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Diagonal Flips of Triangulations on Closed Surfaces Preserving Specified Properties

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Consider a class $\mathscr{P}$ of triangulations on a closed surface F^2, closed under vertex splitting. We shall show that any two triangulations with the same and sufficiently large number of vertices which belong to $\mathscr{P}$ can be transformed into each other, up to homeomorphism, by a finite sequence of diagonal flips through $\mathscr{P}$. Moreover, if $\mathscr{P}$ is closed under homeomorphism, then the condition "up to homeomorphism" can be replaced with "up to isotopy."

1. INTRODUCTION

A *triangulation* G on a closed surface F^2 is a simple graph embedded in F^2 such that each face of G is a triangle and any two faces share at most one edge. It is easy to see that no two faces of G share more than one vertex if they do not share an edge. If there were two such faces, then G would not be simple. Let abc and acd be two faces of G which share an edge ac. A *diagonal flip* of ac consists of deleting the edge ac and adding the edge bd in the quadrilateral $abcd$ unless the edge bd already exists, in which case we do not perform the deformation (see Fig. 1). Two triangulations G_1 and G_2 are said to be *equivalent* if they can be transformed into each other by a finite sequence of diagonal flips.

In 1936, Wagner [18] proved that any two triangulations on the sphere with the same number of vertices are equivalent. He actually showed that any triangulation on the sphere with $n+3$ vertices is equivalent to the

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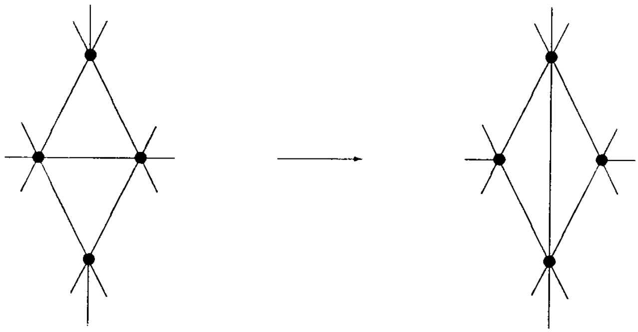

FIG. 1. A diagonal flip.

triangulation in Fig. 2. It is called the *standard spherical triangulation* Δ_n. By our definition of triangulations, the triangulation Δ_1 which is isomorphisc to K_4 is the smallest one on the sphere.

Extensions of this result to other surfaces came afterward. Dewdney [3] proved that any two triangulations on the torus with the same number of vertices are equivalent. Negami and Watanabe [15] showed that the same holds for the projective plane and for the Klein bottle.

For general closed surfaces, Negami has shown the following theorem in [13].

THEOREM 1. (Negami [13]). *For any closed surface F^2, there exists a positive integer $N(F^2)$ such that if G_1 and G_2 are two triangulations on F^2 with $|V(G_1)| = |V(G_2)| \geqslant N(F^2)$, then G_1 and G_2 are equivalent, up to homeomorphism.*

The results of Wagner, Dewdney, Negami, and Watanabe gives us $N(S^2) = 4$, $N(P^2) = 6$, $N(T^2) = 7$, and $N(K^2) = 8$ for the sphere S^2, the projective plane P^2, the torus T^2 and the Klein bottle K^2. Indeed, with respect to S^2, P^2, T^2 and K^2, the number $N(\cdot)$ coincides with the number

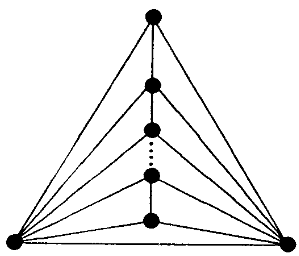

FIG. 2. The standard spherical triangulation Δ_n.

of vertices of minimal triangulations on each surface, but it does not hold in general. For example, Lawrencenko, Negami, and White [6] have constructed three inequivalent triangulations of K_{19} on the orientable closed surface of genus 20. (We can find a survey and some observations on these topics in [14], which also presents arguments for quadrangulations on closed surfaces.)

Notice that the above theorem includes the condition "up to homeomorphism." Consider the unique embedding of K_7 into the torus, which is a triangulation on the torus. Let C be an *essential* cycle of length 3 in the triangulation, that is, one which does not bound a 2-cell on the torus. Cutting the torus along C, we obtain the annulus triangulated. Now twist one boundary several times and identify again. The resulting triangulation is also K_7 on the torus. The former and the latter have the same combinatorial structure, that is, there is one-to-one correspondence between their face-sets. Hence they represent the same embedding, *up to homeomorphism*. In the above theorem, we need such an identification. If we focus only on the combinatorial structure, we cannot distinguish such a difference of homeomorphism of a closed surface and cannot remove this condition "up to homeomorphism" so easily.

Let F^2 be a closed surface. An *isotopy* $H\colon F^2 \times I \to F^2$ is a continuous map such that the map $h_t\colon F^2 \to F^2$, defined by $h_t(x) = H(x, t)$, is a homeomorphism of F^2 for each $t \in I$ with h_0 the identity map of F^2, where I stands for the interval $[0, 1]$. Two graphs G_1 and G_2 are said to be *isotopic* to each other or the same one, *up to isotopy*, if there is an isotopy $H\colon F^2 \times I \to F^2$ such that $h_1(G_1) = G_2$. Roughly speaking, when G_1 and G_2 are isotopic, they can be transformed into each other, moving continuously on F^2.

Nakamoto and Ota [11] proved a stronger theorem, which improves the condition "up to homeomorphism" into "up to isotopy." This theorem guarantees that two triangulations can be transformed directly.

THEOREM 2. (Nakamoto and Ota [11]). *For any closed surface F^2, there exists a positive integer $n(F^2)$ such that if G_1 and G_2 are two triangulations on F^2 with $|V(G_1)| = |V(G_2)| \geqslant n(F^2)$, then G_1 and G_2 are equivalent to each other, up to isotopy.*

In this paper, we will extend Theorems 1 and 2 to show that two triangulations with a prescribed property can be transformed into each other, preserving the property.

Let G be a triangulation on F^2 and ac an edge of G. The *shrinking* of ac consists of contracting the edge ac, identifying the vertices a and c into one vertex $[ac]$ and removing one edge in each of the 2-bonds hence created. We denote the resulting triangulation by G/ac. The edge ac is said to be *shrinkable* if its shrinking results in a triangulation on F^2. The inverse operation is called the *splitting* of the vertex $[ac]$ (or simply *vertex splitting*). A class $\mathscr{P}$ of triangulations on F^2 is said to be *splitting-closed* if it is closed under vertex splittings. Any member of $\mathscr{P}$ is called a *$\mathscr{P}$-triangulation* (or a triangulation with the property $\mathscr{P}$).

Let G be a $\mathscr{P}$-triangulation and ac an edge of G. Then ac is said to be $\mathscr{P}$-shrinkable if ac is shrinkable and if $G/ac \in \mathscr{P}$. Clearly, ac is $\mathscr{P}$-shrinkable if and only if G/ac is a $\mathscr{P}$-triangulation. A $\mathscr{P}$-triangulation G is *$\mathscr{P}$-irreducible* if no edge of G is $\mathscr{P}$-shrinkable. A *$\mathscr{P}$-diagonal flip* in a $\mathscr{P}$-triangulation G is a diagonal flip such that the resulting graph is also a $\mathscr{P}$-triangulation. Two triangulations G_1 and G_2 are said to be *$\mathscr{P}$-equivalent* to each other if they can be transformed into each other by a finite sequence of $\mathscr{P}$-diagonal flips.

The following two are our main theorems in this paper. We shall prove the first one in Section 2.

THEOREM 3. *For any closed surface F^2 and for any splitting-closed class $\mathscr{P}$ of triangulations on F^2, there exists a positive integer $N_{\mathscr{P}}(F^2)$ such that if G_1 and G_2 are any two $\mathscr{P}$-triangulations with $|V(G_1)| = |V(G_2)| \geqslant N_{\mathscr{P}}(F^2)$, then G_1 and G_2 are $\mathscr{P}$-equivalent, up to homeomorphisms.*

A class $\mathscr{P}$ of triangulations on a closed surface F^2 is said to be *closed under homeomorphism* if $h(G) \in \mathscr{P}$ for any member $G \in \mathscr{P}$ and for any homeomorphism $h: F^2 \to F^2$. In Section 3, we will give the following stronger form of Theorem 2, carrying out more technical arguments.

THEOREM 4. *For any closed surface F^2 and for any splitting-closed class $\mathscr{P}$ of triangulations on F^2 which is closed under homeomorphism, there exists a positive integer $n_{\mathscr{P}}(F^2)$ such that G_1 and G_2 are any two $\mathscr{P}$-triangulations with $|V(G_1)| = |V(G_2)| \geqslant n_{\mathscr{P}}(F^2)$, then G_1 and G_2 are $\mathscr{P}$-equivalent, up to isotopy.*

A graph G embedded in a closed surface F^2 is said to be *k-representative* if G intersects any essential simple closed curve in at least k points. (A simple closed curve γ on F^2 is called *essential* if γ does not bound a 2-cell on F^2.) The minimum value k such that G is k-representative is called the *representativity* of G and is usually denoted by $\rho(G)$. (These definitions can be found in [17].) The property of being k-representative is often discussed with graph-minor arguments.

Although our theorems, Theorems 3 and 4, are quite abstract, we can create many theorems, assigning concrete classes to $\mathscr{P}$. For example, the classes of triangulations with each of the following properties are splitting-closed and closed under homeomorphism, and hence they can be used as $\mathscr{P}$ not only in Theorem 3, but also in Theorem 4:

- Being k-representative.
- Intersecting any nonseparating simple closed curve in at least k points.
- Containing at least k disjoint homotopic cycles.
- Containing at least k disjoint cycles.
- Containing at least k distinct spanning trees.

Readers will be able to list many more splitting-closed classes of triangulations than those which we listed above.

2. PROOF OF THEOREM 3

To avoid confusion between abstract graphs and embedded graphs, we refer to an embedded graph as an *embedding*. If G is an embedding, then a *minor* H of G is an embedding obtained from G by a sequence of the following three operations: *single edge contraction*, *single edge deletion*, and *single isolated vertex removal*. Note that minor embeddings are defined up to isotopy. Notice that we never contract a loop (the deletion of its edge achieves the same result). Also, since the graphs we deal with are connected, we never need to use the third operation under careful use of the first two. We shall therefore only use the first two as minor operations.

Let G_1 and G_2 be two embedded graphs on a closed surface F^2. We say that G_1 is *minor-included* in G_2 (which we denote by $G_1 \leqslant_m G_2$) if G_1 is isomorphic to a minor of G_2, up to homeomorphism. Consider now a class $\mathscr{M}$ of embeddings that is closed under minor inclusion. Denote the complement of $\mathscr{M}$ in the set of all embeddings of F^2 by $\mathscr{U}$. Clearly, if $G_1 \leqslant_m G_2$ and $G_1 \in \mathscr{U}$, then $G_2 \in \mathscr{U}$. We call such a class $\mathscr{U}$ an *upward-closed class* of embeddings of F^2. The set of minor-minimal embeddings in $\mathscr{U}$ (i.e., minimal under minor inclusion) is called the *obstacle set* of $\mathscr{U}$ and is denoted by $\Omega(\mathscr{U})$. An important consequence of the work of Robertson and Seymour [16] is that $|\Omega(\mathscr{U})|$ is finite up to homeomorphism.

The next lemma is an "embedding form" of Wagner's Conjecture (see [16]).

LEMMA 5. *Let S be a set of embeddings on a closed surface F^2. Then either S is finite up to homeomorphism or there are two distinct embeddings G_1 and G_2 of S such that $G_1 \leqslant_m G_2$.*

Proof. Let $\mathscr{U}$ be the set of all embeddings G in F^2 for which there exists $G' \in S$ with $G' \leqslant_m G$. It is easy to see that $\mathscr{U}$ is an upward-closed class of embeddings. It is also clear that each element of $\Omega(\mathscr{U})$ is homeomorphic to an element of S. If the converse is also true, then we have that S is also finite up to homeomorphism since so is $\Omega(\mathscr{U})$. If some element G_2 of S is homeomorphic to no element of $\Omega(\mathscr{U})$, then there is another embedding $G_1 \in \Omega(\mathscr{U}) \subset S$ such that $G_1 \leqslant_m G_2$. ▮

The following lemma plays a role to connect edge shrinkings in triangulations and minor operations (contractions and deletions of edges) in general graphs. Obviously, an edge shrinking consists of one contraction and two deletions of edges. However, in order to apply graph-minor arguments to triangulations, we have to show the following lemma.

LEMMA 6. *Let G and G' be two triangulations on F^2 such that $G \leqslant_m G'$. Then G can be obtained from G' by a sequence of edge shrinkings.*

Proof. We may suppose that in the sequence of graphs transformed from G' to G by minor operations, each graph, except G' and G, is not a triangulation. It is clear that we can permute any two minor operations in the sequence that transforms G' into G. We show that we can rearrange these operations into a sequence of shrinkings. Since G and G' are triangulations, we must contract at least one edge, say ac. Let abc and acd be the two faces of G' incident with ac.

First suppose that b and d are the only common neighbors of a and c. Then ac is shrinkable. Since G and G' are triangulations, one edge in each of the 2-bonds resulting from the contraction of ac must be deleted. Therefore, we can start the sequence of minor operations on G' by the shrinking of ac.

Now suppose that a and c have another common neighbor x distinct from b and d. Notice that the cycle axc cannot be essential. For, otherwise, the contraction of ac would lead to a 2-representative embedding and so G would not be a triangulation. Therefore, axc bounds an open 2-cell Δ that is not a face. Let Δ' be the 2-cell obtained from Δ after the contraction of ac. Since the vertices $[ac]$ and x form a 2-cut, Δ' must be reduced to either a vertex or an edge in G. This can be achieved by contracting and deleting edges in Δ'. By Lemma 2 of [13], Δ contains at least one shrinkable edge inside, say e. We can therefore start the sequence of minor operations by contracting e and deleting an edge in each of the 2-bonds resulting.

In both cases, we can perform a shrinking in G' so that we obtain a triangulation G. ▮

LEMMA 7. *If $\mathscr{P}$ is a splitting-closed class of triangulations on F^2, then the number of $\mathscr{P}$-irreducible triangulations on F^2 is finite, up to homeomorphism.*

Proof. Let S be the set of all $\mathscr{P}$-irreducible triangulations on F^2. Suppose that S is not finite, up to homeomorphism. Then, by Lemma 5, there are two $\mathscr{P}$-irreducible triangulations G and G' such that $G \leqslant_m G'$. By

Lemma 6, G can be obtained from G' by a sequence of edge shrinkings. Since $\mathscr{P}$ is splitting closed and $G \in \mathscr{P}$, this sequence must be a sequence of $\mathscr{P}$-shrinkings. But this contradicts the $\mathscr{P}$-irreducibility of G'. ∎

Let $\mathscr{R}_k$ be the class of k-representative triangulations on a closed surface F^2. A $\mathscr{R}_k$-irreducible triangulation is usually called a *k-irreducible triangulation.* Although the above lemma implies the finiteness of the number of k-irreducible triangulations, Barnette and Edelson [1] have given an elementary proof of this fact for $k=3$, and Malnic and Nedela [10] have done in general cases. An explicit bound on the number of edges $|E(T)|$ of the $\mathscr{R}_k$-irreducible triangulations was given by Gao, Richter, and Seymour [4]. More precisely

$$|E(T)| \leqslant 18k^2 k!(4k \cdot k!)^k \, \xi^2,$$

where ξ is the Euler genus of F^2 and is equal to $2-\chi(F^2)$. ($\chi(F^2)$ denotes the Euler characteristic of F^2.) Furthermore, Nakamoto and Ota [12] have shown a linear bound for 3-irreducible triangulation with respect to the Euler genus.

For the rest of this section, let F^2 be a closed surface and $\mathscr{P}$ a splitting-closed class of triangulations on F^2.

LEMMA 8. *Let G_1 and G_2 be two $\mathscr{P}$-triangulations on F^2. Suppose that there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ of degree 3 such that $G_1 - v_1$ and $G_2 - v_2$ are isomorphic and in $\mathscr{P}$. Then G_1 is $\mathscr{P}$-equivalent to G_2.*

Proof. We only have to show that we can "move" v_1 in the faces of $G_1 - v_1$ by a sequence of $\mathscr{P}$-diagonal flips so that at the end, the resulting graph is isomorphic to G_2. Figure 3 shows how to transfer v_1 through adjacent faces by diagonal flips. Notice that at each stage, the resulting graph is a splitting of $G_1 - v_1$. Since $G_1 - v_1 \in \mathscr{P}$, the same is true for each of these graphs. Repetitions of this process give the result. ∎

The isomorphism of $G_1 - v_1$ and $G_2 - v_2$ in the above lemma can be replaced with their $\mathscr{P}$-equivalence as in the following lemma.

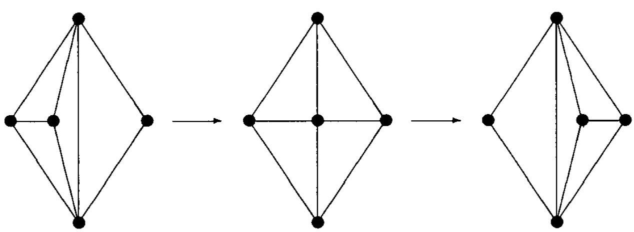

FIGURE 3

LEMMA 9. *Let G_1 and G_2 be two $\mathscr{P}$-triangulations on a closed surface F^2. Suppose that there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ of degree 3 such that $G_1 - v_1$ is $\mathscr{P}$-equivalent to $G_2 - v_2$. Then G_1 is $\mathscr{P}$-equivalent to G_2.*

Proof. We use induction on the length of a sequence of triangulations $G_1 - v_1 = H_0, H_1, ..., H_n = G_2 - v_2$ such that H_i is obtained from H_{i-1} by a $\mathscr{P}$-diagonal flip, for $i = 1, ..., n$. The base of the induction ($n = 0$) is given by Lemma 8.

Now suppose that the first $\mathscr{P}$-diagonal flip switches the edge $a'c' \in E(H_0)$ to the edge $b'd' \in E(H_1)$. Let abc be the face of H_0 that contains v_1. If $a'c'$ is not one of the sides of abc, we can apply this $\mathscr{P}$-diagonal flip to G_1 directly. Otherwise, suppose that $a'c'$ is an edge on the side of abc, say ac. Figure 4 shows how to obtain the corresponding diagonal flips in G_1 so that G_1 and the resulting graph G_1' are both in $\mathscr{P}$ (notice that the bottom graphs in the figure are splitting of their top counterparts, and therefore belong to $\mathscr{P}$). In both cases, G_1 is $\mathscr{P}$-equivalent to G_1'. By the induction hypothesis and Lemma 8, G_1' is $\mathscr{P}$-equivalent to G_2, and therefore G_1 is $\mathscr{P}$-equivalent to G_2. ∎

Now let T be a $\mathscr{P}$-triangulation on F^2 and n a positive integer. We amalgamate the graphs T and Δ_n in the following way: we identify the outer triangle of Δ_n with the boundary of a face f of T so that Δ_n is contained in the closure of f. Denote the resulting triangulation by $T + \Delta_n$. It is easy to see that $T + \Delta_n$ can be obtained by doing n vertex splittings inside the closure of f, and so $T + \Delta_n$ is a $\mathscr{P}$-triangulation. Since $T + \Delta_n$ can be also obtained by adding to f the interior vertices of Δ_n, it follows from

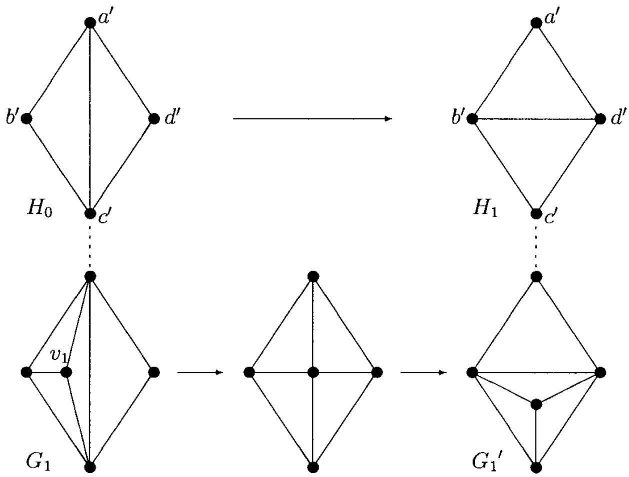

FIGURE 4

Lemma 9 that the equivalence class of the amalgamation of T and Δ_n is independent of the face of T where it is made.

LEMMA 10. *Let G and T be two $\mathscr{P}$-triangulations on a closed surface F^2. If G is $\mathscr{P}$-shrinkable to T, then G is $\mathscr{P}$-equivalent to $T+\Delta_n$ with $n=|V(G)|-|V(T)|$.*

Proof. Let $G=G_0, ..., G_n=T$ be a sequence of $\mathscr{P}$-shrinkings. We prove the result by induction on the length n of the given sequence. Suppose that G_1 is obtained from G by $\mathscr{P}$-shrinking the edge uv. Let $v, v_1, ..., v_s$ be the neighbors of u in G in clockwise order around u. Since the edge uv is $\mathscr{P}$-shrinkable, v and v_{s-1} are not adjacent. (For, if $vv_{s-1}\in E(G)$, then G_1 would have multiple edges.) So we can $\mathscr{P}$-flip uv_s to $v_{s-1}v$. In the same way, we can $\mathscr{P}$-flip $uv_{s-1}, ..., uv_3$ so that at the end, u is adjacent only to v, v_1, and v_2, and each of $v_1, ..., v_s$ is adjacent to v. The resulting graph with u removed is therefore isomorphic to $G_1=G/uv$. Notice that shrinking of uv in each of the triangulations obtained through this sequence yields G_1 and hence they belong to $\mathscr{P}$. Thus, G is $\mathscr{P}$-equivalent to $G_1+\Delta_1$. By the induction hypothesis, G_1 is $\mathscr{P}$-equivalent to $T+\Delta_{n-1}$. Since G_1 and $T+\Delta_{n-1}$ can be obtained respectively from $G_1+\Delta_1$ and $T+\Delta_n$ by deleting a vertex of degree 3 in each, $G_1+\Delta_1$ is $\mathscr{P}$-equivalent to $T+\Delta_n$ by Lemma 9. Hence G is $\mathscr{P}$-equivalent to $T+\Delta_n$. ∎

Let G be a triangulation on F^2. A *refinement* of G is a triangulation on F^2 that contains a subdivision of G as a subgraph.

LEMMA 11. *Any refinement G of a $\mathscr{P}$-triangulation T is $\mathscr{P}$-shrinkable to T.*

Proof. By Lemma 10 of [13], G is shrinkable to T. So G can be obtained from T by a sequence of vertex splittings. Since $\mathscr{P}$ is splitting-closed and $T\in\mathscr{P}$, the result follows. ∎

LEMMA 12. *For any two $\mathscr{P}$-triangulations G_1 and G_2 on a closed surface F^2, there are integers m_1 and m_2 such that $G_1+\Delta_{m_1}$ is $\mathscr{P}$-equivalent to $G_2+\Delta_{m_2}$.*

Proof. Embed both G_1 and G_2 on F^2 so that G_1 and G_2 intersect only at their edges and put a vertex at each edge crossing, and if necessary, subdivide regions to obtain a triangulation. The resulting triangulation G is a refinement of both G_1 and G_2, so by Lemma 11, it is a $\mathscr{P}$-triangulation that $\mathscr{P}$-shrinks to both G_1 and G_2. By Lemma 10, G is $\mathscr{P}$-equivalent to both $G_1+\Delta_{m_1}$ and $G_2+\Delta_{m_2}$, where $m_i=|V(G)|-|V(G_i)|$. Therefore, $G_1+\Delta_{m_1}$ is $\mathscr{P}$-equivalent to $G_2+\Delta_{m_2}$. ∎

We can now prove Theorem 3.

Proof of Theorem 3. Let $T_1, ..., T_n$ be the complete list of $\mathscr{P}$-irreducible triangulations on F^2, up to homeomorphism (we know that such a finite list exists by Lemma 7). By Lemmas 12 and 11, there is an integer $N = N_{\mathscr{P}}(F^2)$ such that for any i and j, $T_i + \Delta_{m_i}$ is $\mathscr{P}$-equivalent to $T_j + \Delta_{m_j}$ and both have precisely N vertices (here $m_l = N - |V(T_l)|$ for $l = 1, ..., n$).

Let G and G' be any two $\mathscr{P}$-triangulations with $|V(G)| = |V(G')| \geqslant N$. Then each of G and G' is $\mathscr{P}$-shrinkable to a $\mathscr{P}$-irreducible triangulation, say, without loss of generality, to T_1 and T_n, respectively. By Lemma 10, G is $\mathscr{P}$-equivalent to $T_1 + \Delta_m$ and G' is $\mathscr{P}$-equivalent to $T_n + \Delta_{m'}$, where $m = |V(G)| - |V(T_1)|$ and $m' = |V(G')| - |V(T_n)|$. Now $m \geqslant m_1$ and $m' \geqslant m_n$, and clearly $T_1 + \Delta_m$ $\mathscr{P}$-shrinks to $T_1 + \Delta_{m_1}$, and $T_n + \Delta_{m'}$ $\mathscr{P}$-shrinks to $T_n + \Delta_{m_n}$. Therefore, by Lemma 10 again, $T_1 + \Delta_m$ is $\mathscr{P}$-equivalent to $T_n + \Delta_{m'}$, and so G is $\mathscr{P}$-equivalent to G'. ∎

3. PROOF OF THEOREM 4

Let S_g and N_g denote the orientable and nonorientable closed surfaces of genus g, respectively. First we shall describe the group of all the auto-homeomorphisms of those surfaces and discuss how to construct any homeomorphic image of a given triangulation by diagonal flips.

Let F^2 be a closed surface and let $\Gamma(F^2)$ denote the group of all the homeomorphisms from F^2 onto itself. Then the set consisting of homeomorphisms isotopic to the identity map, denoted by $\Phi(F^2)$, forms a normal subgroup in $\Gamma(F^2)$. The quotient group $\Gamma(F^2)/\Phi(F^2)$ is called the *mapping class group* (or the *homeotopy group*) of F^2 and is denoted by $\Lambda(F^2)$. The mapping class group $\Lambda(F^2)$ is the set of isotopy classes of auto-homeomorphisms of F^2 and contains at most countably many elements.

Let γ be a 2-*sided* essential simple closed curve on F^2, that is, one with an annular neighborhood. Cut open F^2 along γ and identify the resulting two boundary components after full-twisting one of them. This deformation of F^2 naturally induces an auto-homeomorphism of F^2. We call this homeomorphism (or its isotopy class) the *Dehn twist* along γ and denote it by τ_γ.

It has been shown in [7–9] that the isotopy classes of Dehn twists generate a normal subgroup $\Lambda_0(F^2)$ of index 2 in $\Lambda(F^2)$. (We call each element of $\Lambda_0(F^2)$ a *twist-homeomorphism.*) Moreover, Humphries [5] has determined a finite set of generators of $\Lambda_0(S_g)$, which consists of the Dehn twists along the $2g+1$ simple closed curves given in Fig. 5. To generate the whole of $\Lambda(S_g)$, we need to add a fixed orientation-reversing homeomorphism as another generator. In other words, any orientation-preserving homeomorphism of S_g is a twist-homeomorphism.

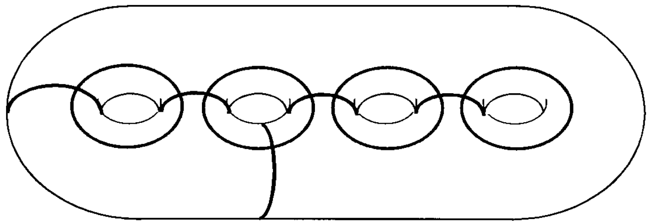

FIG. 5. Humphries' generators of $\Lambda(S_g)$.

On the other hand, Chillingworth [2] has determined a finite set of generators of $\Lambda(N_g)$, which consists of Dehn twists along specified simple closed curves and another homeomorphism called a "Y-homeomorphism." However, we shall omit here the description of their concrete forms as all that we require is the fact that $\Lambda_0(N_g)$ is of index 2 in $\Lambda(N_g)$; similarly $\Lambda_0(S_g)$ is of index 2 in $\Lambda(S_g)$,

Let γ be a 2-sided essential simple closed curve on a closed surface F^2. Consider a triangulation G on F^2. Suppose that G has two cycles $v_0 v_1 \cdots v_{2n-1}$ and $u_0 u_1 \cdots u_{2n-1}$ of length $2n$ along γ which together bound an annulus on F^2 triangulated by edges $v_i u_i$ and $v_i u_{i+1}$ ($i \equiv 0, 1, ..., 2n-1 \bmod 2n$). Let L be the subgraph of G consisting of these two cycles and the $4n$ edges in the annulus and γ the center line of the annulus. We call L a *ladder strip* along γ if L is an induced subgraph and there is no vertex inside the annulus. The edges $v_i u_i$ and $v_i u_{i+1}$ are called *rungs* of L.

Let G_k be one obtained from G by replacing all the rungs of the ladder strip L with $v_{i-k} u_{i+k}$ and $v_{i-k} u_{i+k+1}$ ($i \equiv 0, 1, ..., 2n-1 \bmod 2n$). That is, $G_k = G - \bigcup_{i=0}^{n-1} \{v_i u_i, v_i u_{i+1}\} + \bigcup_{i=0}^{n-1} \{v_{i-k} u_{i+k}, v_{i-k} u_{i+k+1}\}$. Equivalently, G_{k+1} can be obtained from G_k by a sequence of diagonal flips which first flips $v_{i-k} u_{i+k}$ to $v_{i-k-1} u_{i+k+1}$ for $i \equiv 0, 1, ..., 2n-1 (\bmod 2n)$, and next $v_{i-k} u_{i+k+1}$ to $v_{i-k-1} v_{i+k+2}$ for $i \equiv 0, 1, ..., 2n-1 (\bmod 2n)$, in this order. The sequence of these diagonal flips from $G = G_0$ to G_n is called the *Dehn sequence* along L or along γ. It is clear that this Dehn sequence realizes the Dehn twist along γ, that is, $\tau_\gamma(G) = G_n$. Actually, the edges $v_{n+i} u_{n+i}$ of G are carried to $v_{n+i-n} u_{n+i+n} = v_i u_i$ in G_n by this Dehn sequence and these new $v_i u_i$'s are running around γ. The inverse τ_γ^{-1} also can be realized by a similar sequence of diagonal flips. This fact is a key to prove Theorem 4. (The essence of our arguments in this section can be found in [11].)

LEMMA 13. *Given a splitting-closed class $\mathscr{P}$ of triangulations on a closed surface F^2 which is closed under homeomorphism, then there exists a triangulation H on F^2 such that $h(H)$ is $\mathscr{P}$-equivalent to H, up to isotopy, for any twist-homeomorphism $h: F^2 \to F^2$.*

Proof. Let $\{\gamma_1, \gamma_2, ..., \gamma_{2g+1}\}$ be the system of simple closed curves on S_g giving Humphries' generators. (Consider the corresponding system of Chillingworth's generators for N_g.) Let G be one of $\mathscr{P}$-triangulations and put a homeomorphic image of G on F^2 so that each γ_i crosses G transversely at the interiors of edges. Place a ladder strip L_i along γ_i, together with the configuration given in Fig. 6, where the paths $x_i y_i$ are segments of edges of G. The top and the bottom have to be identified to obtain an annulus and the central vertical part forms the ladder strip. Subdivide it suitably to triangulate F^2. We shall show that the resulting triangulation H is the required one.

One can easily see from Fig. 6 that for each integer k, there is a collection of pairwise disjoint paths $\{P_i\}$ such that each P_i joins x_i and v_{i+k} (and similarly for y_i and u_{i+k}), where subscripts are taken modulo $2n$. This structure guarantees that all the triangulations obtained through the Dehn sequence along this ladder strip contains a subdivision of G and hence they are $\mathscr{P}$-shrinkable to G by Lemma 11. Since $\mathscr{P}$ is splitting-closed, they belong to $\mathscr{P}$. Thus, the Dehn sequence along each L_i transforms H into $\tau_{\gamma_i}(H)$, preserving the property $\mathscr{P}$.

Let $h\colon F^2 \to F^2$ be any twist-homeomorphism of F^2. Then h can be expressed as a word $h = w_1 w_2 \cdots w_s$, where each w_j is either τ_{γ_i} or $\tau_{\gamma_i}^{-1}$ for some $i \in \{1, 2, ..., 2g+1\}$. (The Dehn twists $w_1, w_2, ..., w_s$ have to be applied in this order.) We shall show that H is $\mathscr{P}$-equivalent to $h(H)$, using induction on the length s of the word $w_1 w_2 \cdots w_s$. When h is expressed as an empty word, that is, when h is isotopic to the identity map, then we need to do nothing.

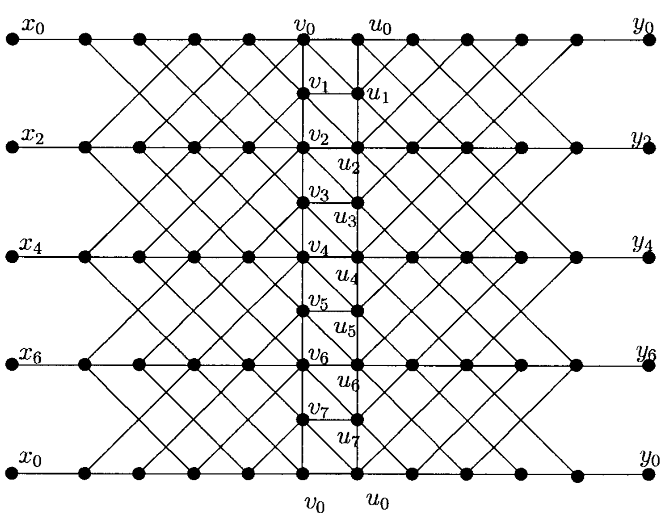

FIGURE 6

Put $W = w_2 \cdots w_s$ and suppose that $w_1 = \tau_{\gamma_i}$. (The same argument works for $\tau_{\gamma_i}^{-1}$.) Notice that $f^{-1}\tau_\gamma f = \tau_{f(\gamma)}$ for any homeomorphism $f\colon F^2 \to F^2$ and for any simple closed curve γ on F^2. Thus we have

$$h = WW^{-1}w_1 W = W \cdot (W^{-1}\tau_{\gamma_i} W) = W \cdot \tau_{W(\gamma_i)}.$$

This means that $h(H)$ can be obtained as the image of $W(H)$ by the Dehn twist $\tau_{W(\gamma_i)}$ along $W(\gamma_i)$. Since $W(H)$ is a homeomorphic image of H, $\tau_{W(\gamma_i)}$ can be realized by the Dehn sequence along the ladder strip $W(L_i)$ and each of the triangulations arising in the sequence from $W(H)$ to $h(H)$ contains a subdivision of $W(G)$. Since $\mathscr{P}$ is closed under homeomorphism, $W(G)$ belongs to $\mathscr{P}$, and so do the triangulations. This implies that $W(H)$ is $\mathscr{P}$-equivalent to $h(H)$. By our induction hypothesis, H is $\mathscr{P}$-equivalent to $W(H)$ and hence $\mathscr{P}$-equivalent to $h(H)$. ∎

We can now prove Theorem 4.

Proof of Theorem 4. In the proof of Theorem 3, replace the finite set $\{T_1, ..., T_n\}$ of $\mathscr{P}$-irreducible triangulations with $\{T_1, T'_1, T_2, T'_2, ..., T_n, T'_n\}$, where each T'_i is the image of T_i by an orientation-reversing homeomorphism for S_g (or a Y-homeomorphism for N_g). Then the same argument concludes with the statement of Theorem 3 with the condition "up to twist-homeomorphism" instead of "up to homeomorphism." Furthermore, we have the corresponding lower bound $N'_{\mathscr{P}}(F^2)$ for the number of vertices.

Let H be the triangulation constructed in the proof of Lemma 13 and put $n_{\mathscr{P}}(F^2) = \max\{N'_{\mathscr{P}}(F^2), |V(H)|\}$. Let G_1 and G_2 be any two $\mathscr{P}$-triangulations on F^2 with $|V(G_1)| = |V(G_2)| \geqslant n_{\mathscr{P}}(F^2)$. Subdivide H, preserving the structure of ladder strips $L_1, ..., L_{2g+1}$, so that $|V(H)| = |V(G_1)| = |V(G_2)|$. Then the statement of Lemma 13 still holds for this new H.

By Theorem 3, G_i $(i = 1, 2)$ and H are $\mathscr{P}$-equivalent up to homeomorphism since $|V(G_i)| = |V(H)| \geqslant N'_{\mathscr{P}}(F^2)$. That is, G_i is $\mathscr{P}$-equivalent to $h_i(H)$ up to isotopy, for some twist-homeomorphism $h_i\colon F^2 \to F^2$. Since $h_i(H)$ is $\mathscr{P}$-equivalent to H up to isotopy, G_1 and G_2 are $\mathscr{P}$-equivalent to each other, up to isotopy (via $h_1(H)$, H and $h_2(H)$). ∎

4. REMARKS

As described in Section 1, we can create many theorems from Theorems 3 and 4, assigning various splitting-closed classes to $\mathscr{P}$. One can easily see that all the splitting-closed classes listed in Section 1 are closed under homeomorphism of surfaces. So, we can apply Theorem 4 to these classes.

Are there splitting-closed classes which are not closed under homeomorphism of surfaces? The answer is "yes."

Consider the property $\mathscr{C}$ that a triangulation G on a closed surface F^2 contains k disjoint cycles homotopic to a fixed essential simple closed curve γ on F^2. It is clear that this property is preserved by vertex-splittings, but not by homeomorphisms of F^2 in general. That is, some homeomorphism of F^2 carries the k parallel disjoint cycles to ones not homotopic to γ.

Even in this case, the number of $\mathscr{C}$-irreducible triangulations on F^2 is finite, up to homeomorphism, by Lemma 7. So one might expect that a similar argument works for Theorem 4 without the assumption of a property, "closed under homeomorphism." To prove such a theorem, we need to find a finite set of Dehn twists preserving the property $\mathscr{C}$ so that they generate the group of homeomorphisms preserving $\mathscr{C}$. However, we have never known such a set of generators.

Does there exist a splitting-closed class of triangulations that includes a pair of triangulations with the same but arbitrarily large number of vertices which are not equivalent, up to isotopy?

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