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EXTREMAL CRITICALLY CONNECTED MATROIDS

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Abstract. A connected matroid M is called a critically connected matroid if the deletion of any one element from M results in a disconnected matroid. We show that a critically connected matroid of rank n, $n \geq 3$, can have at most $2n-2$ elements. We also show that a critically connected matroid of rank n on $2n-2$ elements is isomorphic to the forest matroid of $K_{2,n-2}$.

1. Introduction

A non-separable graph G is called a critically non-separable graph if the deletion of any edge from G results in a graph which is separable. Dirac [1] and Plummer [3] have characterized critically non-separable graphs. Dirac [1] has shown that a critically non-separable graph on n vertices has at most $2n-4$ edges. In this paper we generalize this result to matroids. The relevant definitions and theorems from matroid theory are given in Section 2. The results of this paper are proved in Section 3.

2. Matroids

A *matroid* $M = (E, \mathcal{I})$ is a finite set E of *elements* together with a non-empty family $\mathcal{I}$ of subsets of E, called *independent sets*, such that:

(I1) Every subset of an independent set is independent.

(I2) For every $A \subseteq E$, all maximal independent subsets of A have the same cardinality, called the *rank* $r(A)$ *of* A.

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An example of a matroid is obtained as follows: Let G be a graph. Let E denote the set of edges of G and let I denote the family of edge sets of forests contained in G. Then $M = (E, I)$ is a matroid called the *forest matroid of G*. It is denoted by $F(G)$. A matroid is called a *graphic* matroid if it is isomorphic to the forest matroid of some graph.

A maximal independent subset of A, where $A \subseteq E$, is called a *basis of* A (or M-basis of A if we wish to specify the matroid being considered). An M-basis of E is called a *basis of M*; $r(M)$, the rank of M, equals $r(E)$. A subset of E is called dependent if it is not independent. Minimal dependent sets are called *circuits*. The family of circuits of M determines a matroid. Indeed Whitney [5] showed that a family $\mathcal{C}$ of non-empty subsets of a finite set E is the family of circuits of a matroid M on E if and only if the following conditions (to be called circuit axioms) are satisfied:

(C1) A proper subset of a member of $\mathcal{C}$ is not a member of $\mathcal{C}$.

(C2) (Exange axiom). If $a \in C_1 \cap C_2$ and $b \in C_1 - C_2$, where $C_1, C_2 \in \mathcal{C}$ and $a, b \in E$, then there exists a $C_3 \in \mathcal{C}$ such that $b \in C_3 \subseteq C_1 \cup C_2 - \{a\}$.

The ciruits of the forest matroid $F(G)$ of a graph G are the edge sets of cycles of G, the bases are the edge sets of maximal spanning forests of G, and the rank of $F(G)$ is equal to the number of vertices minus the number of connected components of G.

Let $M = (E, I)$ be a matroid. If $X \subseteq E$, then the matroid on $E-X$ whose circuits are those M-circuits which are contained in $E-X$ is called the restriction of M to $E-X$ (or the matroid obtained by deleting X from M) and is denoted by M'_X. There is another derived matroid of importance introduced by Tutte [4]. If $X \subseteq E$, then the family of minimal non-empty intersections of $E-X$ with M-circuits is the family of circuits of a matroid on $E-X$ called the *contraction* of M to $E-X$ (or the matroid obtained by contracting X out of M) and is denoted by M''_X. If $X = \{e\}$, we shall simply write M'_e and M''_e for restriction and contraction to $E-\{e\}$, respectively. Deletion and contraction of elements in the forest matroid $F(G)$ of a graph G correspond to deletion and contraction of edges in G. Tutte [4] has shown that the operations of deletion and contraction of elements from a matroid commute. More precisely we have the following:

Lemma 2.1. *If M is a matroid and if X and Y are two disjoint sets of elements of M, then*

$$(M'_X)''_Y = (M''_Y)'_X .$$

A subset S of the set of elements E of a matroid M is called a *separator* of M if every circuit of M is either contained in S or $E-S$. Tutte [4] has shown that a subset S of E is a separator if and only if $r(S)+r(E-S) = r(E)$. Union and intersection of two separators of M is also a separator of M. If $\emptyset$ and E are the only separators of M, then M is said to be *connected*. The minimal non-empty separators of M are called the *components* of M. An element e of a connected matroid M is said to separate elements x and y if x and y belong to different components of M'_e. If G is a graph, the components of $F(G)$ are blocks of G and $F(G)$ is connected if and only if G is non-separable.

We now state two lemmas which are used in the proofs in Section 3. Lemma 2.2 is due to Whitney [5], a proof of Lemma 2.3 can be found in [2].

Lemma 2.2. *A matroid M is connected if and only if for every pair e_1, e_2 of distinct elements of M, there is a circuit containing both e_1 and e_2.*

Lemma 2.3. *If M is a connected matroid, then for every $e \in E$, either M'_e or M''_e is also connected.*

A matroid is called *simple* if it does not contain any one or two element circuits. Simple matroids are also known as combinatorial geometries.

Lemma 2.4.[1] *A simple matroid M of rank 3 is connected if and only if it contains a circuit of cardinality 4.*

Proof. Let M be a simple matroid of rank 3 which contains a circuit C of cardinality 4. If possible, let S, $S \neq \emptyset, E$, be a separator. Then $r(S) + r(E-S) = 3$. Without loss of generality, we may assume that $C \subseteq S$. Then $r(S) \geq r(C) = 3$. But this implies that $r(E-S) = 0$. Since M

[1] This lemma is, in fact, true for all matroids. The proof given here can be extended to prove this more general statement.

is a simple matroid, we have $E - S = \emptyset$, which implies that $S = E$. This is contradictory to our assumption. Thus M is connected.

Conversely, let M be a simple connected matroid of rank 3. Suppose, if possible, that M does not contain any circuit of cardinality 4. Then all circuits of M are of cardinality 3. Let $\{e, f, g\}$ be a basis of M. By Lemma 2.2, there exist circuits $C_1 = \{e, f, h\}$ and $C_2 = \{e, g, i\}$. If $h = i$, then, by applying the exchange axiom for circuits, we obtain that $\{e, f, g\}$ is a circuit. Therefore $h \neq i$. Now by the exchange axiom, $C = C_1 \cup C_2 - \{e\} = \{f, g, h, i\}$ contains a circuit containing f. If C itself is not a circuit, then at least one of $C_3 = \{f, g, h\}$, $C_4 = \{f, g, i\}$ and $C_5 = \{f, h, i\}$ is a circuit. If C_3 is a circuit, then, by the exchange axiom, $C_1 \cup C_3 - \{h\} = \{e, f, g\}$ contains a circuit. But this is impossible because $\{e, f, g\}$ is a basis of M. Similarly, we may establish that both C_4 and C_5 are also not circuits thereby establishing that C is a circuit. The proof is complete.

Let $M = (E, \mathcal{I})$ be a matroid. Let X be a subset of E. The elements of X are said to be *in series* in M if any circuit of M which contains an element of X contains all the elements of X.

Let e_1 be an element not in E and let e be an element of E. Define

$$E_1 = E \cup \{e_1\},$$
$$\mathcal{I}_1 = \mathcal{I} \cup \{I \cup \{e\} : e \notin I, I \in \mathcal{I}\} \cup \{I \cup \{e_1\} : I \in \mathcal{I}\}.$$

Then it may be verified that $(E_1, \mathcal{I}_1)$ is a matroid. In this matroid, e and e_1 are in series — thus it is said to have been obtained by *extending M by series at e*. In case of graphic matroids, series extension corresponds to subdivision of an edge by inserting a vertex.

A matroid M is called a *one element extension* of another matroid N if

(i) $r(M) = r(N)$, and

(ii) there is an element e of M such that $M'_e = N$.

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B}$ denote the family of bases of M. Let $\mathcal{B}^*$ denote the family of complements of members of $\mathcal{B}$ in E. Then Whitney [5] showed that $\mathcal{B}^*$ is the family of bases of a matroid, denoted by M^*, called the *dual* of M. If G is a graph, then the dual of $F(G)$ is called the *co-forest* matroid of G. The circuits of this matroid are minimal cutsets or bonds of G.

The circuits of M^* are called the *co-circuits* of M. A maximal subset of E that is of M-rank $r(M) - 1$ is called a *hyperplane* of M. The following lemma follows from the definition of the dual.

Lemma 2.5. *A subset Y of E is a co-circuit of M if and only if $E - Y$ is a hyperplane of M.*

The following lemma is due to Tutte [4].

Lemma 2.6. *If $X \subseteq E$, then*

$$(M''_X)^* = (M^*)'_X .$$

Thus a co-circuit of M''_X is a co-circuit of M.

We are now ready to state and prove the results of this paper.

3. Critically connected matroids

Let $M = (E, \mathcal{I})$ be a connected matroid. An element e of M is called an *essential* element if M'_e is disconnected. Otherwise it is called an *inessential* element. A connected matroid each of whose elements is essential is called a *critically connected* matroid or simply a critical matroid. As mentioned in the introduction, Dirac [1] and Plummer [3] have characterized critical graphic matroids. However, not every critical matroid is graphic. By series extension of each of the elements of any simple connected matroid of rank ≥ 2, one obtains a critical matroid. No useful characterization of critical matroids is known. Here by a useful characterization I mean one which suggests a recursive construction of all critical matroids.

The following lemma generalizes the fact that every critically non-separable graph on 3 or more vertices contains a vertex of degree 2.

Lemma 3.1. *A critical matroid of rank ≥ 2 contains a co-circuit of cardinality two.*

Proof. If $r(M) = 2$, it is easy to observe that M has a co-circuit of cardinality two. If the lemma is false, let n be the least integer ≥ 3 for which there is a critical matroid of rank n with no co-circuits of cardinality two.

Let e be an element of M. Since M is critical, M'_e is disconnected. Hence, by Lemma 2.3, M''_e is connected. Now if x is any element of $E - \{e\}$, the matroid $(M'_x)''_e$ is disconnected unless $\{e\}$ is a component of M'_x. But if $\{e\}$ is a component of M'_x, then $\{e, x\}$ is a co-circuit of M; which is a contradiction. It follows that $(M'_x)''_e$ is disconnected for each $x \in E - \{e\}$. Since, by Lemma 2.1, $(M''_e)'_x = (M'_x)''_e$, we have that $(M''_e)'_x$ is disconnected for all $x \in E - \{e\}$. It follows that M''_e is a critical matroid. But $r(M''_e) = n - 1$. Therefore, by the minimality of n, M''_e has a co-circuit of cardinality two. But a co-circuit of M''_e is also a co-circuit of M. This contradiction proves the lemma.

Of all the critical matroids of a given rank, one with the largest number of elements is called an *extremal* critical matroid or simply an extremal matroid. In the sequel we shall characterize extremal matroids. Our characterization is a generalization of a theorem due to Dirac [1] for graphs.

We observe that the forest matroid of $K_{2,n-1}$ $(n \geq 3)$ is a critical matroid of rank n. Thus, if we define $f(n)$ as the number of elements in an extremal matroid of rank n, we have for $n \geq 3$,

(1) $$f(n) \geq 2n - 2 .$$

Theorem 3.2. *If $M = (E, I)$ is an extremal matroid of rank n, $n \geq 3$, then*

(2) $$f(n) = |E| \leq 2n - 2 .$$

Proof. It follows from Lemma 2.4 that the only critical matroid of rank 3 is a circuit of cardinality 4. Thus the theorem is valid when $n = 3$. We shall prove it in general by induction on n: Let n be an integer ≥ 4 and assume that $f(k) \leq 2k - 2$ for all k such that $3 \leq k < n$. Consider an extremal matroid M of rank n.

By Lemma 3.1, M has a co-circuit of cardinality two. Let $\{a, b\}$ be a co-circuit of M.

Consider M''_a. Since M'_a is disconnected, M''_a is connected. If M''_a is critically connected, then, by induction hypothesis, we have

$$|E - \{a\}| \leq f(n-1) \leq 2(n-1) - 2 ,$$

from which it follows that $f(n) = |E| \leq 2n - 3$. But this contradicts (1). Thus M''_a is not critical.

We shall now show that if $e \in E - \{a, b\}$, then e is essential in M''_a.

Case (i) Suppose that a, b and e are in series. Then M'_e has at least three components; $\{a\}$ and $\{b\}$ are two of the components of M'_e. Thus $(M'_e)''_a$ is disconnected. But, by Lemma 2.1, $(M''_a)'_e = (M'_e)''_a$ it follows that $(M''_a)'_e$ is disconnected. Thus e is essential in M''_a.

Case (ii) Now suppose that a, b and e are not in series. In this case, a and b would have to be in the same component of M'_e. For if a and b were to be indifferent components in M'_e, then every circuit that contains a and b would contain e as well, and a, b and e would be in series. If a and b are in the same component of M'_e, then $(M'_e)''_a$ would be disconnected. Thus $(M''_a)'_e = (M'_e)''_a$ is disconnected. It follows that e is essential in M''_a.

But since M''_a is not critical, the only inessential element in M''_a is b; and $(M''_a)'_b$ is connected.

Now we shall show that $(M''_a)'_b = M'\{a, b\}$. The circuits of M''_a are the circuits of M which are contained in $E - \{a, b\}$, together with sets of form $Y \cup \{b\}$, where $Y \cup \{a, b\}$ is a circuit of M. Thus the circuits of $(M''_a)'_b$ are the circuits of M which are contained in $E - \{a, b\}$. It now follows that $(M''_a)'_b = M'\{a, b\}$.

Now we shall show that $M'\{a, b\}$ is critically connected; or that $(M'\{a, b\})'_e$ is disconnected for all $e \in E - \{a, b\}$.

Since e is essential in M, e separates two elements of M. If e separates a and b, then we know that a, b and e are in series, in which case M''_a would be critical (because in this case $(M''_a)'_b = (M'_b)''_a$ is disconnected implying that b is essential in M''_a). Thus a and b are in the same component of M'_e.

Now there are two possibilities.

Case (I) There are two elements $x, y \in E - \{a, b, e\}$ such that x and y are separated by e. Since $(M'\{a,b\})'_e = (M'_e)'\{a,b\}$, x and y belong to different components of $(M'\{a,b\})'_e$ implying that e is essential in $M'\{a,b\}$.

Case (II) There do not exist $x, y \in E - \{a, b, e\}$ which are separated by e. In this case, $\{a, b\}$ is component of M'_e which implies that $\{a, b\}$ is a circuit of M. This is a contradiction because a critical matroid cannot have two element circuits.

Therefore $M'\{a, b\}$ is critically connected. By Lemma 2.5, the rank of $M'\{a, b\}$ is $n - 1$. It now follows from the induction hypothesis that

$$(3) \qquad |E - \{a, b\}| \le f(n-1) \le 2(n-1) - 2 .$$

It follows from (3) that

$$f(n) = |E| \le 2n - 2 .$$

The proof of the theorem now follows from the principle of induction.

Corollary 3.3. *If $n \ge 3$, then $f(n) = 2n - 2$.*

Theorem 3.4. *An extremal critical matroid of rank n ($n \ge 3$) is isomorphic to the forest matroid of $K_{2,n-1}$.*

Proof. It follows from Lemma 2.4 that the only critical matroid of rank 3 is isomorphic to the forest matroid of $K_{2,2}$. Assume inductively that the theorem is valid for all k, $3 \le k < n$. Consider an extremal critical matroid $M = (E, I)$ of rank n. It follows from the arguments used in the proof of the previous theorem that there is a co-circuit $\{a, b\}$ in M, b is inessential in M''_a and that $M'_{\{a,b\}}$ is an extremal critical matroid of rank $n-1$. By the induction hypothesis, $M'_{\{a,b\}}$ is isomorphic to the forest matroid of a $K_{2,n-2}$. Denote $M'_{\{a,b\}}$ by M_1 and M''_a by M_2. We have shown during the proof of Theorem 3.2 that $(M''_a)'_b = M'_{\{a,b\}}$. Thus M_2 is a one element extension of M_1. We also note that M is obtained from M_2 by extending it in series at b. Let C_1 and C_2, respectively denote the circuit sets of M_1 and M_2. It follows that $C_1 \subset C_2$. In fact, each circuit of M_2 in $C_2 - C_1$ contains the element b.

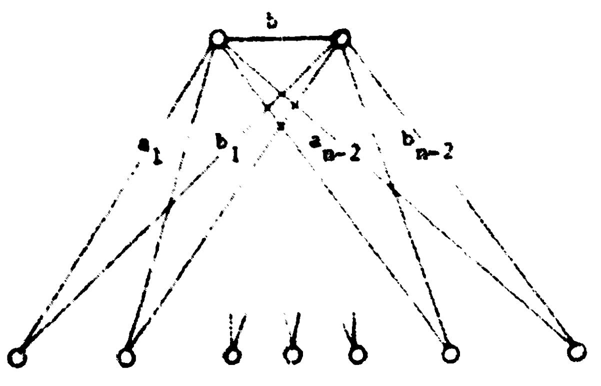

Fig. 1.

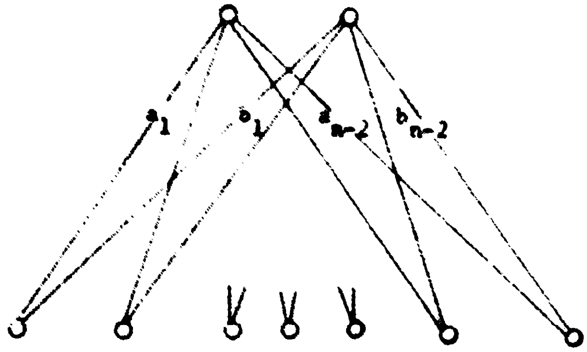

Fig. 2.

Lemma 3.5. *M_2 is isomorphic to the forest matroid of the graph in Fig.1.*

Proof. The case $n = 4$ has to be dealt with separately. We shall first deal with the case $n > 4$. Consider the $K_{2,n-2}$ with the labels as shown in Fig. 2. Denote this graph by G_1.

We may assume without loss of generality that M_1 is the same as $F(G_1)$. Now $(M_2)'_{a_1}$ is disconnected since b is the only inessential element of M_2. If we write $R = \{a_2, b_2, ..., a_{n-2}, b_{n-2}\}$, then the restriction of M_1 to R is connected, being the forest matroid of $K_{2,n-3}$. (Note that this assertion is false if $n = 4$.) Thus R is contained in one of the components of $(M_2)'_{a_1}$. Now since the ranks of $(M_2)'_{a_1}$ and R are, respectively, $n-1$ and $n-2$, it follows that there is only one component of $(M_2)'_{a_1}$ other than the component that contains R. This component will have to be a singleton. If this component were $\{b\}$, then the other component would be $R \cup \{b_1\}$. But if $R \cup \{b_1\}$ were a component of $(M_2)'_{a_1}$, then it would be a component of $(M_1)'_{a_1}$, which is not the case. It follows that $\{b_1\}$ is a component of $(M_2)'_{a_1}$. It means that a_1 separates b_1 and b in M_2. Thus every circuit of M_2 that contains b and b_1 also contains a_1. Similarly, every circuit that contains a_1 and b contains b_1. In general, a circuit containing b contains a_i if and only if it contains b_i, $i = 1, 2, ..., n-2$. Now if a circuit C containing b contains a_i for some i, then it cannot contain a_j if $j \neq i$. For if $b, a_i, a_j \in C$, then $\{b, a_i, b_i, a_j, b_j\} \subseteq C$. This contradicts the first circuit axiom because $\{a_i, b_i, a_j, b_j\}$ is a circuit. We may now conclude that

$$\mathcal{C}_2 - \mathcal{C}_1 = \{\{a_i, b_i, b\} : 1 \leq i \leq n-2\} .$$

This in turn implies that M_2 is isomorphic to the forest matroid of the graph in Fig. 1.

We shall now deal with the case $n = 4$. In this case, $r(M_2) = 3$ and b is the only inessential element of M_2. Let $\{e_1, e_2, e_3, e_4, b\}$ be the set of elements of M_2. Then $C_1 = \{e_1, e_2, e_3, e_4\}$ is a circuit of M_2. Now since $(M_2)'_{e_4}$ is disconnected, it contains a component of rank 2 and cardinality 3. This component will have to be a circuit. Without loss of generality, let $C_1 = \{e_1, e_2, b\}$ be this circuit. By arguing with $(M_2)'_{e_1}$, we may now show that $C_3 = \{e_3, e_4, b\}$ is a circuit. (Both $\{e_i, e_j, b\}$ and $\{e_i, e_k, b\}$, $j \neq k$, cannot be circuits. For if both of them were circuits, then, by the exchange property, it implies that $\{e_i, e_j, e_l\}$ is a circuit. But this contradicts the first circuit axiom because $\{e_i, e_j, e_k\}$ is a proper subset of C_1.)

It now follows that the only circuits of M_2 are C_1, C_2 and C_3. We may now conclude that when $n = 4$, the lemma is valid.

Returning to the proof of Theorem 3.4, we observe that series extension of the forest matroid of the graph in Fig. 1 is isomorphic to the forest matroid of $K_{2,n-1}$. The theorem now follows from the principle of induction.

Acknowledgments

At first I did not see the simple proof of Lemma 3.1 that appears here. My first proof was based on a theorem of W.R. Richardson which will appear in his thesis. Using a theory of decomposition of two connected but not three connected matroids into three connected components, Richardson has obtained a characterization of critical matroids. He will presumably present it in his thesis. Thanks are due to the refree for suggesting many improvements in the presentation of the paper.

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