for } n \geq 1, \quad t_{p,n+1} = t_{p,n} \lambda_p^n(1) t_{p,n}^{-1} \lambda_p^n(2) \ldots \lambda_p^n(p - 1) t_{p,n},
\end{cases} \\
& \text{with } t_{p,n}^{t_p} = \begin{cases} 
& \overline{t_{p,n}}^R & \text{if } p \text{ is even;} \\
& t_{p,n} & \text{otherwise.}
\end{cases}
\end{align*}
\]
Examples. For \( p \) odd

\[
\begin{align*}
t_{3,1} &= 12, \\
t_{3,2} &= 12321412, \\
t_{3,3} &= 123214125 \bar{2}14123216 12321412, \\
t_{3,4} &= 123214125 \bar{2}14123216 123214127 123214125 \bar{2}14123218 123214125 \bar{2}14123216 12321412, \\
&\text{and so on.}
\end{align*}
\]

For \( p \) even

\[
\begin{align*}
t_{4,1} &= 123, \\
t_{4,2} &= 123432151236321, \\
t_{4,3} &= 1234321512363217 1236321512343218 1236321512343219 123632151234321; \\
&\text{and so on} \ldots
\end{align*}
\]

Proposition 2. Let \( p \geq 2 \) and \( T_p \) the \( p \)-Toeplitz word. Then \( T_p = \lim_{n \to \infty} t_{p,n} \).

4. Main results

In the sequel, if no confusion is possible, we will denote \( \lambda_p \) by \( \lambda \) and \( t_{p,n} \) by \( t_n \).

4.1. The genericity of the \( p \)-Toeplitz word \( T_p \)

We first define an another (abuse of) notation: for all \( r \in \Sigma \cup \bar{\Sigma} \), we note

\[
\begin{align*}
&\{ l \leq r \leq m \text{ instead of } r \in \{ l, l + 1, \ldots, m \} \cup \{ \bar{l}, \bar{l} + 1, \ldots, \bar{m} \}, \\
&\{ r > m \text{ instead of } r \in \{ m + 1, m + 2, \ldots \} \cup \{ \bar{m} + 1, \bar{m} + 2, \ldots \}. 
\end{align*}
\]

Definition 5. Let \( p \geq 2 \) and let \( h_p : (\Sigma \cup \bar{\Sigma})^* \to (\Sigma \cup \bar{\Sigma})^* \) be the morphism which is \( p \)-uniform and expansive on the letter 1 and defined

for \( p \) odd, by:

\[
h_p(r) = \begin{cases} 
\bar{1}^R \lambda(r) & \text{if } r \text{ is odd,} \\
1^R \lambda(r) & \text{otherwise,}
\end{cases}
\]

for \( p \) even, by:

\[
h_p(r) = \begin{cases} 
\bar{1}^R \lambda(r) & \text{if } r \text{ is odd and } r < p, \\
1^R \lambda(r) & \text{otherwise.}
\end{cases}
\]
We are ready to state our first result:

**Theorem 1.** Let \( p \geq 2 \).

1. The \( p \)-Toeplitz word \( T_p \) is \( p \)-morphic of base \( \eta_p \), i.e., \( T_p = \eta_p^p(1) \).
2. Furthermore, if \( T_p \) is \( \kappa \)-morphic of base \( \eta_\kappa \) then there exists an integer \( \nu > 0 \) such that \( \kappa = p^\nu \) and, in this case, \( \eta_\kappa = (\eta_p)^\nu \).

Condition 2 is based on the following result which is interesting in itself.

**Proposition 3.** Let \( p \geq 2 \). The \( p \)-Toeplitz word is not ultimately periodic.

### 4.2. The \( p \)-automaticity of \( p \)-paperfolding sequences

An immediate consequence of Proposition 1 and Condition 1 of Theorem 1 is that each \( p \)-paperfolding sequence (they are uncountably many) is image by a literal morphism of a \( p \)-morphic word of base \( \eta_p \); more precisely, we have \( w = \sigma_f[\eta_p^p(1)] \). However, this does not imply that all \( p \)-paperfolding sequences are \( p \)-automatic because this process of generation is not a tag system since the morphism \( \eta_p \) is defined over an infinite set. In fact, as in the case \( p = 2 \), using a famous theorem of Cobham [13], it follows:

**Proposition 4.** Let \( p \geq 2 \). A \( p \)-paperfolding sequence is \( p \)-automatic if and only if it is associated with an ultimately periodic sequence of \( p \)-unfolding instructions.

It seems that this well-known property has been apparently never written down until the paper of Mendès France and Shallit [20] has been published. Note that Razafy Andriamampianina in [23] only gives the automaticity for the iterated \( p \)-paperfolding sequences.

As a consequence of Proposition 4, we will only work in what follows with those of \( p \)-paperfolding sequences which are associated with an ultimately periodic sequence of \( p \)-unfolding instructions. This implies that we consider particular cases of general Definition 1 and, thus, of Proposition 1 (see Definition 1' and Proposition 1' below).

For a given pair of integers \( i, \tau > 0 \), let \( f_{p,i,\tau} = f_{p,0} \cdots f_{p,i-1}(f_{p,i} \cdots f_{p,i+\tau-1})^\omega \) be an ultimately periodic sequence of words of length \( (p - 1) \) over \( \{a, a\} \) and let \( \sigma_{p,i,\tau} \) be its associated morphism (see Definition 2).

Of course, each \( p \)-paperfolding sequence associated with \( f_{p,i,\tau} \) is entirely determined by the \((i + \tau)\) subsequences of \( p \)-unfolding instructions:

\[
\begin{align*}
    f_{p,0} &= (w(k))_{1 \leq k < p}, \\
    f_{p,1} &= (w(pk))_{1 \leq k < p}, \\
    &\quad \ldots, \\
    f_{p,i+\tau-1} &= (w(p^{i+\tau-1}k))_{1 \leq k < p}.
\end{align*}
\]
Note that if $p$ is odd, we can define the sequence $f_{p,i,r}$ with $i = 0$. But we have chosen $i \neq 0$ since every periodic sequence can be written as an ultimately periodic sequence: for instance, $(f_{p,0} \ldots f_{p,z-1})^0 = f_{p,0}(f_{p,1} \ldots f_{p,z-1}f_{p,0})^0$.

In other respects, since $f_{p,s+k} = f_{p,s}$ for $i < s < i + \tau$ and $k > 0$, then: $\sigma_{p,i,\tau}$ is defined over the alphabet $\{1, \ldots, (i + \tau)(p - 1)\} \cup \{1, \ldots, (i + \tau)(p - 1)\}$ (that we denote $\Sigma_{p,i,\tau} \cup \bar{\Sigma}_{p,i,\tau}$); $\sigma_{p,i,\tau}(T(p^{s+k}(pj + k))) = \sigma_{p,i,\tau}(T(p^{s+k}(pj + k)))$ for $j \geq 0$ and $1 \leq k < p$.

Consequently, we have the particular case of Definition 1.

**Definition 1'.** The $p$-paperfolding sequence associated with the ultimately periodic sequence of $p$-unfolding instructions, $f_{p,i,\tau}$ is defined for all $j > 0$ and $1 \leq k < p$ by

$$w(p^j(pj + k)) = \begin{cases} f_{p,s}(k) & \text{if } j \text{ is even}, \\ f_{p,s}(p - k) & \text{otherwise}, \end{cases}$$

for $0 \leq s < i + \tau$, $\sigma_{p,i,\tau}(T(p^{s+k}(pj + k))) = \sigma_{p,i,\tau}(T(p^{s+k}(pj + k)))$ for $i < s < i + \tau$ and $k > 0$.

Now, as in Section 3.2, we search for an infinite word such that its image by $\sigma_{p,i,\tau}$ is the $p$-paperfolding associated with $f_{p,i,\tau}$. But here, this word must be defined over the alphabet $\Sigma_{p,i,\tau} \cup \bar{\Sigma}_{p,i,\tau}$.

We first define the following morphism.

**Definition 6.** Let $\rho : (\Sigma \cup \bar{\Sigma})^* \rightarrow (\Sigma_{p,i,\tau} \cup \bar{\Sigma}_{p,i,\tau})^*$ be the morphism such that

$$\rho(r) = \begin{cases} r & \text{for } r \in \Sigma_{p,i,\tau} \cup \bar{\Sigma}_{p,i,\tau}, \\ \rho(r - \tau(p - 1)) & \text{otherwise}. \end{cases}$$

We then have the following particular case of Proposition 1.

**Proposition 1'.** Let $p \geq 2$. For a given pair of integers $i, \tau > 0$, let $f_{p,i,\tau} = f_{p,0} \ldots f_{p,i-1}(f_{p,1} \ldots f_{p,i+\tau-1})^0$ be an ultimately periodic sequence over $\{a, \bar{a}\}$. Then the following assertions are equivalent:

(i) the sequence $w$ is the $p$-paperfolding sequence associated with $f_{p,i,\tau}$,

(ii) there exists a morphism $\sigma_{p,i,\tau}$ such that $w = \sigma_{p,i,\tau}([\rho(T_p)])$.

Note that for a given pair of integers $i, \tau > 0$, $\rho(T_p)$ generates $2^{(p-1)(i+\tau)}$ different $p$-paperfolding sequences (which is the number of $p$-automatic $p^\tau$-paperfolding sequences).

Now, we describe the tag system which generates the word $\sigma_{p,i,\tau}([\rho(T_p)])$.

We first define a "restriction" of $h_p$ over $(\Sigma_{p,i,\tau} \cup \bar{\Sigma}_{p,i,\tau})^*$. 
Definition 7. Let $h_{p,i,r} : (\Sigma_{p,i,r} \cup \Sigma_{p,i,r})^* \rightarrow (\Sigma_{p,i,r} \cup \Sigma_{p,i,r})^*$ be the morphism defined by:

For $p$ odd,

$$h_{p,i,r}(r) = h_p(r) = \begin{cases} t_1 \lambda(r) & \text{if } r \text{ is odd and } r \in \Sigma_{p,i,r}, \\ t_1^{-1} \lambda(r) & \text{if } r \text{ is even and } r \in \Sigma_{p,i,r} \\ t_1 \lambda^l(k) & \text{if } k \text{ is odd and } 1 \leq k < p, \\ t_1^{-1} \lambda^l(k) & \text{if } k \text{ is even and } 1 \leq k < p, \\ 
\end{cases}$$

For $p$ even,

$$h_{p,i,r}(r) = h_p(r) = \begin{cases} t_1 \lambda(r) & \text{if } r \text{ is odd and } r < p, \\ t_1^{-1} \lambda(r) & \text{if } r \text{ is even or } p \leq r < (i + \tau - 1)(p - 1), \\ t_1 \lambda^l(k) & \text{for } 1 \leq k < p. \\ \end{cases}$$

Example. For $p$ odd:

$$h_{3,1,3}(1) = h_{3,1,3}(7) = 123; \quad h_{3,1,3}(2) = h_{3,1,3}(8) = 214;$$

$$h_{3,1,3}(3) = 125; \quad h_{3,1,3}(4) = 216; \quad h_{3,1,3}(5) = 127; \quad h_{3,1,3}(6) = 218;$$

$$h_{3,1,3}(7) = 125; \quad h_{3,1,3}(8) = 216; \quad h_{3,1,3}(9) = 127; \quad h_{3,1,3}(10) = 218.$$

For $p$ even:

$$h_{4,1,3}(1) = 1234; \quad h_{4,1,3}(2) = 3215; \quad h_{4,1,3}(3) = 1236;$$

$$h_{4,1,3}(7) = 1237; \quad h_{4,1,3}(8) = 3215; \quad h_{4,1,3}(9) = 1236;$$

$$h_{4,1,3}(4) = 3217; \quad h_{4,1,3}(5) = 3218; \quad h_{4,1,3}(6) = 3219;$$

$$h_{4,1,3}(7) = 32110; \quad h_{4,1,3}(8) = 32111; \quad h_{4,1,3}(9) = 32112;$$

We then establish that $\rho(T_p)$ is $p$-morphic of base $h_{p,i,r}$ using the following relations between the morphism $h_p$ and $h_{p,i,r}$.

Lemma 2. (1) For all $r \in \Sigma \cup \Sigma$, $h_{p,i,r} \circ \rho(r) = \rho \circ h_p(r)$;

(2) for all $n \in \mathbb{N} \setminus \{0\}$, $h_{p,i,r}^n(1) = \rho[h_p^n(1)];$

(3) $h_{p,i,r}^n(1) = \rho[h_p^n(1)].$
In other words, the word \( h_{p,i,r}^{0}(1) \) is obtained from the word \( h_{p}^{0}(1) \) by replacing

\[
\begin{align*}
\lambda^{i+1}(k), \lambda^{i+2}(k), \ldots, \lambda^{i+\tau}(k), \ldots \\
\lambda^{i+\tau+1}(k), \lambda^{i+2\tau+1}(k), \ldots, \lambda^{i+\tau+\tau+1}(k), \ldots \\
\ldots \\
\lambda^{i+2\tau-1}(k), \lambda^{i+3\tau-1}(k), \ldots, \lambda^{i+(\tau+1)\tau-1}(k), \ldots
\end{align*}
\]

for \( 1 \leq k < p \) by \( \lambda^{i}(k) \), \( \lambda^{i+1}(k) \), \( \ldots \), \( \lambda^{i+\tau-1}(k) \).

We are now ready to state our second main result.

**Theorem 2.** Let \( p \geq 2 \). For each pair of integers \( i, \tau > 0 \), the \( p \)-uniform tag system

\[
\mathcal{S}_{p,i}(f_{p,i}) = (\Sigma_{p,i}, \Sigma_{p,i}, 1, h_{p,i}, \sigma_{p,i})
\]

generates the \( p \)-paperfolding sequence associated with the ultimately periodic sequence of \( p \)-unfolding instructions \( f_{p,i} = f_{p,0} \ldots f_{p,i-1} \). Conversely, every \( p \)-automatic \( p \)-paperfolding sequence is generated by such a \( p \)-uniform tag system.

**5. Proofs**

The proof of Proposition 2 use the following technical result.

**Lemma 3.** \( \forall n \in \mathbb{N}, \)

\[
\bar{r}_{n+1} = \begin{cases} 
\bar{t}_{n}^{\lambda^{n}(p-2)} \lambda^{n}(p-2) \ldots \lambda^{n}(1) \bar{r}_{n}^{R} & \text{if } p \text{ is even}, \\
\bar{r}_{n}^{R} \lambda^{n}(p-2) \ldots \lambda^{n}(1) \bar{t}_{n}^{R} & \text{otherwise}.
\end{cases}
\]

This is an easy consequence of Fact. \( \Box \)

**Proof of Proposition 2.** We prove, by induction, that \( (T(r))_{1 \leq k < p^{n}} = t_{n} \) for all \( n \geq 1 \). If \( n = 1 \), we have \( (T(r))_{1 \leq k < p^{n}} = t_{1} \), since \( r = p^{0}(p \times 0 + r) \) and \( \lambda^{0} \) is the identity morphism.

Suppose that for \( n \geq 2 \) and for \( 1 \leq \alpha < p^{n} \), we have \( T(\alpha) = t_{n}(\alpha) \).

- We first have \( T(kp^{n}) = \lambda^{n}(k) \) since, on the one hand, \( kp^{n} = p^{n}(p \times 0 + k) \) and the other, \( t_{n+1}(kp^{n}) = \lambda^{n}(k) \) for all \( 1 \leq k < p \) (because \( t_{n+1} = t_{n} \lambda^{n}(1) \bar{r}_{n}^{R} \lambda^{n}(2) \ldots \lambda^{n}(p-1) t_{n} \) (where \( t_{n} = t_{n} \) or \( \bar{r}_{n}^{R} \)) and \( |\bar{r}_{n}| = |r_{n}| = p^{n} - 1 \).

- Now, let \( r \neq kp^{n} \). Write \( r = mp^{n} + \alpha \) where \( 0 \leq m < p \) and \( 1 \leq \alpha < p^{n} \); then \( 1 \leq mp^{n} + \alpha < p^{n+1} \) with \( p^{n+1} - 1 = |t_{n+1}| \).

So, we must prove that

\[
T(mp^{n} + \alpha) = \begin{cases} 
t_{n}(\alpha) & \text{if } m \text{ even}, \\
\bar{r}_{n}^{R}(\alpha) & \text{otherwise}.
\end{cases}
\]

For, write \( \alpha = p^{s}(pj + k) \). Hence, \( 0 \leq s < m \) and \( mp^{n} + \alpha = p^{s}(pi + k) \) with \( i = mp^{n-s-1} + j \).

- Let \( m \) be even. Since \( i \) and \( j \) have the same parity, \( T(mp^{n} + \alpha) = T(\alpha) \) and \( T(\alpha) = t_{n}(\alpha) \) by the induction hypothesis.
Let \( m \) be odd. Using Fact, we easily prove that:
for \( p \) even, \( \tilde{t}_n^R = t_{n-1} \lambda^{n-1}(p - 1) \tilde{t}_{n-1}^R \) and
for \( p \) odd, \( \tilde{t}_n^R = \tilde{t}_{n-1}^R \lambda^{n-1}(p - 1) \tilde{t}_{n-1}^R \).
It follows that \( \tilde{t}_n^R(kp^n - 1) = \lambda^{n-1}(p - k) \) and \( T(mp^n + x) = T(p^{n-1}(pm + k)) = \lambda^{n-1}(p - k) \), since \( m \) is odd. Thus, if \( x = kp^n - 1 \), we have \( T(mp^n + x) = \tilde{t}_n^R(x) \).
But if \( x \neq kp^n - 1 \), we consider two cases:
- if \( p \) is even, we have: on the one hand, \( \tilde{t}_n^R(x) = t_n(x) \) and \( T(x) = T(mp^n + x) \) (since \( i \) and \( j \) have the same parity), and the other, using the induction hypothesis, \( t_n(x) = T(x) \).
Thus we have: \( T(mp^n + x) = \tilde{t}_n^R(x) \).
- if \( p \) is odd, then \( i \) and \( j \) differ in parity. So,
\[
T(mp^n + x) = \begin{cases} 
\lambda^i(k) & \text{if } j \text{ is odd}, \\
\lambda^i(p - k) & \text{otherwise}, 
\end{cases}
\]
and from the induction hypothesis, \( \tilde{t}_n^R(x) = T(p^n - x) \).
Since \( p^n + x = p^n(pm + p - k) \) where \( \mu = p^{n-1} - j - 1 \) has the same parity as \( j \) (because \( p \) odd implies \( p^{n-1} - 1 \) even), we have
\[
T(p^n - x) = \begin{cases} 
\lambda^j(p - k) & \text{if } j \text{ is even}, \\
\lambda^j(k) & \text{otherwise}. 
\end{cases}
\]
and
\[
\tilde{t}_n^R(x) = \begin{cases} 
\lambda^j(p - k) & \text{if } j \text{ is even}, \\
\lambda^j(k) & \text{otherwise}. 
\end{cases}
\]
Finally, we have proved that, in every case, \( T(mp^n + x) = t_{n+1}(x) \). \( \square \)

**Proof of Theorem 1.** Condition 1 is easily deduced from the two technical lemmas.

**Lemma 4.** For all \( n \geq 1 \), we have \( t_{n+1} = h_p(t_n) \tilde{t}_1^R \) where
\[
t_1' = \begin{cases} 
\tilde{t}_1^R & \text{if } p \text{ is even}, \\
t_1 & \text{otherwise}, 
\end{cases}
\]
and \( t_{n+1}^R = h_p(t_n^R) \tilde{t}_1^R \).

**Proof.** This is shown by induction on \( n \), for \( p \) even and for \( p \) odd.
- Let \( n = 1 \).
  - If \( p = 2m \), then \( t_1 = 12 \ldots 2m - 1, t_2 = t_1 \lambda(1) \tilde{t}_1^R \ldots t_1 \lambda(2m - 1) \tilde{t}_1^R \) and \( h_{2m}(t_1) = t_1 \lambda(1) \tilde{t}_1^R \lambda(2) \ldots t_1 \lambda(2m - 1) \tilde{t}_1^R \); it follows that: \( t_2 = h_{2m}(t_1) \tilde{t}_1^R \).
  - In the same way, we have: \( \tilde{t}_1^R = 2m - 1 \ldots 2 \tilde{t}_1, \tilde{t}_2^R = t_1 \lambda(2m - 1) \tilde{t}_1^R \ldots t_1 \lambda(1) \tilde{t}_1^R \) and \( h_{2m}(\tilde{t}_1^R) = t_1 \lambda(2m - 1) \tilde{t}_1^R \lambda(2m - 2) \ldots t_1 \lambda(1) \tilde{t}_1^R \); it follows that: \( \tilde{t}_2^R = h_{2m}(\tilde{t}_1^R) \tilde{t}_1^R \).
  - If \( p = 2m + 1 \), then \( t_1 = 12 \ldots 2m, t_2 = t_1 \lambda(1) \tilde{t}_1^R \ldots \tilde{t}_1^R \lambda(2m) t_1 \) and \( h_{2m+1}(t_1) = t_1 \lambda(1) \tilde{t}_1^R \lambda(2) \ldots \tilde{t}_1^R \lambda(2m) \); it follows that: \( t_2 = h_{2m+1}(t_1) t_1 \).
In the same way, we have: 
\[\overline{t}_1^R = \overline{\lambda}_R \overline{t}_1 \overline{t}_2 \overline{\lambda}(\overline{t}_2) \ldots \overline{t}_n \overline{\lambda}(\overline{t}_n) \overline{t}_1^R\]
and
\[h_{2m+1}(\overline{t}_1^R) = \overline{\lambda}(\overline{t}_1) \overline{t}_2 \overline{\lambda}(\overline{t}_2) \ldots \overline{t}_1 \overline{\lambda}(\overline{t}_n) \overline{t}_1^R;\]
it follows that: 
\[\overline{t}_2^R = h_{2m+1}(\overline{t}_1^R)\overline{t}_1^R.\]

- Suppose that 
\[t = \sum \overline{t}_n \lambda^n(\overline{t}_n^R) \overline{t}_n \lambda^n(2m - 1)\overline{t}_n^R\]
and 
\[\overline{t}^R = h_p(\overline{t}_n^R)\overline{t}_1^R \text{ for some } n \geq 2.\]

- If \(p = 2m\), we successively have:

\[t_{n+1} = t_n \lambda^n(1)\overline{t}_n^R \ldots t_n \lambda^n(2m - 1)\overline{t}_n^R.\]

Since \(n \geq 2\) and \(1 \leq r < 2m\), we have \(\lambda^{n-1}(r) \geq p\) and 
\[\overline{t}_1^R \lambda^n(r) = \overline{\lambda}_R(\lambda^{n-1}(r)) = h_{2m}(\lambda^{n-1}(r)).\]
it follows that
\[t_{n+1} = h_{2m}(t_n)\overline{t}_1^R.\]

In the same way, we successively have:
\[\overline{t}_{n+1} = t_n \lambda^n(1)\overline{t}_n^R \ldots t_n \lambda^n(2m - 1)\overline{t}_n^R.\]

Since \(h_{2m}(\lambda^{n-1}(r)) = \overline{\lambda}(\lambda^{n-1}(r))\), it follows that
\[t_{n+1} = h_{2m}(t_n)\overline{t}_1^R.\]

- If \(p = 2m + 1\), we successively have

\[t_{n+1} = t_n \lambda^n(1)\overline{t}_n^R \ldots \overline{t}_n \lambda^n(2m)\overline{t}_n,\]

\[= h_{2m+1}(t_n)\overline{t}_1^R \overline{t}_2 \overline{\lambda}(\overline{t}_2) \ldots \overline{t}_1 \overline{\lambda}(\overline{t}_n) \overline{t}_1^R \times \lambda^n(2m)h_{2m+1}(t_n).\]

Since
\[h_{2m+1}(\lambda^{n-1}(r)) = \begin{cases} 
\overline{t}_1 \lambda(\lambda^{n-1}(r)) & \text{if } r \text{ is odd,} \\
\overline{t}_1^R \lambda(\lambda^{n-1}(r)) & \text{otherwise,}
\end{cases}\]
it follows that
\[t_{n+1} = h_{2m}(t_n)\overline{t}_1^R \ldots \overline{t}_n \lambda^n(2m)\overline{t}_n^R.\]
In the same way, we successively have
\[
\begin{align*}
t_{n+1}^{R} &= t_{n}^{R} \lambda^{n}(2m) t_{n-1} \ldots t_{n} \lambda^{n}(1) t_{n}^{R} \\
&= h_{2m+1}(t_{n-1}^{R}) t_{1}^{R} \lambda^{n}(2m) h_{2m+1}(t_{n-1}) t_{1} \ldots h_{2m+1}(t_{n-1}) t_{1} \lambda^{n}(1) \\
&= h_{2m+1}(t_{n-1}^{R}) t_{1}^{R} \\
&= h_{2m+1}(t_{n-1}^{R}) \lambda^{n}(2m) t_{n-1} \ldots t_{n-1} \lambda^{n}(1) t_{n-1}^{R} \\
&= h_{2m+1}(t_{n-1}^{R}) t_{1}^{R}.
\end{align*}
\]

Lemma 5. For all \( n \geq 1 \) we have \( h_{n}^{R}(1) = t_{n} \lambda^{n}(1) \).

Proof. By induction on \( n \), for \( p \) even and \( p \) odd, using Lemma 4, we have
- If \( p = 2m \), then \( h_{2m+1}(1) = t_{1} \lambda^{n}(1) \). Suppose that \( h_{2m+1}(1) = t_{n} \lambda^{n}(1) \) for \( n \geq 2 \). We then have \( h_{2m+1}^{R+1}(1) = h_{2m+1}(t_{n} \lambda^{n}(1)) = h_{2m+1}(t_{n}) h_{2m+1}(\lambda^{n}(1)) = h_{2m+1}(t_{n}) h_{2m+1}(t_{n-1}) \lambda^{n}(1) \) since \( \lambda^{n}(1) \) is odd and \( \lambda^{n}(1) \geq p \). It follows that: \( h_{2m+1}^{R+1}(1) = t_{n+1} \lambda^{n+1}(1) \).
- If \( p = 2m + 1 \), then \( h_{2m+1}(1) = t_{1} \lambda^{n}(1) \). Suppose that \( h_{2m+1}^{R+1}(1) = t_{n} \lambda^{n}(1) \) for \( n \geq 2 \). We then have \( h_{2m+1}^{R+1}(1) = h_{2m+1}(t_{n} \lambda^{n}(1)) = h_{2m+1}(t_{n}) h_{2m+1}(\lambda^{n}(1)) = h_{2m+1}(t_{n}) t_{1} \lambda^{n}(1) \) since \( \lambda^{n}(1) \) is odd. It follows that \( h_{2m+1}^{R+1}(1) = t_{n+1} \lambda^{n+1}(1) \).

Proof of Condition 1 of Theorem 1. Evidently, we successively have
\[
\begin{align*}
h_{p}^{R}(1) &= \lim_{n \to \infty} h_{p}^{n}(1) = \lim_{n \to \infty} t_{n} \lambda^{n}(1) = \lim_{n \to \infty} t_{n} = T_{p}. \quad \square
\end{align*}
\]

In the sequel, we need the following lemma:

Lemma 6. Let \( T_{p} = (T(r))_{r \geq 1} \). If \( T(r + \alpha) = T(r) \) for a given integer \( r > 0 \), then \( \alpha = 2mp^{s} + 1 \), where \( m \in \mathbb{N} \) and where \( s \) is determined by the expansion in base \( p \) of \( r = p^{s}(pj + k) \) with \( 1 \leq k < p \) and \( j \in \mathbb{N} \).

Proof. Suppose \( \alpha = p^{s}(pj + k) \) with \( 1 \leq \gamma < p, \beta \in \mathbb{N} \).

Then
\[
r + \alpha = \begin{cases} 
p^{s}(pj + k + p\beta + \gamma) & \text{if } v = s, 
p^{s+1}(pj + k + p\beta + \gamma) & \text{if } v < s, 
p^{s}(pj + k + p^{s}(p\beta + \gamma)) & \text{if } v > s.
\end{cases}
\]

Thus, there exist \( \mu, i \in \mathbb{N} \) and \( 1 \leq \kappa < p \), such that
\[
r + \alpha = \begin{cases} 
p^{s+i}(p\mu + \kappa) & \text{if } v = s, 
p^{s}(p\mu + \gamma) & \text{if } v < s, 
p^{s}(p\mu + k)) & \text{if } v > s.
\end{cases}
\]
Using Definition 3, the equation $T(r + x) = T(r)$ implies

\[
\begin{align*}
&\lambda^{x+1}(k) = \begin{cases} 
\lambda^x(k) & \text{if } (\mu, j) \text{ even}, \\
\lambda^x(p-k) & \text{if } (\mu \text{ even, } j \text{ odd),} \\
\lambda^x(p-k) & \text{if } (\mu \text{ odd, } j \text{ even),} \\
\lambda^x(p-k) & \text{if } (\mu, j) \text{ odd;}
\end{cases} \\
&\lambda^y(p-\gamma) = \begin{cases} 
\lambda^y(k) & \text{if } (\mu, j) \text{ even,} \\
\lambda^y(p-k) & \text{if } (\mu \text{ even, } j \text{ odd),} \\
\lambda^y(p-k) & \text{if } (\mu \text{ odd, } j \text{ even),} \\
\lambda^y(p-k) & \text{if } (\mu, j) \text{ odd;}
\end{cases}
\end{align*}
\]

But $\Sigma \cap \Sigma = \emptyset$ implies that the cases where $\mu$ and $j$ differ in parity are impossible. So, we consider the other cases:

- If $v = s$, then $\lambda^{x+1}(k) = k + (s + i)(p - 1) = \lambda^x(k) = k + s(p - 1)$ implies $k + i(p - 1) = k(\ast)$ and $\lambda^{x+1}(p-k) = p - \kappa + (s + i)(p - 1) = \lambda^x(p-k) = p - k + s(p - 1)$ implies $\kappa + i(p - 1) = k(\ast\ast)$.
  
  If $k + \gamma < p$, then $\kappa = k + \gamma$ and $i = 0$. Hence, $(\ast)$ implies $\gamma = 0$ and $(\ast\ast)$ implies $\gamma = 0$, and this is impossible because $1 < k < p$ and $1 < \kappa < p$ by hypothesis.

- If $v < s$, then $\lambda^y(\gamma) = \gamma + v(p - 1) = \lambda^x(k) = k + s(p - 1)$ implies $\gamma - k = (s - v)(p - 1)$ and $\lambda^y(p-\gamma) = p - \gamma + v(p - 1) = \lambda^x(p-k) = p - k + s(p - 1)$ implies $\gamma - k = (s - v)(p - 1)$; thus $\gamma = (s - v)(p - 1)$.

So, we have $|\gamma - k| > p$ (since $v < s$), and this is impossible because $1 < k < p$ and $1 < \kappa < p$ by hypothesis.

- Thus, we have $v > s$. But $p^y(pj + k) + p^y(p\beta + \gamma) = p^y[p^y(pj + p^{v-s-1}(p\beta + \gamma)) + k] = p^y(pu + k)$ with $u = j + p^{v-s-1}(p\beta + \gamma)$. Since $\mu$ and $j$ have the same parity, hence $\mu - j$ is even and also $p^{v-s-1}(p\beta + \gamma)$ for all $p$. It follows that $(v = s + 1, \beta = 0$ and $\gamma$ even) i.e., $x = 2mp^{s+1}$ with $m \in \mathbb{N}$. \[\square\]

Proof of Proposition 3. If $T_p = (T(r))_{r \geq 1}$ is ultimately periodic then there exist a unique integer $x > 0$ and an integer $N > 0$ such that $T(r + x) = T(r)$ for all $r > N$. Thus, in particular, the last equality is true for $r = p^{x+1}k$ (where $1 \leq k < p$). Using Lemma 6, we have $x = 2mp^{s+1}$ with $m \in \mathbb{N}$, hence $r + x = p^{x+1}(p \times 0 + k + 2m)$. But $T(r + x) = \lambda^{x+1}(k + 2m) = T(r) = \lambda^{x+1}(k)$ implies $(m = 0$ and $x = 0$) and this is a contradiction. \[\square\]
Condition 2 of theorem 1 can be deduced from a well-known result of Cobham [12]:  
Let \( x \) be an automatic sequence and \( p, q \) be multiplicatively independent positive integers (i.e., such that \( p^aq^b = 1 \) for integers \( a, b \) implies \( a = b = 0 \)). Then \( x \) is both \( p \)-automatic and \( q \)-automatic if and only if \( x \) is ultimately periodic.

However, in order to make the present paper self-contained, we directly show that the \( p^v \)-Toeplitz word is \( p^{v+n} \)-morphic of base \( h^p_n \), where \( n \geq 0 \) but it cannot be \( p^n \)-morphic with \( n < v \). In particular, if \( p \) is not a power of another integer, the \( p^v \)-Toeplitz word is not \( p \)-morphic of base \( h_p \).

**Proof of Condition 2 of Theorem 1.** Let \( \kappa \) an integer \( \kappa \geq 2 \). If \( T_p \) is \( \kappa \)-morphic of base \( h_\kappa \), then \( h_\kappa(T_p) = h_\kappa(\lim_{n \to \infty} h_\kappa^n(1)) = \lim_{n \to \infty} h_\kappa^{n+1}(1) = h_\kappa^n(1) = T_p \). Thus, there exists \( r > 0 \) such that \( T(r + z) = T(r) \) with \( z = 2mp^{p+1} \) if \( r = p^s(pj + k), 1 < k < p \) and since \( h_\kappa \) is a morphism, we also have \( h_\kappa(T(r)) = h_\kappa(T(r + z)) \).

Now we consider the following scheme with \( |h_\kappa(r)| = \kappa \geq 2 \) for all \( r > 0 \).

**Proof of Proposition 4.** If \( w \) is \( p \)-automatic then \( (f_{p,i})_{i \geq 0} = ((w(p^i k))_{k < p})_{i \geq 0} \) must be ultimately periodic (see [1, 11]). Conversely, let \( f_{p,0} \ldots f_{p,i-1} (f_{p,i} \ldots f_{p,i+\tau-1})^\omega \) with \( i, \tau > 0 \) an ultimately periodic sequence. Hence, the set of all subsequences \( (f_{p,i})_{i \geq k} \) is finite and it is well known that in this case (see [13]), the sequence \( w \) is \( p \)-automatic. \( \square \)
Proof of Lemma 2.

(1) If \( r \leq (i + \tau - 1)(p - 1) \), then \( h_{p,i,z}(r) = h_{p,i,z}(r) = h_p(r) = \rho \circ h_p(r) \) (the last equality holds because \( h_p(\lambda^s(k)) = t_i \lambda^{s+1}(k) \) with \( 1 \leq k < p \) and \( s < i + \tau \)).

- If \( r = \lambda^{i+z-1}(1) \), \( 1 \leq k < p \), using the definition of \( \rho \) (Definition 6), we have
  \[ h_{p,i,z} \circ \rho(\lambda^{i+z-1}(1)) = h_{p,i,z}(t_i \lambda^i(k)) \]
  and \( \rho \circ h_p(\lambda^{i+z-1}(k)) = \rho(t_i \lambda^{i+z-1}(k)) = t_i \lambda^i(k) \).

- If \( r \geq \lambda^{i+z}(1) \), write \( r = \lambda^{i+z}(k) \) with \( i \leq s < i + \tau \), \( \kappa > 0 \) and \( 1 \leq k < p \).

Then

\[
h_{p,i,z} \circ \rho(r) = h_{p,i,z}(\lambda^s(k)) = \begin{cases} h_p(\lambda^s(k)) & \text{if } s < i + \tau - 1, \\ t_i \lambda^i(k) & \text{if } s = i + \tau - 1, \end{cases}
\]

and since \( \rho(\lambda^{i+z}(k)) = \lambda^s(k) \), using \( \kappa \) times the relation: \( \rho(r) = \rho(r - \tau(p - 1)) \), we have

\[
\rho \circ h_p(r) = \rho(t_i \lambda^{i+z-1}(k)) = \begin{cases} t_i \lambda^{i+1}(k) & \text{if } s < i + \tau - 1, \\ t_i \lambda^i(k) & \text{if } s = i + \tau - 1. \end{cases}
\]

So, in every case, we have \( h_{p,i,z} \circ \rho(r) = \rho \circ h_p(r) \).

(2) This relation will be shown by induction on \( n \). We have \( h_{p,i,z}(1) = h_p(1) = t_i \lambda(1) = \rho \circ h_p(1) \) for all \( i, \tau > 0 \), since \( i + \tau - 1 \geq 1 \). Suppose that \( h_{p,i,z}(n) = \rho \circ h_p(n) \) for \( n > 2 \). We then have

\[
h_{p,i,z}(n+1) = h_{p,i,z}(n) \circ \rho \circ h_p(1) = \rho \circ h_p \circ h_p(1) = \rho \circ h_{p+1}(1).
\]

(3) This relation is an immediate consequence of the above result. \( \square \)

Proof of Theorem 2. Using the above result, the first part is clear: for every integers \( i, \tau > 0 \), we have \( \sigma_{p,i,z}[\rho(T_p)] = \sigma_{p,i,z}[h_p^{n_z}(1)] \) i.e., the \( p \)-paperfolding sequence \( \sigma_{p,i,z}[\rho(T_p)] \) associated with \( f_{p,i,z} \) is \( p \)-automatic (because \( h_{p,i,z} \) is a morphism \( p \)-uniform and expansive on letter 1 since \( h_p \) is, and \( \sigma_{p,i,z} \) is a literal morphism since \( \sigma_f \) is).

The converse is deduced from the following arguments:

- A \( p \)-paperfolding sequence is \( p \)-automatic if and only if its associated sequence of \( p \)-unfolding instructions is ultimately periodic;
- Each pair of integers \( i, \tau > 0 \) determines one ultimately periodic sequence \( f_{p,i,z} \).

Thus we obtain all ultimately periodic sequences if we consider every pair \( (i, \tau) \); two different sequences \( f_{p,i,z} \) give two different \( p \)-uniform tag systems \( G_{i,\tau}(f_{p,i,z}) \). \( \square \)

6. Conclusion

This paper gives an answer to a question of Allouche [3] who asked whether the 2-paperfolding tag systems for 2-paperfolding sequences which we have described in [7] can be extended to the case of \( p \)-paperfolding sequences with \( p > 3 \). After giving a general definition of \( p \)-paperfolding sequences for \( p \geq 2 \), we have shown that this
generalization is possible if one distinguishes between two cases: \( p \) even and \( p \) odd. The very simple form of these tag systems allow us to implement an algorithm to generate the associated \( p \)-dragon curves. It would be interesting to know the geometrical properties of these curves.

It is clear that all the \( p \)-automatic \( p \)-paperfolding sequences are \( p' \)-automatic for all integers \( r > 0 \) and some \( p' \)-paperfolding sequences are \( p \)-automatic, but the question arises as to whether a \( p \)-paperfolding sequence can be \( \kappa \)-automatic with \( \kappa \neq p' \)?

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