The (mod, integral) sum numbers of fans and $K_{n,n} - E(nK_2)$

Wenqing Dou$^{a,b}$, Jingzhen Gao$^b$

$^a$Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China
$^b$Department of Mathematics, Shandong Normal University, Jinan 250014, PR China

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Abstract

Let $N(Z)$ denote the set of all positive integers (integers). The sum graph $G^+(S)$ of a finite subset $S \subset N(Z)$ is the graph $(S, E)$ with $uv \in E$ if and only if $u + v \in S$. A graph $G$ is said to be an (integral) sum graph if it is isomorphic to the sum graph of some $S \subset N(Z)$. The (integral) sum number $\sigma(G)(\zeta(G))$ of $G$ is the smallest number of isolated vertices which when added to $G$ result in an (integral) sum graph. A mod sum graph is a sum graph with $S \subset \mathbb{Z}_m \setminus \{0\}$ and all arithmetic performed modulo $m$ where $m \geq |S| + 1$. The mod sum number $\rho(G)$ of $G$ is the least number $\rho$ of isolated vertices $\rho K_1$ such that $G \cup \rho K_1$ is a mod sum graph.

In this paper, we prove that for $n \geq 3$, the $n$ spoked fan $F_n$ is an integral sum graph, $\sigma(F_4) = 1$, $\sigma(F_n) = 2$ for $n \neq 4$, and

$$\sigma(F_n) = \begin{cases} 2, & n = 4, \\ 3, & n = 3 \text{ or } n \geq 6 \text{ and } n \text{ even,} \\ 4, & n \geq 5 \text{ and } n \text{ odd.} \end{cases}$$

We also show that for $K_{n,n} - E(nK_2)$, $n \geq 6$, $\rho = n - 2$, $\sigma = 2n - 3$ and $\zeta = 2n - 5$.

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1. Introduction

The concept of the (integral) sum graph was introduced by Harary [4,5]. Let $N(Z)$ denote the set of all positive integers (integers). The sum graph $G^+(S)$ of a finite subset $S \subset N(Z)$ is the graph $(S, E)$ with $uv \in E$ if and only if $u + v \in S$. A graph $G$ is said to be an (integral) sum graph if it is isomorphic to the sum graph of some $S \subset N(Z)$. We say that $S$ give an (integral) sum labelling for $G$. The (integral) sum number $\sigma(G)(\zeta(G))$ is the smallest number of isolated vertices which when added to $G$ result in an (integral) sum graph. It is obvious that $\zeta(G) \leq \sigma(G)$ for any graph $G$. The relevant results about the (integral) sum number of graphs can be found in [2,3,5–13,15]. Harary [5] raised the following open problem:

Open Problem 1. What is the integral sum number of $K_{n,n} - E(nK_2)$?

The concept of mod sum graph was introduced by Boland et al. [1] in 1990. A mod sum graph is a sum graph with $S \subset \mathbb{Z}_m \setminus \{0\}$ and all arithmetic performed modulo $m$ where $m \geq |S| + 1$. Trees on $n \geq 3$ vertices, cycles on $n \geq 4$

$E$-mail addresses: wenqingdou@yahoo.com.cn (W. Dou), jzhgao@beelink.com (J. Gao).

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vertices, cocktail party graphs and some complete bipartite graphs have been shown to be mod sum graphs [1]. However, complete graphs \( K_n \) for \( n \geq 2 \) are not mod sum graphs. Consequently, Sutton et al. [16] introduced the concept of mod sum number. The mod sum number \( \rho(G) \) of graph \( G \) is the least number \( \rho \) of isolated vertices \( \rho K_1 \) such that \( G \cup \rho K_1 \) is a mod sum graph. It is obvious that \( \rho(G) \leq \sigma(G) \) for any graph \( G \). In the same paper, they proved wheel \( W_n \) (\( n \neq 4 \)) and the symmetric complete bipartite graph \( K_{n,n} \) (\( n \geq 3 \)) are not mod sum graphs. Sutton et al. [14] gave the mod sum number of wheel \( W_n \) and raised the following open problem:

**Open Problem 2.** What is the mod sum number of the \( n \)-spoked fan \( F_n \)?

In this paper, the mod sum number, the sum number and the integral sum number of \( F_n \) (\( n \geq 2 \)) and \( K_{n,n} - E(nK_2) \) are determined.

2. The mod sum number of \( F_n \)

A fan \( F_n \) is a graph \((V, E)\) with vertex set \( V = \{c, a_1, a_2, \ldots, a_n\} \) and edge set \( E = \{ca_1, ca_2, \ldots, ca_n, a_1a_2, a_2a_3, \ldots, a_{n-1}a_n\} \). The vertex \( c \) is called the center of the fan, each edge \( ca_i, i = 1, \ldots, n \), is called a spoke, and the path \( P_n = F_n - c \) is called the rim.

We say that two edges are adjacent if they have an endvertex in common. In a (mod, integral) sum labelling of \( F_n \cup r K_1 \), where \( r = \sigma(F_n) \) (\( r = \rho(F_n) \)) with modulus \( m, r = \zeta(F_n) \), a vertex \( u \) is called a working vertex if \( u = x + y \) holds for some edge \( xy \). A spoke \( ca \) is said to be working if \( c + a \in V \). The term edge sum means the sum of the labels of the two vertices incident with the edge.

It is clear that \( F_1 = K_2 \) and \( F_2 = K_3 \). Since \( \sigma(K_2) = 1, \sigma(K_3) = 2, \zeta(K_2) = \zeta(K_3) = 0 \), and \( \rho(K_2) = \rho(K_3) = 1 \) we only need to consider the case of \( n \geq 3 \). To simplify notations, throughout this paper we may assume that the vertices of \( G \) are identified with their labels, and in this section all arithmetic are performed modulo \( m \) if it is not pointed out specially.

Letting \( n \geq 3 \) and \( r = \rho(F_n) \) with modulus \( m \), we shall give some properties of the mod sum graph \( F_n \cup r K_1 \). Let \( V = V(F_n) = \{a_1, \ldots, a_n, c\} \), where \( c \) is the center of the fan, \( C = \{c_1, \ldots, c_r\} = V(rK_1) \), and \( S = V(F_n \cup r K_1) = V \cup C \).

**Lemma 2.1.** In a mod sum labelling of \( F_n \cup r K_1 (n \geq 3) \) the center is not a working vertex.

**Proof.** By contradiction. If the center is a working vertex we may assume without loss of generality that \( c = a_1 + a_2 \). Since there exist at least one vertex that has two adjacent rim vertices between \( a_1 \) and \( a_2 \) we may assume without loss of generality that \( a_1 \) is the rim vertex which is adjacent to \( a_1 \) and other than \( a_2 \). Then \( a_1, a_2, a_3 \) are three consecutive rim vertices and \( a_1 + a_1 = S, a_1 + c = (a_1 + a_1) + a_2 \in S \). We now aim at proving that \( a_i + a_1 \neq a_2 \).

Suppose to the contrary that \( a_i + a_1 = a_2 \), i.e. \( a_1 = a_2 - a_1 \in V \). Since \( a_1 + c = (a_2 - a_1) + (a_1 + a_2) = 2a_2 \in S \) and \( c + a_2 = a_1 + 2a_2 \in S \), we have that \( 2a_2 \) is adjacent to \( a_1 \) if \( a_1 \neq 2a_2 \), contradicting that \( 2a_2 \neq a_1, a_2 \) and \( c \). Thus, \( a_1 = 2a_2 \) and we have \( a_2 - a_1 = m - a_2, c = a_1 + a_2 = 3a_2 \) and \( c + a_2 = 4a_2 \in S \). Note that \( a_2 + (c + a_2) = 5a_2 = c + a_1 \in S \) and \( a_1 \neq c + a_2 \). Thus, \( a_2 + a_1 = a_2 + a_1 \) are rim vertices. Since \( a_i + (c + a_2) = m - a_2 + 4a_2 = 3a_2 \in S \), we have that \( a_1 = c + a_2 \) or \( a_i \) is adjacent to \( c + a_2 \), contradicting the configuration of \( F_n \). Hence, \( a_1 + a_1 \neq a_2 \).

As \( a_1 + a_1 = S, a_1 + c = (a_1 + a_1) + a_2 \in S, a_1 + a_1 \) is adjacent to \( a_2 \) and \( a_1, a_2, a_3, a_1 + a_1 \) are four consecutive rim vertices. Note that \( (a_1 + a_1) + c = (a_1 + a_1) + (a_1 + a_2) = (a_1 + a_1) + (a_1 + a_2) + a_2 + a_3 \in S \) and \( a_1 + c = a_1 + a_1 + a_2 \in S \). If \( a_1 + a_1 + a_2 \neq m \) then we have that \( a_1 + a_1 + a_2 \) is adjacent to \( a_1 \). However, \( a_1 + a_1 + a_2 \neq a_1, a_2, a_3 \), a contradiction. If \( a_1 + a_1 + a_2 = a_1 \), then \( a_1 + a_2 + a_3 = a_1 + a_2 \in S \). Since \( a_1 \neq a_1, a_1, a_1 \), we have that \( 2a_1 \) is not adjacent to \( a_2 \). So we have that \( 2a_1 = a_2 \) by \( c + a_1 = 2a_1 + a_2 \in S \). Thus, \( a_1 = -2a_1, a_1 + a_1 = -a_1, c = 3a_1, c + a_1 = 4a_1, c + a_2 = 5a_1 = (c + a_1) + a_1 \in S \). Hence, \( c + a_1 \) is adjacent to \( a_1 \). Since \( c + a_1 \neq a_2, a_2 \), we have that \( c + a_1 = a_1, i.e. 6a_1 = 0 \). Note that \( a_1 + (a_1 + a_1) = 3a_1 = 3a_1 = c \in S \). Thus, \( a_1 \) is adjacent to \( a_1 + a_1 \), contradicting to the conclusion of \( F_n \). \( \square \)

**Lemma 2.2.** In a mod sum labelling of \( F_n \cup r K_1 (n \geq 3) \) if all spokes are working then \( \{a_1, a_2, \ldots, a_n\} = \{a_1 + c, a_2 + c, \ldots, a_n + c\} \).

**Proof.** It is obvious that \( a_1 + c, a_2 + c, \ldots, a_n + c \) are \( n \) distinct labels from \( \{1, 2, \ldots, m - 1\} \). By Lemma 2.1 and the spokes are all working we obtain the result. \( \square \)
Lemma 2.3. In a mod sum labelling of $F_n \cup rK_1$ $(n \geq 3)$ if all spokes are working then the labels of the rim vertices can be partitioned into sets of equal size $l$ such that the elements of each set form an $l$-cycle under addition of $c$.

Proof. Select the smallest (mod $m$) label of the set $R = \{a_1, a_2, \ldots, a_n\}$, say $a_t$, remove it from $R$ and place it as the first label of a new set representing the first partition. By Lemma 2.2 we know that there must be a label $a_j \in R$ such that $a_j = a_t + c$. Remove $a_j$ from $R$ and place it in the first partition. Repeat the process until the next label in the series cannot be found in $R$. Since the label must have been in the original set by Lemma 2.2, the label has previously been selected for the first partition and must be $a_t$. This completes the selection of the first partition set.

If $R \neq \emptyset$ then the process is repeated for a second partition. Repeat the selection process until $R = \emptyset$. Thus we obtain a partition of $R$. Suppose that the $i$th partition set contains $l_i$ labels and the first label is $u_i$. Then $u_i + l_i c = u_i$, i.e. $l_i c = 0$ and $j c \neq 0$ for $j = 1, 2, \ldots, l_i - 1$. So all cycles must be of the same size. □

In this paper we shall refer to these partition sets as $l$-cycles. We denote the vertices whose label form $l$-cycle $C_i$ by $a_{i1}, a_{i2}, \ldots, a_{il}$.

Lemma 2.4. The sum of the labels of any two vertices from an $l$-cycle cannot be equal to the label of a vertex from the same $l$-cycle.

Proof. Assume that $a_{si} + a_{sj} = a_{sk}$ for some $i, j, k \in \{1, \ldots, l\}, i \neq j$.

\[
\text{LHS} = [a_{s1} + (i - 1)c] + [a_{c1} + (j - 1)c],
\]
\[
\text{RHS} = [a_{s1} + (k - 1)c].
\]
Therefore, $a_{s1} = tc$, where $t \in \{1, \ldots, l\}$. Then for some $s \in \{1, \ldots, l\}$ we have $a_{s1} = c$, which is a contradiction. □

Lemma 2.5. The sum of labels of a vertex from $C_i$ and a vertex from $C_j$ (where $i \neq j$) must be equal to the label of an isolated vertex.

Proof. Suppose $a_{ir} + a_{js} = a_{kt}$. If $k = i$ then $a_{js} = a_{ir} - a_{ir} = pc$, where $p \in \{1, \ldots, l - 1\}$. Thus, we have $a_{js} = c$ for some $q \in \{1, \ldots, l\}$, which is a contradiction. So $k \neq i$ and similarly $k \neq j$. Then by Lemma 2.1 we have that any vertex from $C_i$ and any vertex from $C_j$ are adjacent, which is a contradiction. □

Lemma 2.6. For $n \geq 3$, $F_n$ is not a mod sum graph for any modulus.

Proof. By contradiction. Suppose that $F_n$ is a mod sum graph. It is obvious that all spokes are working. Hence, we may assume that there exist $r$ distinct $l$-cycles by Lemmas 2.2 and 2.3. If $t \geq 2$ then there must exist $i, j$ ($i \neq j$) such that a vertex from $C_i$ is adjacent to a vertex from $C_j$. The sum of the labels of the two vertices is an $l$-cycle under addition of $c$ by Lemma 2.5, which is a contradiction. If $t = 1$ there exists at least one isolated vertex by Lemmas 2.1 and 2.4, which is a contradiction. □

Lemma 2.7. $\rho(F_n) = 1$ for $n = 4$.

Proof. We consider the following mod labelling of the graph $F_n \cup K_1$:

\[
a_1 = \frac{m}{2} + 2, \quad a_2 = 1, \quad a_3 = \frac{m}{2} + 1, \quad a_4 = 2, \quad c = \frac{m}{2}, \quad c_1 = \frac{m}{2} + 3,
\]

with modulus $m > 10$, where $c$ is the center of the fan.

Let $V = V(F_n) = \{c, a_1, a_2, a_3, a_4\}, C = \{c_1\}$ be the vertex set of $K_1$, and $S = V(F_n \cup K_1) = V \cup C$. It is easy to verify that the following assertions are true.

1. The vertices in $S$ are distinct and $S \subset Z_m\{0\}$.
2. $c_1 + a_i \notin S$ for any $a_i \in V$. 

We have that 

Therefore, we know that the above mod labelling is a mod sum labelling. Hence by Lemma 2.6 we have that $\rho(F_n) = 1$ for $n = 4$. \hfill \Box

**Lemma 2.8.** In a mod sum labelling of $F_n \cup rK_1$ if a spoke is non-working then the rim edges adjacent to the spoke are also isolated vertices.

**Proof.** By contradiction. Assume that $a$ and $b$ are adjacent rim vertices, the spoke $ca$ is non-working and $a + b$ is not an isolated vertex. By Lemma 2.1 we have that $a + b$ is a rim vertex. Hence $c + (a + b) = (c + a) + b \in S$. So we have that $c + a \in V$, i.e. spoke $ca$ is working, which is a contradiction. \hfill \Box

**Lemma 2.9.** $\rho(F_n) \geq 2$ for $n \geq 3$ and $n \neq 4$.

**Proof.** We will consider two cases.

Case 1: There exist at least one non-working spoke.

Assume that spoke $ca_l$ is non-working then $c + a_l$ is an isolated vertex. Since there is at least one rim vertex, say $a_j$, that is adjacent to $a_l$, we have that $a_i + a_j$ is an isolated vertex by Lemma 2.8. Hence the lemma holds by the distinctness of $c + a_l$ and $a_i + a_j$.

Case 2: All spokes are working.

In this case, we have that Lemmas 2.2–2.5 hold. Assume that there exist $t$ distinct $l$-cycles. We consider three subcases.

Subcase 1: $t = 1$.

We have that there exist three integers $i$, $j$, and $k \ (1 \leq i, j, k \leq n)$ such that $a_i$ is adjacent to $a_j$ and $a_k$. Thus, $a_i + a_j + a_k \in S - V$ by Lemmas 2.1 and 2.4.

Subcase 2: $t = 2$.

In this case, we have that $n$ even, $n \geq 6$ and $l \geq 3$. If some vertex of an $l$-cycle (say $C_1$) is adjacent to two vertices of another $l$-cycle (say $C_2$). Thus, the lemma holds by Lemma 2.5. If not, then any vertex of $C_1$ must be adjacent to some vertex of $C_2$ and if the vertex of $C_1$ is adjacent to two vertices then the other vertex must belongs to $C_1$. So the vertices of $C_1(i = 1, 2)$ must present in pair except at most two. Since $l \geq 3$ there exists at least one rim vertex of $C_1$, say $a_{i1}$, that is adjacent to two rim vertices and the two adjacent vertices are, respectively, $a_{i1} + il, a_{i2} + jlc \ (0 \leq i, j \leq l - 1)$. We have that $a_{i1} + a_{i2} + ic \in S - V$ and $a_{i1} + a_{i2} + jlc \neq 2a_{i1} + ic$.

If $2a_{i1} + ic \in S - V$ the lemma holds. If $2a_{i1} + ic \in V$ we can obtain that $2a_{i1} + ic \in C_2$ by Lemmas 2.1 and 2.4. So there exists an integer $k \ (0 \leq k \leq l - 1)$ such that $2a_{i1} + ic = a_{i1} + kc$. Thus

$$a_{i1} = 2a_{i1} + (i - k)c.$$  

(1)

If $a_{i1}$ is adjacent to two rim vertices then there exists an integer $p \ (0 \leq p \leq l - 1)$ such that $a_{i1}$ is adjacent to $a_{i1} + pc$. So $a_{i1} + (a_{i1} + pc) = 2a_{i1} + pc$ and $2a_{i1} + pc \neq a_{i1} + a_{i2} + jlc$. If $2a_{i1} + pc \in S - V$ then the lemma holds.

If $a_{i1}$ is adjacent to two rim vertices then there exists at least one rim vertex of $C_2$ that is adjacent to two rim vertices by $l \geq 3$. Similar to the discussion of the above we have that the lemma holds.

Subcase 3: $t \geq 3$.

There exists some $l$-cycle (say $C_1$) such that at least one vertex from $C_1$ is adjacent to some vertices of other two $l$-cycles. Therefore, the lemma holds by Lemma 2.5. \hfill \Box

**Lemma 2.10.** $\rho(F_n) \leq 2$ for $n \geq 5$ and $n$ odd.
Proof. We consider the following mod labelling of the graph \( F_n \cup 2K_1 \):

\[
a_i = (i - 1)c + 1, \quad i = 1, 2, \ldots, n, \quad c_1 = 2, \quad c_2 = (n - 2)c + 2,
\]

with modulus \( m = nc \), where \( c > 4 \) is the center of the fan.

Let \( V = V(F_n) = \{c, a_1, a_2, \ldots, a_n\} \), \( C = \{c_1, c_2\} \) be the vertex set of \( 2K_1 \), and \( S = V(F_n \cup 2K_1) = V \cup C \). It is easy to verify that the following assertions are true.

1. The vertices in \( S \) are distinct and \( S \subset \mathbb{Z}_m \setminus \{0\} \).
2. \( c_1 + c_2 \notin S \).
3. \( c_i + a_j \notin S \) for any \( c_i \in C \) and any \( a_j \in V \).
4. \( c + c_i \notin S \) for any \( c_i \in C \).
5. \( c + a_i \notin S \) for any \( a_i \in V \).
6. \( a_1 a_{n-1} a_3 a_{n-3} a_5 a_{n-5} \ldots a_{n-4} a_4 a_{n-2} a_2 a_n \) is the rim of \( F_n \).

Thus, we know that the above mod labelling is a mod sum labelling, and \( \rho(F_n) \leq 2 \) for \( n \geq 5 \) and \( n \) odd. \(\square\)

**Lemma 2.11.** \( \rho(F_n) \leq 2 \) for \( n \geq 6 \) and \( n \) even.

Proof. We consider the following mod labelling of the graph \( F_n \cup 2K_1 \):

\[
a_i = 8(i - 1) + 5, \quad i = 1, 2, \ldots, \frac{n}{2},
\]

\[
b_i = 8(i - 1) + 1, \quad i = 1, 2, \ldots, \frac{n}{2},
\]

\[
c = 8, \quad c_1 = 8\left(\frac{n}{2} - 1\right) + 2, \quad c_2 = 8\left(\frac{n}{2} - 1\right) + 6,
\]

with modulus \( m = 4n \), where \( c \) is the center of the fan.

Let \( V = V(F_n) = \{c, a_1, a_2, \ldots, a_{n/2}, b_1, b_2, \ldots, b_{n/2}\} \), \( C = \{c_1, c_2\} \) be the vertex set of \( 2K_1 \), and \( S = V(F_n \cup 2K_1) = V \cup C \). It is easy to verify that the following assertions are true.

1. The vertices in \( S \) are distinct and \( S \subset \mathbb{Z}_m \setminus \{0\} \).
2. \( c_1 + c_2 \notin S \).
3. \( c_i + a_j \notin S \) for any \( c_i \in C \) and any \( a_j \in V \).
4. \( c_i + b_j \notin S \) for any \( c_i \in C \) and any \( b_j \in V \).
5. \( c + c_i \notin S \) for any \( c_i \in C \).
6. \( c + a_i \notin S \) for any \( a_i \in V \).
7. \( c + b_i \notin S \) for any \( b_i \in V \).
8. If \( n/2 \) odd then \( a_1 a_2 b_1 b_2 a_1 a_{n/2-1} b_2 b_{n/2-1} a_{n/2-2} \ldots a_{n-6} a_{n-6} a_{n-6} a_{n-6} a_{n-6} \) is the rim of \( F_n \).

If \( n/2 \) even then \( a_1 a_2 b_1 b_2 a_1 a_{n/2-1} b_2 b_{n/2-1} a_{n/2-2} \ldots b_{n/4-1} b_{n/4+1} a_{n/4-1} a_{n/4+1} a_{n/4+1} b_{n/4+1} a_{n/4+1} b_{n/4+1} \) is the rim of \( F_n \).

Thus, we know that the above mod labelling is a mod sum labelling, and \( \rho(F_n) \leq 2 \) for \( n \geq 6 \) and \( n \) even. \(\square\)

**Lemma 2.12.** \( \rho(F_n) \leq 2 \) for \( n = 3 \).

Proof. We consider the following mod labelling of the graph \( F_n \cup 2K_1 \):

\[
a_1 = 2, \quad a_2 = \frac{m}{2} + 1, \quad a_3 = \frac{m}{2} + 2, \quad c = 1, \quad c_1 = 3, \quad c_2 = \frac{m}{2} + 3,
\]

with modulus \( m > 12 \), where \( c \) is the center of the fan.
Let \( V = V(F_n) = \{c, a_1, a_2, a_3\} \) and \( C = \{c_1, c_2\} \) be the vertex set of \( 2K_1 \), and \( S = V(F_n \cup 2K_1) = V \cup C \). It is easy to verify that the following assertions are true.

1. The vertices in \( S \) are distinct and \( S \subset \mathbb{Z}_m \setminus \{0\} \).
2. \( c_1 + c_2 \notin S \).
3. \( c_i + a_j \notin S \) for any \( c_i \in C \) and any \( a_j \in V \).
4. \( c + c_i \notin S \) for any \( c_i \in C \).
5. \( c + a_i \in S \) for any \( a_i \in V \).
6. \( a_1a_2a_3 \) is the rim of \( F_n \).

Thus, we know that the above mod labelling is a mod sum labelling, and \( \rho(F_n) \leq 2 \) for \( n = 3 \). □

From Lemmas 2.7 and 2.9–2.12 we obtain the following theorem.

**Theorem 2.1.** \( \rho(F_4) = 1 \) and \( \rho(F_n) = 2 \) for \( n = 3 \) and \( n \geq 5 \).

3. The sum number and integral sum number of \( F_n \)

Lemmas 3.1 and 3.2 have been established for mod sum graph labelling of the \( n \) spoked fan in Section 2. We quote them here in their sum graph form.

**Lemma 3.1.** In a sum labelling of \( F_n \cup rK_1 \) (\( n \geq 3 \)) the center is not a working vertex.

**Lemma 3.2.** In a sum labelling of \( F_n \cup rK_1 \) (\( n \geq 3 \)) if a spoke is non-working then the rim edges adjacent to the spoke are also isolated vertices.

**Lemma 3.3.** In a sum labelling of \( F_n \cup rK_1 \) (\( n \geq 3 \)) if a spoke is non-working then the rim vertex adjacent to this spoke is not the edge sums of spokes.

**Proof.** By contradiction. Suppose that a spoke \( ca \) is non-working and \( b \) is a rim adjacent vertex of \( a \). If there exists a spoke \( cd \) such that \( b = d + c \) then \( a + b = a + (d + c) = (a + c) + d \in S \), contradicting the fact that \( a + c \in S - V \). □

**Lemma 3.4.** \( \sigma(F_n) \leq 3 \) for \( n \geq 6 \) and \( n \) even.

**Proof.** We consider the following labelling of the graph \( F_n \cup 3K_1 \):

\[
a_i = (i - 1)c + 1, \quad i = 1, 2, \ldots, n, \quad c_1 = nc + 1, \quad c_2 = (n - 1)c + 2, \quad c_3 = (n - 3)c + 2,
\]

where \( c > 4 \) is the center of the fan.

Let \( V = V(F_n) = \{c, a_1, a_2, \ldots, a_n\} \), \( C = \{c_1, c_2, c_3\} \) be the vertex set of \( 3K_1 \), and \( S = V(F_n \cup 3K_1) = V \cup C \). It is easy to verify that the following assertions are true.

1. The vertices in \( S \) are distinct.
2. \( c_i + c_j \notin S \) for any \( c_i, c_j \in C \) (\( i \neq j \)).
3. \( c_i + a_j \notin S \) for any \( c_i \in C \) and any \( a_j \in V \).
4. \( c + c_i \notin S \) for any \( c_i \in C \).
5. \( c + a_i \in S \) for any \( a_i \in V \).
6. \( a_na_{n-2}a_3a_{n-4}a_5 \ldots a_{n-5}a_4a_{n-3}a_2a_{n-1} \) is the rim of \( F_n \).

Thus, we know that the above labelling is a sum labelling, and \( \sigma(F_n) \leq 3 \) for \( n \geq 6 \) and \( n \) even. □
Lemma 3.5. \( \sigma(F_n) \leq 4 \) for \( n \geq 5 \) and \( n \) odd.

**Proof.** We consider the following labelling of the graph \( F_n \cup 4K_1 \):

\[
a_i = 8i + 5, \quad i = 1, 2, \ldots, r, \quad b_j = 8j + 1, \quad j = 1, 2, \ldots, r + 1,
\]

\[
c_1 = 8(r + 1) + 5, \quad c_2 = 8(r + 2) + 1, \quad c_3 = 8(r + 1) + 6, \quad c_4 = 8(r + 2) + 2,
\]

where \( c = 8 \) is the center of the fan and \( r = (n - 1)/2 \).

Let \( V = V(F_n) = \{c, a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_{r+1}\}, C = \{c_1, c_2, c_3, c_4\} \) be the vertex set of \( 4K_1 \) and \( S = V(F_n \cup 4K_1) = V \cup C \). It is easy to verify that the following assertions are true.

(1) The vertices in \( S \) are distinct.
(2) \( c_i + c_j \notin S \) for any \( c_i, c_j \in C \) (\( i \neq j \)).
(3) \( c_i + a_j \notin S \) for any \( c_i \in C \) and any \( a_j \in V \).
(4) \( c_i + b_j \notin S \) for any \( c_i \in C \) and any \( b_j \in V \).
(5) \( c + c_i \notin S \) for any \( c_i \in C \).
(6) \( c + a_i \in S \) for any \( a_i \in V \).
(7) \( c + b_i \in S \) for any \( b_i \in V \).
(8) If \( r \) odd then \( b_{r+1} + b_1 a_1 b_1 b_2 a_{r-1} b_{r-1} \ldots b_{r+1}/2 + 2b_{r+1}/2 - 1a_{r+1}/2 + 1a_{r+1}/2 - 1b_{r+1}/2 + 1b_{r+1}/2 \) is the rim of \( F_{n+1} \).

If \( r \) even then \( b_1 a_1 b_1 b_2 a_{r-1} b_{r-1} \ldots b_{r+1}/2 + 2b_{r+1}/2 - 1a_{r+1}/2 + 1a_{r+1}/2 - 1b_{r+1}/2 + 1b_{r+1}/2 \) is the rim of \( F_{n+1} \).

Thus, we know that the above labelling is a sum labelling, and \( \sigma(F_n) \leq 4 \) for \( n \geq 5 \) and \( n \) odd. \( \square \)

Lemma 3.6. \( \sigma(F_n) \geq 3 \) for \( n \geq 3 \) and \( n \neq 4 \).

**Proof.** Letting \( \sigma = \sigma(F_n) \). In a sum labelling of \( F_n \cup \sigma K_1 \) there exists at least one non-working spoke. Let \( V = V(F_n) = \{a_1, \ldots, a_n, c\} \), where \( c \) is the center of the fan, \( a_1 a_2 a_3 \ldots a_n \) is the rim of \( F_n \), \( C = V(\sigma K_1) \), and \( S = V(F_n \cup \sigma K_1) = V \cup C \).

If there exist \( i \) (\( 2 \leq i \leq n - 1 \)) such that \( ca_i \) is a non-working spoke then \( c + a_1, a_1 + a_i - 1, a_i + a_{i+1} \) are three distinct vertices by Lemma 3.2. So the lemma holds. If not, there exist at most two non-working spokes \( ca_1 \) and \( ca_n \). We may assume without loss of generality that \( ca_1 \) is a non-working spoke. Then \( c + a_1 \) and \( a_1 + a_2 \) are isolated vertices. We will consider two cases.

**Case 1:** \( a_2 + a_3 \in V \).

Assume that there exists \( j \) (\( 1 \leq j \leq n \)) such that \( a_2 + a_3 = a_j \). We have that \( c + a_j = (c + a_3) + a_2 \in S \). Hence \( c + a_3 \) is adjacent to \( a_2 \). So \( c + a_3 = a_1 \) and \( c + a_j = a_1 + a_2 \in C \). Therefore, \( j = n \) and \( ca_n \) is a non-working spoke. Since the size of the set of all non-working spokes is the same as the size of the set of all rim vertices that are not the edge sums of spokes and \( a_2, a_{n-1} \) are not the edge sums of spokes we have that \( a_3, a_n \) are the edge sums of some spokes. Assume that \( a_n = c + a_1, a_3 = c + a_2 \). So \( a_k = a_1 \) by \( a_2 + a_3 = a_n \). Since \( a_k \neq a_1 \) and \( a_k \neq a_3 \) we have that \( a_k = a_2 \), i.e. \( a_3 = c + a_2 \) and \( a_3 + a_4 = (c + a_4) + a_2 \in S \). So \( c + a_4 \) is adjacent to \( a_2 \). However, \( c + a_4 \neq a_1 \) and \( c + a_4 \neq a_3 \), which is a contradiction.

**Case 2:** \( a_2 + a_3 \notin V \).

If \( a_2 + a_3 \neq c + a_1 \) then \( c + a_1, a_1 + a_2, a_2 + a_3 \) are distinct isolated vertices. So the lemma holds. If \( a_2 + a_3 = c + a_1 \) we will prove it by contradiction. Suppose that \( \sigma(F_n) = 2 \). We consider two subcases.

**Subcase 1:** \( ca_n \) is a non-working spoke.

We have that \( \{c + a_n, a_{n-1} + a_n\} = \{c + a_1 = a_2 + a_3, a_1 + a_2\} \). So \( c + a_n = a_1 + a_2, a_{n-1} + a_n = a_2 + a_3 \). Hence \( a_1 + a_{n-1} = c + a_3 \in S \). So \( a_1 \) is adjacent to \( a_{n-1} \) by \( n \geq 3 \). Thus, we have that \( a_1 + a_2 = c + a_3 \) and \( a_2 + a_3 = c + a_1 \). Hence \( 2a_2 = 2c \), i.e. \( a_2 = c \), which is a contradiction.

**Subcase 2:** \( ca_n \) is a working spoke.

We have that \( \{a_1, a_3, \ldots, a_n\} = \{c + a_2, \ldots, c + a_n\} \). Hence \( a_1 = (n - 1)c + a_2, c + a_1 = nc + a_2, a_1 + a_2 = (n - 1)c + 2a_2 \) and \( a_3 = nc \). Thus, \( V - \{c\} = \{a_1, a_2, \ldots, a_n\} = \{a_2, c + a_2, \ldots, (n - 1)c + a_2\} \). There exist an
integer \( i (1 \leq i \leq n-2) \) such that \( a_3 = ic + a_2 \). So \( a_2 = (n-i)c, c + a_1 = (2n-i)c, a_1 + a_2 = (3n-2i-1)c \) and \( V - \{c\} = (n-i)c, (n-i+1)c, \ldots, (2n-i-1)c \).

1. If \( i \leq n-4 \) then \( a_1 = (2n-i-1)c \geq (n+3)c \). Since \((n-i)c \leq (n-1)c\) we have that \((n+1)c, (n+2)c, (n+3)c \in V - \{c\} \)
and \((n+1)c, (n+2)c, (n+3)c \) are the rim vertices that is adjacent to one rim vertex, which is a contradiction.

2. If \( i = n-3 \) then \( n \geq 5 \) and \( V - \{c\} = \{3c, 4c, \ldots, (n+2)c\} \). So \( 3c \) is adjacent to \((n+2)c, nc \) and \( 4c \), which is a contradiction.

3. If \( i = n-2 \) then \( c + a_1 = (n+2)c, a_1 + a_2 = (n+3)c = c + (c + a_1) \). So \( c + a_1 \in V \), which is a contradiction.

From the above discussion we have that \( \sigma(F_n) \geq 3 \) for \( n \geq 3 \) and \( n \neq 4 \). \( \square \)

**Lemma 3.7.** \( \sigma(F_n) \geq 4 \) for \( n \geq 5 \) and \( n \) odd.

**Proof.** All rim vertices are distinct and hence the edge sums of all spokes are distinct. It follows that the size of the set of all working spokes is the same as the size of the set of all rim vertices that are the edge sums of spokes. Conversely, the size of the set of all non-working spokes is the same as the size of the set of all rim vertices that are not the edge sums of spokes.

Letting \( \sigma = \sigma(F_n) \). In a sum labelling of \( F_n \cup \sigma K_1 \), there exists at least one non-working spoke. Let \( V = V(F_n), C = V(\sigma K_1) \) and \( S = V(F_n \cup \sigma K_1) = V \cup C \). We will consider two different cases.

**Case 1:** At least one non-working spoke is incident on a rim vertex that is not the edge sum of a spoke.

Consider a non-working spoke incident on a rim vertex that is not the edge sum of a spoke. The rim vertex adjacent to this non-working spoke is not the edge sum of a spoke (Lemma 3.3) so that two or three consecutive rim vertices are not the edge sum of a spoke. Thus, there must also be two or three non-working spokes. An additional non-working spoke may not contribute an extra rim vertex that is not the edge sum of a spoke but all additional non-working spokes contribute at least one extra rim vertex that is not the edge sum of a spoke. So there must be fewer non-working spokes than rim vertices that are not the edge sum of some spoke unless all spokes are non-working or \((n-1)/2\) working spokes and \((n+1)/2\) non-working spokes alternate on the rim.

If all spokes are non-working then \( \sigma(F_n) \geq n > 4 \). The lemma holds. If \((n-1)/2\) working spokes and \((n+1)/2\) non-working spokes alternate on the rim then \( \sigma(F_n) \geq (n+1)/2 \). If \( n \geq 7 \) then \( \sigma(F_n) \geq 4 \). If \( n = 5 \) then there exist two different structures (Black dot denotes the rim vertex that is not the edge sum of a spoke, circle denotes the rim vertex that is the edge sum of a spoke.).

**Structure 1:**

\[ a_1 \circledast a_2 \circledast a_3 \circledast a_4 \circledast a_5 \]

If \( \sigma(F_n) = 3 \) then \( \{a_1, a_5\} = \{c + a_2, c + a_4\} \) and \( c + a_1, c + a_3, c + a_5 \) are isolated vertices. So \( \{a_2 + a_3, a_3 + a_4\} = \{c + a_1, c + a_5\} \) by Lemma 3.2. We will consider three stations.

**Subcase 1:** \( a_1 = c + a_2, a_5 = c + a_4, a_2 + a_3 = c + a_1, a_3 + a_4 = c + a_5 \).

We have that \( a_2 + a_3 = 2c + a_2 \). So \( a_3 = 2c \). If \( a_1 + a_2 = c + a_3 \) then \( a_3 = 2a_2 \). Hence \( 2a_2 = 2c \), i.e. \( a_2 = c \), which is a contradiction. If \( a_1 + a_2 = c + a_5 \) then \( a_5 = 2a_2 \). In a similar way we have that \( a_1 = 2a_4 \). Hence \( a_1 + a_5 = 2a_2 + 2a_4 = 2c + a_2 + a_4 \). Thus \( a_2 + a_4 = 2c = a_3 \), which is a contradiction.

**Subcase 2:** \( a_1 = c + a_2, a_5 = c + a_4, a_2 + a_3 = c + a_5, a_3 + a_4 = c + a_1 \).

We have that \( a_2 + a_3 = 2c + a_4 \) and \( a_3 + a_4 = 2c + a_2 \). So \( 2a_2 = 2a_4 \), i.e. \( a_2 = a_4 \), which is a contradiction.

**Subcase 3:** \( a_1 = c + a_4, a_5 = c + a_2 \).

If \( a_1 + a_2 = c + a_3 \) then \( a_3 = a_2 + a_4 \), which is a contradiction. If \( a_1 + a_2 = c + a_5 \) then \( a_5 = a_2 + a_4 \), which is a contradiction.

**Structure 2:**

\[ a_1 \circledast a_2 \circledast a_3 \circledast a_4 \circledast a_5 \]

If \( \sigma(F_n) = 3 \) then \( \{a_3, a_5\} = \{c + a_2, c + a_4\} \) and \( c + a_1, c + a_3, c + a_5 \) are isolated vertices. So \( \{a_2 + a_3, a_3 + a_4\} = \{c + a_1, c + a_5\} \) by Lemma 3.2. We will consider two stations.
We may assume without loss of generality that 

This implies that two non-working spokes cannot be adjacent to each other by hypothesis. Using the simple counting

Let 

Assume that 

So

Subcase 1: 

Subcase 2: 

We have that 

Subcase 1: 

We have that 

Subcase 2: 

We have that 

If 

Subcase 1: 

We have that 

Subcase 2: 

We have that 

Case 2: Every non-working spoke is incident on a rim vertex that is the edge sum of a spoke.

In this case, we have that the vertex adjacent to the non-working spoke is not the edge sum of a spoke by Lemma 3.3. This implies that two non-working spokes cannot be adjacent to each other by hypothesis. Using the simple counting argument in Case 1 we have that the construction of the rim is where non-working spokes alternate with working spokes in one or two side of the rim. The others are working spokes and the rim vertices incident on these working spokes are also working.

Assume that 

We have that 

If 

If 

Subcase 1: 

Subcase 2: 

We have that 

Subcase 2: 

We have that 

Thus we have that 

If 

Subcase 2: 

We have that 

Subcase 2: 

We have that 

Subcase 2: 

We have that 

However,

We may assume without loss of generality that 

We have that 

Subcase 1: 

Subcase 2: 

We have that 

Subcase 2: 

We have that 

Subcase 2: 

We have that 

Subcase 3: 

We have that 

Subcase 3: 

We have that 

We may assume without loss of generality that 

(1) It is obvious that 

(2) If 

(3) If 

(4) If 

(5) Similar to the proof of (4) we have that 

Subcase 3: 

We have that 

Subcase 3: 

We have that 

Subcase 3: 

We have that 

We may assume without loss of generality that 

(1) If $l_1 > l_2$ then $l_1c + b_1$ is adjacent to exactly one vertex $b_1$. We will prove that $l_1c + 2b_1$ is the fourth distinct isolated vertex. It is obvious that $l_1c + 2b_1 \neq (l_1 + 1)c + b_1$. If $l_1c + 2b_1 = (l_1 + 1)c + b_1$ then $b_2 = (l_1 - 1)c + 2b_1$, $c + b_2 = (l_1 - 1)c + 2b_1 \in V$ and $2c + b_2 = (l_1 - 1)c + 2b_1 \in S$. So $b_1$ is adjacent to $(l_1 - l_2)c + b_1$ and $(l_1 - 2)c + b_1$. If $l_2 > l_1$ then $(l_1 - l_2)c + b_1 < (l_1 - l_2 + 1)c + b_1 < l_1c + b_1$ and $b_1$ is also adjacent to $l_1c + b_1$, which is a contradiction. If $l_2 = 1$ then $l_2 = 1$, $b_1$ is adjacent to $l_1c + b_1$ and $(l_1 - 1)c + b_1$. Since there exists some rim vertex of $C_1$ adjacent to some rim vertex of $C_j$ $(2 \leq j \leq 3)$, we assume that the edge sum is $kc + b_1 + b_j$ $(1 \leq k \leq l_1)$. If $k > 1$ then $kc + b_1 + b_j = (2c + b_1 + (k - 2)c + b_1 \in S$, contradicting the fact that $2c + b_1$ is an isolated vertex. If $k = 1$ then $b_1$ is adjacent to $c + b_j$, contradicting the fact that $b_1$ is adjacent to $l_1c + b_1$ and $(l_1 - 1)c + b_1$. So $l_1c + 2b_1 \neq (l_1 + 1)c + b_2$. Similar to the above proof we have that $l_1c + 2b_1 \neq (l_3 + 1)c + b_3$.

(2) If $l_1 = l_2$ then $l_1 = l_2 \geq l_3 \geq 1$. We will consider the following stations.

If $l_1c + b_1$ is adjacent to $b_2$ we will prove that $l_1c + b_1 + b_2$ is the fourth distinct isolated vertex. It is obvious that $l_1c + b_1 + b_2 \neq (l_1 + 1)c + b_1$ and $l_1c + b_1 + b_2 \neq (l_2 + 1)c + b_2$. If $l_1c + b_1 + b_2 = (l_1 + 1)c + b_3$ then $c + b_3 = (l_1 - l_3)c + b_1 + b_2 \in V$ and $2c + b_3 = (l_1 - l_3)c + b_1 + b_2 \in S$. So $b_1$ is adjacent to $(l_1 - l_3)c + b_3$ and $(l_1 - l_3 + 1)c + b_2$, $b_2$ is adjacent to $(l_2 - l_3)c + b_1$ and $(l_2 - l_3 + 1)c + b_1$. Since there exists some rim vertex of $C_3$ adjacent to some rim vertex of $C_j$ $(1 \leq j \leq 2)$, we may assume without loss of generality that $j = 1$ and the edge sum is $kc + b_1 + b_3 (k \leq l_3)$. So $b_1$ is adjacent to $kc + b_3$, contradicting the fact that $b_1$ is adjacent to $(l_2 - l_3)c + b_2$ and $(l_2 - l_3 + 1)c + b_2$. Hence $(l_1 + 1)c + b_1, (l_2 + 1)c + b_2, (l_3 + 1)c + b_3, l_1c + b_1 + b_2$ are four distinct isolated vertices.

If there exist $i$ $(1 \leq i \leq 2)$ such that $l_1c + b_1$ is adjacent to $b_2$ then from the above proof we have that the lemma holds for $l_2 = l_3$. If $l_2 \geq l_3$ then $(l_1 + b_1) + b_3 = [(l_1 + 1)c + b_3] + [(l_1 - l_3 - 1)c + b_1] \in S$, contradicting the fact that $(l_1 + 1)c + b_3$ is an isolated vertex.

If $l_1c + b_1$ is not adjacent to $b_2$ and $l_1c + b_1$ is not adjacent to $b_3$ for $1 \leq i \leq 2$ then $l_1c + b_1$ is adjacent to exactly one vertex $b_1$ and $l_1c + b_2$ is adjacent to exactly one vertex $b_2$. Since there exists some rim vertex of $C_3$ adjacent to some rim vertex of $C_j$ $(1 \leq j \leq 2)$, we may assume without loss of generality that $j = 1$ and the edge sum is $kc + b_1 + b_3 (k \leq l_3)$. So $b_1$ is adjacent to $kc + b_3$. We will prove that $l_1c + 2b_1$ is the fourth distinct isolated vertex. It is obvious that $l_1c + 2b_1 \neq (l_1 + 1)c + b_1$. If $l_1c + 2b_1 = (l_2 + 1)c + b_2$ then $c + b_2 = 2b_1 \in V$ and $2c + b_2 = c + 2b_1 \in S$. So if $l_1 > 1$ then $b_1$ is adjacent to $c + b_1, l_1c + b_1$ and $kc + b_3$, which is a contradiction, if $l_1 = 1$ then $n = 6$, which is also a contradiction. So $l_1c + 2b_1 \neq (l_2 + 1)c + b_2$. Similar to the above proof we have that $l_1c + 2b_1 \neq (l_3 + 1)c + b_3$.

From the above discussion we have that $\sigma(F_n) \geq 4$ for $n \geq 5$ and $n$ odd.

**Lemma 3.8.** $\sigma(F_n) = 2$ for $n = 4$.

**Proof.** We consider the following labelling of the graph $F_n \cup 2K_1$:

$$a_1 = 6, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 5, \quad c_1 = 1, \quad c_1 = 7, \quad c_2 = 9,$$

where $c_1$ is the center of the fan, $a_1a_2a_3a_4$ is the rim of $F_n$ and $\{c_1, c_2\}$ be the vertex set of $2K_1$. It is easy to verify that the above labelling is a sum labelling. So $\sigma(F_n) \leq 2$ for $n = 4$. Since the degree of every vertex is at least two we have that $\sigma(F_n) \geq 2$ for $n = 4$. So the lemma holds.

**Lemma 3.9.** $\sigma(F_n) = 3$ for $n = 3$.

**Proof.** We consider the following labelling of the graph $F_n \cup 3K_1$:

$$a_1 = 2c, \quad a_2 = 1, \quad a_3 = c + 1, \quad c_1 = 3c, \quad c_2 = 2c + 1, \quad c_3 = c + 2,$$

where $c > 4$ is the center of the fan, $a_1a_2a_3$ is the rim of $F_n$ and $\{c_1, c_2, c_3\}$ be the vertex set of $3K_1$. It is easy to verify that the above labelling is a sum labelling. So $\sigma(F_n) \leq 3$ for $n = 3$. So the lemma holds by Lemma 3.6.

From Lemmas 3.4–3.9 we establish the following theorem.
Theorem 3.2. For $n \geq 3$,
\[
\sigma(F_n) = \begin{cases} 
2, & n = 4, \\
3, & n = 3 \text{ or } n \geq 6 \text{ and } n \text{ even}, \\
4, & n \geq 5 \text{ and } n \text{ odd}.
\end{cases}
\]

Theorem 3.2. $F_n$ is an integral sum graph for $n \geq 3$.

Proof. We label the rim of fan $F_n$ using the integral sequence $(a_1, a_2, a_3, a_4, a_5, \ldots) = (-1, 1, -2, 3, -5, \ldots)$ satisfy that $a_0 = a_{n-2} - a_{n-1}$. Letting $S = \{c = 0, a_1, a_2, a_3, a_4, \ldots, a_n\}$. It is easy to verify that $|a_1| = |a_2| < |a_3| < \cdots < |a_n|, c + a_i \in S$ for any $a_i \in S$ and $a_i + a_{i+1} \in S (1 \leq i \leq n - 1)$. For any $j > i + 1$ if $a_i, a_j$ have the same signs then $|a_j| < |a_i + a_j| = |a_i| + |a_j| \leq |a_{j+1}|$. The equality holds if and only if $i = 1, j = 3$. However, $a_1 + a_3 = -3 \notin S$. If $a_i, a_j$ have different signs then $j > i + 2$ and $|a_{j-1}| = |a_j| - |a_{j-2}| \leq |a_j| - |a_i| = |a_i + a_j| < |a_j|$. The equality holds if and only if $i = 1, j = 4$. However, $a_1 + a_4 = 2 \notin S$. So we have that $a_i$ is not adjacent to $a_j$ for any $j > i + 1$. Thus \{0, a_1, \ldots, a_n\} is an integral sum labelling of $F_n$, where $c$ is the center of the fan and $a_1 a_2 \ldots a_n$ is the rim of $F_n$. So the theorem holds. \(\square\)

4. The mod sum number of $K_{n,n} - E(nK_2)$

It is clear that $K_{n,n} - E(nK_2)$ is two independent edges for $n = 2$ and $K_{n,n} - E(nK_2)$ is a 6-cycle for $n = 3$. Since $\sigma(K_{2,2} - E(2K_2)) = 1$, $\zeta(K_{2,2} - E(2K_2)) = \rho(K_{2,2} - E(2K_2)) = 0$, $\sigma(C_6) = 2$ and $\zeta(C_6) = \rho(C_6) = 0$ we only need to consider the case of $n \geq 4$. In this paper, we only consider the case of $n \geq 6$. In this section, all arithmetic are performed modulo $m$ if it is not pointed out specially.

Let $\rho = \rho(K_{n,n} - E(nK_2)) (n \geq 6)$. We will give some properties of the mod sum graph $(K_{n,n} - E(nK_2)) \cup \rho K_1$ for $n \geq 6$. Let $V(K_{n,n} - E(nK_2)) = (A, B)$ be the bipartition of $K_{n,n} - E(nK_2)$, and $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}, \{a_1b_1, a_2b_2, \ldots, a_nb_n\} = E(nK_2), C = V(\rho K_1)$ be the set of isolated vertices, $S = V((K_{n,n} - E(nK_2)) \cup \rho K_1)$ and the modulus be $m$.

Lemma 4.1. If there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_1 + b_q \in B$ for any $a_i \in A (i \neq q)$.

Proof. Since $a_p + b_q \in B$ we may assume without loss of generality that $b_r = a_p + b_q$. It is obvious that $a_1 + b_q \in S$ for any $a_i \in A$ and $i \neq q$. Since $b_r = a_p + b_q \in B$ and $a_i \in A$, we have that $a_i + b_r = a_p + (a_i + b_q) \in S$ for $i \neq r, q$. Thus we can obtain that $a_i + b_q \in B \cup \{a_p\}$ for $i \neq r, q$. We will prove that $a_i + b_q \notin A$ for any $a_i \in A$ and $i \neq q$.

If $a_1 + b_q \in A$ we may assume without loss of generality that $a_1 = a_i + b_q$. Then $a_s + a_j = (a_j + b_q) + a_i \notin S$ for $j \neq s, q$. So $a_i + b_q \in A \cup \{b_q\} \cup C$ for $j \neq s, q$. Hence we have that $a_1 + b_q \in (A \cup \{a_i\} \cup C) \cap (B \cup \{a_p\}) = \{a_p, b_1\}$ for $j \neq r, s, q$. So $n \leq 5$, contradicting the fact that $n \geq 6$. Therefore, $a_i + b_q \notin A$ for any $a_i \in A$ and $i \neq q$. Thus we have that $a_r + b_q \in B \cup C$, $a_1 + b_q \in B$ for $i \neq r, q$. So we only need to prove that $a_i + b_q \notin C$.

If $a_1 + b_q \in C$ then $a_1 + (a_1 + b_q) = (a_1 + b_q) + a_r \notin S$ for $i \neq q$. So $a_i + b_q \in A \cup \{b_r\} \cup C$ for $i \neq q$. Hence we have that $a_i + b_q = b_r$ for $j \neq r, q$. So $n \leq 3$, contradicting the fact that $n \geq 6$.

From the above we have that the lemma holds. \(\square\)

In a similar way we have the following lemma.

Lemma 4.2. If there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in A$ then $a_p + b_i \in A$ for any $b_i \in B (i \neq p)$.

Lemma 4.3. If there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_p + b_i \in B \cup C$ for any $b_i \in B (i \neq p)$.

Proof. By contradiction. If there exists $r \neq p$ such that $a_r + b_r \in A$ then we have that $a_p + b_i \in A$ for any $b_i \in B$ and $i \neq p$ by Lemma 4.2. So $a_p + b_q \in A$, contradicting the fact that $a_p + b_q \in B$. \(\square\)

Lemma 4.4. $K_{n,n} - E(nK_2)$ is not a mod sum graph for $n \geq 6$. 

**Proof.** By contradiction. If $K_{n,n} - E(nK_2)$ is a mod sum graph then $C = \emptyset$ and $a_i + b_j \in A \cup B$ for any $a_i \in A$ and any $b_j \in B(i \neq j)$. We may assume without loss of generality that there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in B$. Then $a_1 + b_1 \in B$ for any $a_1 \in A(i \neq q)$ by Lemma 4.1. Hence $a_1 + b_1 \in B$ for any $a_1 \in A(i \neq q)$ and any $b_j \in B(j \neq i)$ by Lemma 4.3. If there exists $b_r \in B(r \neq q)$ such that $a_q + b_r \in A$ then $a_q + b_r \in A$ for any $b_r \in B$ and $k \neq q$. Thus $A = \{a_q, a_q + b_1, \ldots, a_q + b_{q-1}, a_q + b_{q+1}, \ldots, a_q + b_n\}$. So there exists an integer $l_i \neq q$ such that $a_i = a_q + b_i$ for any $i \neq q$. Since $a_i + a_k = (a_k + b_i) + a_q \notin S$ for any $k \neq i$, $l_i$ we have that $a_q + b_i \in A \cup \{b_q\} \cup C$, $k \neq i, l_i$. From the above, we know that $a_q + b_{k_1} \in \{b_q\}$. $k \neq q, i, l_i$. So $n \leq 4$, contradicting the fact that $n \geq 6$. Hence $a_i + b_j \in B$ for any $a_i \in A$ and any $b_j \in B(j \neq i)$. Let $R = \{a_1 + b_1, \ldots, a_i + b_{i-1}, a_i + b_{i+1}, \ldots, a_i + b_n\} \subset B$. Since $|B \setminus R| = 1$ we may assume without loss of generality that $B \setminus R = \{b_1\}$. Thus $B = \{b_1, a_i + b_1, \ldots, a_i + b_{i-1}, a_i + b_{i+1}, \ldots, a_i + b_n\}$. Hence $b_1 + (n-1)a_i = b_i$. If $ka_i \neq 0$ for any $k(2 \leq k \leq n-1)$ then $B = \{b_1, b_i + a_i, b_i + 2a_i, \ldots, b_i + (n-1)a_i\} = \{a_1 + b_i, \ldots, a_{i-1} + b_i, a_{i+1} + b_i, \ldots, a_n + b_i\}$. Hence $\{a_1, a_{i-1}, a_{i+1}, \ldots, a_n\} = \{a_1, 2a_i, \ldots, (n-1)a_i\}$. Since $n \geq 6$ we have that $a_1, 2a_i, 3a_i$ are distinct vertices of $A$ and $a_1 + 2a_i = 3a_i$. So $a_i$ is adjacent to $2a_i$, which is a contradiction.

If there exists an integer $l_i(2 \leq l_i \leq n-1)$ such that $a_1 a_i = 0$. Suppose that there exist integers $s_i$ and $r_i(0 \leq r_i < l_i)$ such that $n - 1 = s_i l_i + r_i$. We have that the elements of $B$ can be partitioned into $\{b_1, b_2 + a_i, \ldots, b_i + r_i a_i\}$ and $s_i(\geq 1)$ sets of equal size $l_i$ such that the elements of each set form an $l_i$-cycle under addition of $a_i$. Assume that one of the $l_i$-cycles is $\{b_k, b_{k+1}, \ldots, b_k + (l_i - 1)a_i\}$. Since $B = \{a_1 + b_k, \ldots, a_k - 1 + b_k, a_k + 1 + b_k, \ldots, a_n + b_k\}$ we have that $\{a_1, 2a_i, \ldots, (l_i - 1)a_i\} \subset A$. If $l_i \geq 4$ then $a_1, 2a_i, 3a_i$ are three distinct vertices of $A$ and $a_1 + 2a_i = 3a_i$. So $a_i$ is adjacent to $2a_i$, which is a contradiction. Hence $2 \leq l_i \leq 3$. If $l_i = 2$ then $a_i = m/2$. If $l_i = 3$ then $a_i = m/3$ or $2m/3$. So $n = |A| \leq 3$, contradicting the fact that $n \geq 6$. \[\square\]

**Lemma 4.5.** $\rho(K_{n,n} - E(nK_2)) \geq n - 2$ for $n \geq 6$.

**Proof.** By Lemma 4.4 we have that there exists $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in C$. If $a_i + b_j \in C$ for any $a_i \in A(i \neq q)$ then the lemma holds by the distinctness of $a_1 + b_q, \ldots, a_q - 1 + b_q, a_q + 1 + b_q, \ldots, a_n + b_q$. If there exists $a_r \in A(r \neq p, q)$ such that $a_r + b_q \notin C$ then $a_r + b_q \in A \cup B$. If $a_r + b_q \in B$ then by Lemma 4.1 we have that $a_1 + b_q \in B$ for any $a_1 \in A(i \neq q)$, contradicting the fact that $a_p + b_q \in C$. If $a_i + b_q \in A$ then $a_i + b_j \in A$ for any $b_j \in B(j \neq r)$ by Lemma 4.2. Hence $a_i + b_j \notin B$ for any $b_j \in B(j \neq r)$ and any $a_i \in A(i \neq j)$. If there exists $a_i \in A(s \neq j)$ such that $a_s + b_j \in B$ then we have that $a_s + b_j \in B$ for any $a_s \in A(k \neq j)$, contradicting the fact that $a_r + b_j \in A$. So $a_i + b_j \in A \cup C$ for any $a_i \in A$ and any $b_j \in B(j \neq r, i)$. If there exists $a_p + b_r \in A$ then $a_p + b_j \in A$ for any $b_j \in B(j \neq p)$ by Lemma 4.2, contradicting the fact that $a_p + b_q \in C$. Hence $a_p + b_j \in C$ for any $b_j \in B(j \neq r, i)$. So $\{a_p + b_1, \ldots, a_p + b_{p-1}, a_p + b_{p+1}, \ldots, a_p + b_n\} - \{a_p + b_r\} \subset C$. The lemma holds. \[\square\]

**Theorem 4.1.** $\rho(K_{n,n} - E(nK_2)) = n - 2$ for $n \geq 6$.

**Proof.** By Lemma 4.5 we only need to prove that $\rho(K_{n,n} - E(nK_2)) \leq n - 2$. We consider the following mod labelling of the graph $(K_{n,n} - E(nK_2)) \cup (n-2)K_1$:

- $a_i = (i-1)N + 7, \quad i = 1, \ldots, n-1, \quad a_n = 3,$
- $b_j = (j-1)N + 4, \quad j = 1, \ldots, n-1, \quad b_n = m - 3,$
- $c_k = (k-1)N + 11, \quad k = 1, \ldots, n-2,$

and take the modulus $m = (n-1)N$, where $N \geq 30$ is an integer.

Let $V(K_{n,n} - E(nK_2)) = (A, B)$ be the bipartition of $K_{n,n} - E(nK_2)$. $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}, C = V((n-2)K_1) = \{c_1, c_2, \ldots, c_{n-2}\}, S = V(K_{n,n} - E(nK_2) \cup (n-2)K_1) = A \cup B \cup C$. It is easy to verify that the following assertions are true.

1. $S \subset Z_m \setminus \{0\}$,
2. $a_i + a_j \notin S$ for any $a_i, a_j \in A(i \neq j)$,
3. $b_i + b_j \notin S$ for any $b_i, b_j \in B(i \neq j)$,
4. $c_i + c_j \notin S$ for any $c_i, c_j \in C(i \neq j)$. 

\[\square\]
(5) $a_i + c_j \notin S$ for any $a_i \in A$ and any $c_j \in C$.
(6) $b_i + c_j \notin S$ for any $b_i \in B$ and any $c_j \in C$.
(7) $a_i + b_j \notin S$ if and only if $i + j = n$ or $i = j = n$.

So $\{a_1 b_{n-1}, a_2 b_{n-2}, \ldots, a_{n-1} b_1, a_n b_n\}$ is $E(nK_2)$.

Thus the above labelling is a mod sum labelling of $(K_{n,n} - E(nK_2)) \cup (n-2)K_1$ for $n \geq 6$. □

5. The sum number and integral sum number of $K_{n,n} - E(nK_2)$

Lemmas 5.1–5.3 have been established for mod sum graph labelling of $K_{n,n} - E(nK_2)$ in Section 4. We quote them here in their (integral) sum graph form. Let $r = \sigma(K_{n,n} - E(nK_2))(\xi(K_{n,n} - E(nK_2)))(n \geq 6)$.

Lemma 5.1. In a (integral) sum labelling of $(K_{n,n} - E(nK_2)) \cup s K_1 (s \geq r)$ if there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_i + b_j \in B$ for any $a_i \in A (i \neq q)$.

Lemma 5.2. In a (integral) sum labelling of $(K_{n,n} - E(nK_2)) \cup s K_1 (s \geq r)$ if there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_p + b_i \in A$ for any $b_i \in B (i \neq p)$.

Lemma 5.3. In a (integral) sum labelling of $(K_{n,n} - E(nK_2)) \cup s K_1 (s \geq r)$ if there exist $a_p \in A$ and $b_q \in B (q \neq p)$ such that $a_p + b_q \in B$ then $a_p + b_i \in B \cup C$ for any $b_i \in B (i \neq p)$.

Lemma 5.4. $\xi(K_{n,n} - E(nK_2)) \geq 2n - 5$, $\sigma(K_{n,n} - E(nK_2)) \geq 2n - 3$ for $n \geq 6$.

Proof. If $a_i + b_j \in C$ for any $a_i \in A$ and any $b_j \in B (j \neq i)$ we assume that $b_1 < b_2 < \ldots < b_n$ and $a_i$ is the least integer of $A - \{a_i\}$. Then we have that $\sigma(K_{n,n} - E(nK_2)) \geq \xi(K_{n,n} - E(nK_2)) \geq 2n - 3$ by the distinctness of $a_1 + b_1, a_1 + b_2, \ldots, a_1 + b_{i-1}, a_1 + b_{i+1}, \ldots, a_1 + b_n, a_1 + b_2, \ldots, a_{i-1} + b_n, a_{i+1} + b_n, \ldots, a_{n-1} + b_n$.

If there exist $a_p \in A$ and $b_q \in B$ such that $a_p + b_q \in A \cup B$ we may assume without loss of generality that $a_p + b_q \in B$. Then we have that $a_i + b_q \in B$ for any $a_i \in A (i \neq q)$, $a_p + b_i \in B \cup C$ for any $b_i \in B (i \neq p)$ and $B = \{b_1, a_1 + b_2, \ldots, a_1 + b_q, a_{q+1} + b_q, \ldots, a_{i-1} + b_q, a_{i+1} + b_q, \ldots, a_{n-1} + b_q\}$. Suppose that $a_1 < a_2 < \ldots < a_n$ and $b_1$ is the largest integer of $B - \{b_q\}$. Then $b_i = a_n + b_q$. Firstly we will prove that $a_i + b_j \in C$ for any $a_i \in A (i \neq q)$ and any $b_j \in B (j \neq i, q)$.

If there exists $a_r \in A$ and $b_s \in B (s \neq r, q)$ such that $a_r + b_s \in B$ then $a_i + b_s \in B$ for any $a_i \in A (i \neq s)$ and $B = \{b_1, a_1 + b_2, \ldots, a_{r-1} + b_s, a_{r+1} + b_s, \ldots, a_{n-1} + b_s\}$. We may assume without loss of generality that $b_q < b_s$. Then since $a_1 + b_q < a_r + b_s \leq a_1 + b_q$ for $k \neq s$ then $a_1 + b_q = b_s$. So $0 < a_1 < \cdots < a_n$ and $b_q < a_1 + b_q = b_s < a_1 + b_q \leq a_1 + b_s$ for $k \neq s$. So $b_q \notin B = \{b_1, a_1 + b_2, \ldots, a_{r-1} + b_s, a_{r+1} + b_s, \ldots, a_{n-1} + b_s\}$, which is a contradiction. If there exists $a_r \in A (r \neq q)$ and $b_s \in B (s \neq r, q)$ such that $a_r + b_s \in A$ then $a_r + b_s \in A$ for any $b_i \in B$ and $i \neq r$. So $a_r + b_q \in A$, contradicting the fact that $a_r + b_q \in B$. So $a_i + b_q \in C$ for any $a_i \in A (i \neq q)$ and any $b_j \in B (j \neq i, q)$ and $a_q + b_j \notin B$ for any $b_j \in B (j \neq i, q)$.

Hence if $q \neq 1$ then we have that $\xi(K_{n,n} - E(nK_2)) \geq 2n - 5$ by the distinctness of $\{a_1 + b_2, \ldots, a_1 + b_{q-1}, a_1 + b_{q+1}, \ldots, a_1 + a_n, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_2\}$. Hence $\xi(K_{n,n} - E(nK_2)) \geq 2n - 5$ by the distinctness of $\{a_2 + b_3, \ldots, a_2 + b_n, a_3 + b_3, \ldots, a_n + b_2\}$. Thus, we only need to prove that $\sigma(K_{n,n} - E(nK_2)) \geq 2n - 3$. Hence we may assume that $a_i > 0$ for any $1 \leq i \leq n$ in the following proof.

If there exists $b_i \in B (s \neq q)$ such that $a_q + b_i \in A$ then $a_q + b_i \in A$ for any $b_i \in B (i \neq q)$ and $A = \{a_q, a_1 + b_1, \ldots, a_q + b_{q-1}, a_q + b_{q+1}, \ldots, a_q + b_n\}$. Hence $B = \{a_q + b_1 + b_1, \ldots, a_q + b_{q-1}, a_q + b_{q+1}, \ldots, a_q + b_n\}$. We have that $(n-1)(a_q + b_i) = 0$ i.e. $(a_q + b_i) = 0$. However, $a_q + b_i > 0$ for $a_q, b_i > 0$, which is a contradiction. So $a_q + b_i \in C$ for any $b_i \in B (j \neq q)$ and any $a_i \in A (i \neq j)$. Since $a_1 + b_1, a_1 + b_2, \ldots, a_1 + b_{q-1}, a_1 + b_{q+1}, \ldots, a_1 + b_n, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_1$ are distinct let $R = \{a_1 + b_1, a_1 + b_2, \ldots, a_1 + b_{q-1}, a_1 + b_{q+1}, \ldots, a_1 + b_n, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_1\}$. Then $R = \{2a_1 + b_1, a_1 + b_2 + b_2, \ldots, a_1 + b_{q-1} + b_2, a_1 + b_{q+1} + b_2, \ldots, a_1 + b_n + b_2, a_2 + b_2 + b_2, \ldots, a_n + b_1 + b_2\}$, where $2a_1 + b_1 < a_1 + b_2 + b_2 < \cdots < a_1 + b_{q-1} + b_2 < a_1 + b_{q+1} + b_2 < \cdots < a_1 + b_n + b_2 < a_2 + b_2 + b_2 < \cdots < a_n + b_1 + b_2$. Hence $R - \{a_1 + b_1\} \cup \{a_1 + b_1\} \subset C$. Since $a_1 + b_2 \neq b_2$ we have that $a_1 + a_1 + b_1 \in C$. By $a_1 + a_1 + b_1 < a_1 + b_2 + b_2 < a_1 + a_1 + b_1 + b_2$, we can obtain that $a_1 + a_1 + b_1 + b_2 \notin C$. So $|C| > |R - \{a_1 + b_1\} \cup \{a_1 + b_1\}| + 1 \geq 2n - 3$. □
Lemma 5.5. \( \sigma(K_{n,n} - E(nK_2)) \leq 2n - 3 \) for \( n \geq 6 \).

Proof. We consider the following labelling of the graph \( (K_{n,n} - E(nK_2)) \cup (2n - 3)K_1 \):

\[
\begin{align*}
  a_i &= (i - 1)N + 1, \quad i = 1, \ldots, n, \\
  b_j &= (j - 1)N + 3, \quad j = 1, \ldots, n, \\
  c_k &= (k - 1)N + 4, \quad k = 1, \ldots, n - 2, \\
  c_k &= kN + 4, \quad k = n - 1, n, \ldots, 2n - 3,
\end{align*}
\]

where \( N \geq 8 \) is an integer.

Let \( V(K_{n,n} - E(nK_2)) = (A, B) \) be the bipartition of \( K_{n,n} - E(nK_2) \), \( A = \{a_1, a_2, \ldots, a_n\} \), \( B = \{b_1, b_2, \ldots, b_n\} \), \( C = V((2n - 3)K_1) = \{c_1, c_2, \ldots, c_{2n-3}\} \), \( S = V(K_{n,n} - E(nK_2) \cup (2n - 3)K_1) = A \cup B \cup C \). It is easy to verify that the following assertions are true.

(1) \( S \subset Z_m \setminus \{0\} \).
(2) \( a_i + a_j \notin S \) for any \( a_i, a_j \in A \) (\( i \neq j \)).
(3) \( b_i + b_j \notin S \) for any \( b_i, b_j \in B \) (\( i \neq j \)).
(4) \( c_i + c_j \notin S \) for any \( c_i, c_j \in C \) (\( i \neq j \)).
(5) \( a_i + c_j \notin S \) for any \( a_i \in A \) and any \( c_j \in C \).
(6) \( b_i + c_j \notin S \) for any \( b_i \in B \) and any \( c_j \in C \).
(7) \( a_i + b_j \notin S \) if and only if \( i + j = n \) or \( i = j = n \).

So \( a_1b_{n-1}, a_2b_{n-2}, \ldots, a_{n-1}b_1, a_nb_n \) is \( E(nK_2) \).

Thus the above labelling is a sum labelling of \( (K_{n,n} - E(nK_2)) \cup (2n - 3)K_1 \) for \( n \geq 6 \). \( \square \)

Lemma 5.6. \( \zeta(K_{n,n} - E(nK_2)) \leq 2n - 5 \) for \( n \geq 6 \).

Proof. We consider the following integral labelling of the graph \( (K_{n,n} - E(nK_2)) \cup (2n - 5)K_1 \):

\[
\begin{align*}
  a_i &= (i - 1)N + 7, \quad i = 1, \ldots, n - 1, \quad a_n = 3, \\
  b_j &= (j - 1)N + 4, \quad j = 1, \ldots, n - 1, \quad b_n = -3, \\
  c_k &= (k - 1)N + 11, \quad k = 1, \ldots, n - 3, \\
  c_k &= kN + 11, \quad k = n - 2, \ldots, 2n - 5,
\end{align*}
\]

where \( N \geq 30 \) is an integer.

Let \( V(K_{n,n} - E(nK_2)) = (A, B) \) be the bipartition of \( K_{n,n} - E(nK_2) \), \( A = \{a_1, a_2, \ldots, a_n\} \), \( B = \{b_1, b_2, \ldots, b_n\} \), \( C = V((2n - 5)K_1) = \{c_1, c_2, \ldots, c_{2n-5}\} \), \( S = V(K_{n,n} - E(nK_2) \cup (2n - 5)K_1) = A \cup B \cup C \). It is easy to verify that the following assertions are true.

(1) \( S \subset Z_m \setminus \{0\} \).
(2) \( a_i + a_j \notin S \) for any \( a_i, a_j \in A \) (\( i \neq j \)).
(3) \( b_i + b_j \notin S \) for any \( b_i, b_j \in B \) (\( i \neq j \)).
(4) \( c_i + c_j \notin S \) for any \( c_i, c_j \in C \) (\( i \neq j \)).
(5) \( a_i + c_j \notin S \) for any \( a_i \in A \) and any \( c_j \in C \).
(6) \( b_i + c_j \notin S \) for any \( b_i \in B \) and any \( c_j \in C \).
(7) \( a_i + b_j \notin S \) if and only if \( i + j = n - 1 \) or \( n - 1 \leq i = j \leq n \).

So \( a_1b_{n-2}, a_2b_{n-3}, \ldots, a_{n-2}b_1, a_{n-1}b_{n-1}, a_nb_n \) is \( E(nK_2) \).

Thus the above labelling is an integral sum labelling of \( (K_{n,n} - E(nK_2)) \cup (2n - 5)K_1 \) for \( n \geq 6 \). \( \square \)
We have the following theorem by Lemmas 5.4–5.6.

**Theorem 5.1.** \( \sigma(K_{n,n} - E(nK_2)) = 2n - 3, \zeta(K_{n,n} - E(nK_2)) = 2n - 5 \) for \( n \geq 6 \).

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**References**