Complete integrability and the Miura transformation of a coupled KdV equation

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In this letter, a Painlevé integrable coupled KdV equation is proved to be also Lax integrable by a prolongation technique. The Miura transformation and the corresponding coupled modified KdV equation associated with this equation are derived.

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1. Introduction

The coupled KdV equations have many applications in several physical fields, such as those of shallow stratified liquids [1,2], atmospheric dynamical systems [3], Bose–Einstein condensates [4], and so on. In [5], Lou et al. proved that the coupled KdV equation

\[
\begin{align*}
\frac{u_t}{2} - \frac{(7 - 3\alpha)}{2} u_{xxx} - u_x u_x - u_x v - u_x v_x - \frac{(1 - \alpha)}{2} u v_x + \frac{(1 + \alpha)}{2} v v_x &= 0, \\
v_t + v_{xxx} + u_x v + u v_x + v v_x + \frac{(1 + \alpha)}{2} u v_x - \frac{(1 - \alpha)}{2} u u_v_x &= 0,
\end{align*}
\]

which is a new model for describing two-layer fluids with different dispersion relations, is Painlevé integrable by Kruskal’s simplification [6]. Further questions are whether this equation is Lax integrable in the sense of having a Lax pair and how to find its Miura transformation. To answer the two questions, in this letter, we first give the Lax pair of this equation by means of the prolongation technique [7,8] and then derive its Miura transformation [9] and the associated modified equation from the Lax pair obtained.

This letter is organized as follows. In Section 2, the prolongation structures and Lax pair of Eq. (1) are proposed, using the prolongation technique. In Section 3, the Miura transformation and corresponding modified equation of Eq. (1) are given. The conclusions are proposed in the last section.

2. The prolongation structures and the Lax pair of Eq. (1)

In this part, we will propose the prolongation structures and Lax pair of the coupled KdV equation (1). To do this, we first give a basic theorem in the theory of Lie algebra representation [10].
**Theorem.** Let $X$ and $Y$ be two elements of the Lie algebra $g = sl(n+1, C)$ such that $[X, Y] = aY$ ($a \neq 0$) and $X \in \text{range ad } Y$. Then we may identity $Y$ with $e_{\pm}$ and $X$ with $\pm \frac{1}{2} ah$, where $e_{\pm}$ are the nilpotent elements of $g$ and $h$ is the neutral element of $g$.

For Eq. (1), we define a new set of independent variables
\[ p = u_x, \quad q = u_{xx} = p_x, \quad z = v_x, \quad r = v_{xx} = z_x. \] (2)

Then Eq. (1) can be represented by the following set of 2-forms:
\[
\begin{align*}
\alpha_1 &= du \wedge dt - pdx \wedge dt, \\
\alpha_2 &= dp \wedge dt - qdx \wedge dt, \\
\alpha_3 &= dv \wedge dt - zdx \wedge dt, \\
\alpha_4 &= dz \wedge dt - rdx \wedge dt, \\
\alpha_5 &= du \wedge dx + \left( \frac{\alpha - 1}{2} vp + \frac{\alpha + 1}{2} vz - pv - uz - up \right) dt \wedge dx - \frac{3\alpha - 7}{2} dq \wedge dt, \\
\alpha_6 &= dv \wedge dx + \left( -\frac{\alpha + 1}{2} vp + \frac{\alpha - 1}{2} up + \frac{3 + \alpha}{2} pv + \frac{3 + \alpha}{2} uz + vz \right) dt \wedge dx - dr \wedge dt,
\end{align*}
\] (3)

which is closed under the exterior differential, that is
\[ d\alpha_i = \sum_{j=1}^{6} g_j \wedge \alpha^j \]
where $g_j$ ($j = 1, 2, \ldots, 6$) are differential 1-forms.

Next we introduce the system of 1-forms
\[ \omega^i = d\omega^j - F^j(u, v, p, q, z, r, y^j) dx - C^j(u, v, p, q, z, r, y^j) dt, \] (4)

where $y^i$ ($i = 1, 2, \ldots, n$) are called pseudopotentials and we assume that $F^j$ and $C^j$ are of the form
\[ F^j = F^j y^j, \quad C^j = C^j y^j. \]

Next we rewrite $F^j$ as $F$ and $C^j$ as $G$ for simplicity.

We need $\{\alpha_1, \ldots, \alpha_6; \omega^j\}$ to be also a closed ideal, namely
\[ d\omega^j = \sum_{j=1}^{6} f_j^i \alpha^j + \eta^j \wedge \omega^j, \]
which leads to a set of nonlinear partial differential equations concerning $F$ and $G$ as follows:
\[
\begin{align*}
F_p &= F_q = F_z = F_r = 0, \quad \frac{3\alpha - 7}{2} F_u + G_q = 0, \quad F_v + G_r = 0, \\
G_u p + G_q + G_v z + G_z r + F_u &\left[ \frac{\alpha - 3}{2} pv + \frac{\alpha + 1}{2} vz - uz - up \right] \\
&+ F_v \left[ pv + \frac{\alpha - 1}{2} up + \frac{3 + \alpha}{2} uz + vz \right] - [F, G] = 0,
\end{align*}
\] (5)


One solution of this set of equations is
\[
\begin{align*}
F &= uX_1 + vX_2 + X_3, \\
G &= \frac{7 - 3\alpha}{2} qX_1 - rX_2 + \left( \frac{3\alpha - 7}{2} vz - up \right) X_3 + \frac{7 - 3\alpha}{2} pX_5 - zX_6 + X_9 u^2 + v^2 X_9 + uX_{10} + uX_{11} + vX_{12} + X_7
\end{align*}
\] (6)

with
\[
\begin{align*}
X_4 &= [X_1, X_2], \quad X_5 = [X_3, X_1], \quad X_6 = [X_3, X_2], \quad [X_3, X_7] = 0, \\
[X_1, X_8] = [X_2, X_9] = [X_1, X_4] = [X_2, X_4] = 0, \quad [X_1, X_7] + [X_3, X_{11}] = 0, \\
[X_2, X_7] + [X_3, X_{12}] = 0, \quad [X_3, X_9] + [X_1, X_{11}] = 0, \quad [X_1, X_9] + [X_2, X_{10}] = 0, \\
[X_3, X_9] + [X_2, X_{12}] = 0, \quad [X_1, X_{10}] + [X_2, X_9] = 0, \quad [X_3, X_{10}] + [X_1, X_{12}] + [X_2, X_{11}] = 0.
\end{align*}
\] (7)
and
\[ \frac{3 + \alpha}{2} X_2 - X_1 + [X_1, X_6] + [X_3, X_4] + X_{10} = 0, \]
\[ \frac{3\alpha - 9}{2} X_4 = 0, \quad X_{11} = \frac{3\alpha - 7}{2} [X_5, X_3], \quad X_{12} = [X_6, X_3], \]
\[ \frac{\alpha - 3}{2} X_1 + X_2 + \frac{3\alpha - 7}{2} [X_2, X_5] - \frac{3\alpha - 7}{2} [X_3, X_4] + X_{10} = 0, \]
\[ \frac{\alpha - 1}{2} X_2 - X_1 + \frac{3\alpha - 7}{2} [X_1, X_5] + 2X_8 = 0, \quad \frac{\alpha + 1}{2} X_1 + X_2 + [X_2, X_6] + 2X_9 = 0, \]
where \([X_i, X_j] = X_i X_j - X_j X_i\), and the \(X_i (i = 1, 2, \ldots, 12)\) determine an incomplete Lie algebra \(L\), which is called a prolongation algebra.

In order to introduce the spectral parameter of corresponding Lax pairs of Eq. (1), we note that Eq. (1) has the following scale symmetry:
\[ x \to {\lambda}^{-1} x, \quad t \to {\lambda}^{-3} t, \quad u \to {\lambda}^2 u, \quad v \to {\lambda}^2 v, \quad \lambda \neq 0, \]
which leads to the automorphism of the prolongation algebra as follows:
\[ F \to \lambda F, \quad G \to \lambda^3 G \]
and the \(X_i\) must satisfy
\[ X_1 \to {\lambda}^{-1} X_1, \quad X_2 \to {\lambda}^{-1} X_2, \quad X_3 \to {\lambda} X_3, \]
\[ X_4 \to {\lambda}^{-2} X_4, \quad X_5 \to {\lambda} X_5, \quad X_6 \to X_6, \quad X_7 \to {\lambda}^3 X_7, \]
\[ X_8 \to {\lambda}^{-1} X_8, \quad X_9 \to {\lambda}^{-1} X_9, \quad X_{10} \to {\lambda}^{-1} X_{10}, \quad X_{11} \to {\lambda} X_{11}, \quad X_{12} \to {\lambda} X_{12}. \]

To find the matrix representation of \(X_i (i = 1, 2, \ldots, 12)\), we try to embed the prolongation algebra \(L\) represented by (7) and (8) in \(sl(n, C)\). Starting from the cases \(n = 2, 3\), we found that \(sl(2, C)\) and \(sl(3, C)\) cannot be the whole algebra. After some computation, we find \(sl(4, C)\) is just what is needed, i.e. \(n = 4\). So by embedding \(L\) in \(sl(4, C)\) and using the theorem at the beginning of this part, we get the matrix representation of \(X_i (i = 1, 2, \ldots, 12)\) as follows:

\[ X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & \frac{\alpha - 3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ X_3 = \begin{pmatrix} 0 & \frac{1}{4} - \frac{\alpha}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda (2 + \alpha)}{6} & 0 & \frac{2 + \alpha}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ X_5 = [X_3, X_1], \quad X_6 = [X_3, X_2], \quad X_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ X_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{(7 - 3\alpha)(3 + \alpha)}{12} & 0 & 0 & 0 \\ \lambda (3\alpha - 7)(2 + \alpha) & 0 & (7 - 3\alpha)(2 + \alpha) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[ X_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} - \frac{\alpha}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda - \frac{\lambda \alpha^2}{6} & 0 & (2 + \alpha)(\alpha - 3)^2 & 0 \end{pmatrix}. \]
The Miura transformation and a new coupled modified KdV equation

In this part, we will derive the Miura transformation and a new coupled modified KdV equation associated with Eq. (1). To do so, we rewrite the Lax pair (11) with (12) of the coupled KdV equation (1) by setting \( \lambda = 1 \) as follows:

\[
\begin{pmatrix}
y_1' \\ y_2' \\ y_3'
\end{pmatrix}_{xx} = U \begin{pmatrix}
y_1' \\ y_2' \\ y_3'
\end{pmatrix}, \quad \begin{pmatrix}
y_1' \\ y_2' \\ y_3'
\end{pmatrix}_t = M \begin{pmatrix}
y_1' \\ y_2' \\ y_3'
\end{pmatrix} + N \begin{pmatrix}
y_1' \\ y_2' \\ y_3'
\end{pmatrix}_x, \tag{13}
\]

with

\[
U = \begin{pmatrix}
\frac{1}{4} - \frac{\alpha}{12} & 0 & 0 \\
\frac{2 + \alpha}{6} (u + v) & -\frac{2}{3} - \frac{\alpha}{3} & 0 \\
(3 + \alpha) (3u_x - 7u_x + 2v_x) & 0 & 0
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
\frac{v_x (2 + \alpha)}{6} + \frac{u_x (\alpha - 1)}{12} & (2 + \alpha) & 0 \\
\frac{2u - 7v - 3u_x}{6} & \frac{v_x (\alpha - 3) + u_x (3\alpha - 7)}{12} & 0 \\
\frac{u_x + 5u_x + u + 11v}{3(3 + \alpha)} & \frac{3u + u_x + 3v_x + 7v}{3(2 + \alpha)} & 0
\end{pmatrix},
\]

\[
N = \begin{pmatrix}
0 \\
\frac{24}{v_x (2 + \alpha)} & \frac{24}{(3 + \alpha)} & 0 \\
\frac{2u - 7v - 3u_x}{6} & \frac{v_x (\alpha - 3) + u_x (3\alpha - 7)}{12} & 0 \\
\frac{u_x + 5u_x + u + 11v}{3(3 + \alpha)} & \frac{3u + u_x + 3v_x + 7v}{3(2 + \alpha)} & 0
\end{pmatrix}
\]

where \( \lambda \) is a spectral parameter and \( A = \frac{(7 - 3\alpha)}{2} u_x - v_x - \frac{1}{6} (u + v) (3v + v_x + u_x - 3u), B = \frac{(7 - 3\alpha)}{2} u_x - \frac{(\alpha - 3)}{2} v_x - \frac{(\alpha - 1)}{3} (v_x - v)^2, \) and \( C = \frac{(2 + \alpha)}{6} (v_x - 3v_x - 7u_x + 3u_x). \)
Let $\Psi$ be the fundamental matrix of solutions of (13), that is
\[ \Psi_{xx} = U\Psi, \quad \Psi_t = M\Psi + N\Psi_x. \] (14)

Next we define a new matrix as follows:
\[ P = \begin{pmatrix} \left( -\frac{1}{4} - \frac{\alpha}{12} \right) (p + q) & 0 \\ \frac{2 + \alpha}{6} \left( p + q \right) + \left( -\frac{2}{3} - \frac{\alpha}{3} \right) q & \left( -\frac{2}{3} - \frac{\alpha}{3} \right) \left[ p + q \left( \frac{\alpha}{2} - \frac{3}{2} \right) \right] \end{pmatrix} = \Psi_x \Psi^{-1}. \] (15)

Because of
\[ (\Psi^{-1})_x = -\Psi^{-1} \Psi_x \Psi^{-1}, \] (16)
we have
\[ P_x = U - p^2, \] (17)
which just provides the following Miura transformation for Eq. (1):
\[ u = p_x + \frac{pq (\alpha + 1)}{6} - \frac{p^2 (7 + 3\alpha)}{12} - \frac{q^2 (\alpha + 1)}{12}, \]
\[ v = q_x - \frac{pq (2 + \alpha)}{3} + \frac{p^2 (2 + \alpha)}{6} - \frac{q^2}{6}. \] (18)

Differentiating $\Psi_t = M\Psi + N\Psi_x$ about $x$, then differentiating $P = \Psi_x \Psi^{-1}$ about $t$ and making use of (14), we have
\[ P_t = M_x + (N_x - PN)P + NU + [M, P]. \] (19)

Substituting the expressions for $U$, $M$, $N$ and the Miura transformation (18) into Eq. (19), we find that Eq. (19) can be rewritten as
\[
\begin{align*}
p_t &= \frac{7}{2} - 3\alpha p_{xxx} + \frac{1}{3}q^2 q_x - \frac{3 + \alpha}{3}pqp_x + \frac{1 + \alpha}{3}pqq_x + \frac{\alpha - 1}{2}p_x q_x \\
&\quad - \frac{\alpha - 1}{2}q_x^2 - \frac{1 + \alpha}{2}p_x q^2 - \frac{3 + \alpha}{6}p^2 q_x - \frac{\alpha - 1}{2}p_x q_{xx},
\end{align*}
\]
\[
\begin{align*}
q_t &= -q_{xx} + \frac{2 (2 + \alpha)}{3}pq p_x + \frac{3 + \alpha}{3}pq q_x - \frac{3\alpha + 7}{6}p_x p^2 + \frac{1 + \alpha}{2}p_x q_x \\
&\quad + \frac{1 + \alpha}{2}p_x q - \frac{1 + \alpha}{2}pp_{xx} + \frac{2 + \alpha}{3}p^2 q_x + \frac{3 + \alpha}{6}p_{xx} q^2 - \frac{1 + \alpha}{2}p_x q_x,
\end{align*}
\] (20)
which is a new coupled modified KdV equation for $p$ and $q$.

So we have the following proposition immediately.

**Proposition.** Assume $(u, v)$ is a solution of the coupled KdV equation (1); then we can construct a Miura transformation (18) to connect the solution $(u, v)$ of this coupled KdV equation with the solution $(p, q)$ of the associated coupled modified KdV equation, i.e. Eq. (20).

4. Conclusions

In conclusion, by using the prolongation technique the Painlevé integrable coupled KdV equation (1) is proved to be also Lax integrable, which provides the complete integrability of this equation. In addition, the Miura transformation and the coupled modified KdV equation associated with this integrable equation are also derived. In future work, we hope to obtain the new exact solutions of Eq. (1) by using the Lax pair proposed in this letter.

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**References**