Existence and Decay of Global Solutions of Some Nonlinear Degenerate Parabolic Equations

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0. INTRODUCTION

This paper is concerned with the existence and decay of global solution to the initial-boundary value problem for the nonlinear parabolic equation of the form;

\[ u_t - \Delta \beta(u) + \nabla \cdot G(u) + h(u) = 0 \quad \text{on } \Omega \times \mathbb{R}^+ \]

\[ u(x, 0) = u_0 \quad \text{and} \quad u|_{\partial \Omega} = 0 \]

(1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with the (smooth) boundary \( \partial \Omega \). On the functions \( \beta(u) \), \( G(u) = (g_1(u), \ldots, g_n(u)) \) and \( h(u) \) we make the following assumptions;

A.1. \( \beta(u) = |u|^m u \) for some \( m \geq 0 \).

A.2. \( g_i(u) \) (\( i = 1, 2, \ldots, n \)) are continuously differentiable in \( u \in \mathbb{R} \) and satisfy

\[ |g_i'(u)| \leq k_0 |u|^r \quad \text{for some } r \geq 0, k_0 > 0. \]

(2)

A.3. \( h(u) \) is Hölder continuous in \( u \in \mathbb{R} \) and

\[ |h(u)| \leq k_1 |u|^\alpha + 1 \quad \text{for some } \alpha \geq 0. \]

(3)

It should be noted that no monotonicity assumption is made on \( h(u) \) and the so called blowing up term \( -|u|^\alpha u \) is admitted. Though more general functions \( \beta, g_i(u) \) and \( h(u) \) could be treated we restrict ourselves to the above cases for simplicity. Since the case \( m > \alpha \) is easier (at least, concerning the global existence) we assume hereafter that \( \alpha > m \).

As is well known the special case \( m = 0 \), \( G(u) = 0 \) and \( h(u) = -|u|^{\alpha} u \) was studied by Fujita [2] and interesting existence and nonexistence results were established. Recently, such results have been generalized to the...
case: $m > 0$, $G(u) \equiv 0$ and $h(u) = -|u|^2u$ by Galaktionov [3], Nakao [6, 7], Sacks [10] and others. In [3, 6, 7] the so-called "potential well" method is employed, while in [10] comparison theorem is used effectively.

If $G(u) \equiv 0$ multiplication of the equation by $u$, yields

$$\frac{d}{dt} \left\{ \frac{1}{2} \| \nabla \beta(u) \|^2 + H(u) \right\} \leq 0$$  \hspace{1cm} (4)

where $\| \cdot \|$ denotes $L^2(\Omega)$ norm and we set $H(u) = \int_{\Omega} \int_{\Omega} h(s) ds dx$. The inequality (4) is essential for the argument of the method of "potential well." Since such inequality is impossible in the case $G(u) \neq 0$ this method cannot be applied in its original form to our problem.

For the comparison method in [10] it is required that the stationary problem

$$-\Delta \beta(u) + \nabla \cdot G(u) + h(u) = 0, \quad u|_{\partial \Omega} = 0$$  \hspace{1cm} (5)

have a positive solution. Unfortunately, we do not know the existence of a positive solution to the problem (4) except for the case $G(u) \equiv 0$. Thus the method in [10] also does not seem to be applicable to our problem (1).

The object of this paper is to show that a modified method of potential well can be applied to the case $G(u) \neq 0$. By this we shall show that a global solution exists if the initial value $u_0$ is small in a certain sense. This result coincides to the result known to the case $G(u) \equiv 0$. Moreover we derive a decay estimate of such solutions as $t \to \infty$ which is a generalization of our previous work [6, 9].

Our method is also related to that in [8], where the case $h(u) \equiv 0$ in (1) is treated. The Cauchy problem for the case $n = 1$ and $h(u) \equiv 0$ was considered by Gilding and Peletier [4]. There the physical meaning of the degenerate parabolic equation with the term $\nabla \cdot G(u)$ is explained.

At this time we have no nonexistence theorem for (1) with $G(u) \neq 0$. This also seems to be an interesting problem for future research.

1. Preliminaries and Theorems

First we recall the well-known Sobolev and Gagliardo–Nirenberg inequalities.

**Lemma 1.** Let $u \in W^{1,p}_0(\Omega)$. Then $u \in L^q(\Omega)$ and the inequality

$$\| u \|_q \leq C \| u \|_{W^{1,p}_0}$$

holds with a constant $C > 0$ depending on $\Omega, p$, where $1 \leq q \leq np/(n-p)$ if $n > p$, $1 \leq q < \infty$ if $n = p > 1$ and $1 \leq q \leq \infty$ if $n < p$. ($\| \cdot \|_q$ denotes $L^q$ norm).
**Lemma 2.** For all \( u \in W^{1,p}(\Omega) \), \( p \geq 1 \), we have

\[
\| u \|_q \leq C \| u \|_p^{\theta} \| \nabla u \|_p^\theta
\]

with a constant \( C \) independent of \( \Omega \) and \( \theta = (r^{-1} - q^{-1})(n^{-1} - p^{-1} + r^{-1})^{-1} \), where we assume

(i) \( 1 \leq r < \infty, r \leq q \leq \infty \) if \( n < p \),

(ii) \( 1 \leq r \leq q < \infty \) if \( n = p > 1 \),

(iii) \( 1 \leq r \leq q \leq \frac{pn}{n-p} \) if \( n > p \).

We shall use the above lemmas with \( p = 2 \). Following Sattinger [11] and Tsutsumi [12] we define the “potential well” of a functional on \( H_1 = W^{1,2}_0 \) and state some lemmas.

Let \( f(u) \) be a continuous function on \( \mathbb{R} \) such that

\[
| f(u) | \leq k | u |^{\gamma+1}, \quad k > 0
\]

with \( 0 < \gamma \leq 4/(n-2) \) if \( n > 2 \), \( 0 < \gamma < \infty \) if \( n = 1, 2 \), and consider the functional \( J_i(u) \) \((i = 0, 1)\) on \( \hat{H}_1 \):

\[
J_0(u) = \frac{1}{2} \| \nabla u \|^2 + \int_\Omega \int_0^u f(s) \, ds \, dx
\]

and

\[
J_1(u) = \| \nabla u \|^2 + \int_\Omega uf(u) \, dx.
\]

Define

\[
d = \inf_{\| \lambda \|_0 > 0} \sup_{u \in \hat{H}_1} J_0(\lambda u),
\]

which is often called the “depth of the potential well.”

Using the assumption (5) and Sobolev’s lemma we see easily \( d > 0 \). Then the potential well \( \mathcal{W} \) associated with \( J_0(u) \) is defined by

\[
\mathcal{W} = \{ u \in \hat{H}_1 | 0 \leq J_0(\lambda u) < d \text{ for } \forall \lambda \in [0, 1] \}.
\]

When \( f(u) = -k | u |^{\gamma} u \) we denote \( J_0, J_1 \) and \( \mathcal{W} \) by \( \bar{J}_0, \bar{J}_1 \), and \( \bar{W} \), respectively. The following lemmas are known:

**Lemma 3 ([12]).** \( 0 < \bar{d} < \infty \) and \( \bar{W} = \bar{W}^* \cup \{0\} \),

where

\[
\bar{W}^* = \{ u \in \hat{H}_1 | 0 \leq \bar{J}_0(u) < d \text{ and } \bar{J}_1(u) > 0 \}.
\]
**Lemma 4 ([7]).** Let \( u \in \mathcal{W}^\gamma \) and \( \partial - J_0(u) \equiv \varepsilon_0 > 0 \). Then we have

\[
J_1(u) \geq C(\varepsilon_0) \| \nabla u \|^2
\]

where \( C(\varepsilon_0) \) is a positive constant such that \( \lim_{\varepsilon_0 \to 0} C(\varepsilon_0) = 0 \).

In what follows we set \( f(u) = h(|u|^{-(m+1)/(m+1)} u) \) or \( f(u) = -k_1 |u|^{|x-m|/(m+1)} u \), and we consider the functionals and potential wells associated with these functions. Then our results read as follows:

**Theorem 1 (Existence).** Let \( \alpha \) satisfy the condition; \( m < \alpha < (m(n+2)+4)/(n-2) \) if \( n > 2 \) or \( m < \alpha < \infty \) if \( n = 1, 2 \). Assume that \( U_0 \equiv |u_0|^{\alpha} u_0 \in \mathcal{W}^\alpha \), \( J_0(u_0) < \bar{d} \), and \( u_0 \in L^{2r+2} \). Then there exists a positive constant \( d_1 \) independent of \( u_0 \) such that if \( u_0 \) satisfies the additional condition \( J_0(U_0) < d_1 \) the problem (1) admits a global solution \( u(t) \) which satisfies

\[
\beta(u(t)) \in L^{\infty}(R^+; \dot{H}_1), \quad u \in L^{\infty}(R^+; L^{2r+2})
\]

and

\[
\frac{\partial}{\partial t} \left( |u|^{m/2} u \right) \in L^2(R^+; L^2).
\]

The equation is fulfilled in the sense that

\[
\int_0^t \int_{\Omega} \{ -u(x, s) \phi_x + (\nabla \beta(u) - G(u)) \cdot \nabla \phi - f \phi \} \, dx \, ds
\]

\[
= \int_{\Omega} (u_0 \phi(x, 0)) \, dx - \int_{\Omega} u(x, t) \phi(x, t) \, dx \tag{7}
\]

for \( \forall t > 0 \) and \( \forall \phi \in C^0_0([0, \infty); \dot{H}_1) \).

**Remark.** It is easy to see that the solution \( u(t) \) in Theorem 1 is continuous in \( L^1 \) and that \( \lim_{t \to 0} u(t) = u_0 \).

**Theorem 2 (Decay).** The solutions in Theorem 1 satisfy the estimate

\[
\int_{\Omega} \left| \frac{\partial}{\partial t} \left( |u|^{m/2} u \right) \right|^2 \, ds + \| \nabla \beta(u(t)) \|^2 + \| u(t) \|_{2r+2}^2
\]

\[
\leq C(\varepsilon_0) (1 + t)^{-2(\min(m,r)+1)/m} \quad (\text{if } m > 0) \tag{8}
\]

or

\[
\leq C(\varepsilon_0) e^{-\lambda t} \quad \text{for some } \lambda > 0 \quad (\text{if } m = 0)
\]

where \( C(\varepsilon_0) \) is a constant depending on \( \varepsilon_0 = \bar{d} - J_0(U_0) > 0 \).
COROLLARY 1 ($L^\infty$-estimate). In addition to the assumption of Theorem 1 let $U_0 \in L^\infty(\Omega)$. Then $u(t) \in L^\infty(\Omega)$ for all $t \geq 0$, and we have

$$
\|u(t)\|_\infty \leq C(\varepsilon_0)(1 + t)^{-1/m} \quad \text{if } m > 0
$$

or

$$
\leq C(\varepsilon_0) e^{-\lambda t}, \quad \lambda > 0, \text{ if } m = 0.
$$

COROLLARY 2. Under the same assumption as in Corollary 1 the solution is unique if $m = 0$.

2. PROOF OF THEOREM 1

Assume that $u_0 \in C_0^\infty(\Omega)$. This assumption will be removed easily at the last stage. Consider the modified problem:

$$
u_t - \Delta \beta_\varepsilon(u) + \nabla \cdot G(u) + h(u) + \varepsilon |u|^{\alpha'} u = 0, \quad \varepsilon > 0 \tag{10}
$$

$$
u(x, 0) = u_0 \quad \text{and} \quad |u|_{\partial \Omega} = 0
$$

where $\beta_\varepsilon(u) = (u^2 + \varepsilon)^{m/2} u$ and $\alpha' (> \alpha)$ satisfies the same condition as $\alpha$. Since the problem (10) has no singularity and by (3)

$$(h(u) + \varepsilon |u|^{\alpha'} u) u \geq -\text{const.} \quad \text{for } u \in R,$$

there exists a unique classical solution $u_\varepsilon(x, t)$ for all time $t \geq 0$ (cf. [13]). Thus our task is to derive appropriate estimates of $u_\varepsilon$. Meanwhile we write $u$ for $u_\varepsilon$.

Multiplying the equation (10) by $(\partial/\partial t) \beta_\varepsilon(u)$,

$$
\left\{ \int_\Omega \beta'_\varepsilon(u) u_t^2 \, dx + \frac{d}{dt} \left\{ \frac{1}{2} \| \nabla \beta_\varepsilon(u) \|^2 + \varepsilon H^{(1)}_\varepsilon(u) - H_\varepsilon(u) \right\} \right.
$$

$$\left. \leq \int_\Omega |G'(u)| |\nabla u| |\beta_\varepsilon'(u)| |u_t| \, dx \tag{11}
$$

where we set

$$H^{(1)}_\varepsilon(u) = \int_\Omega \int_0^u |s|^{\alpha'} s \beta'_\varepsilon(s) \, ds$$

and

$$H_\varepsilon(u) = -\int_\Omega \int_0^u h(s) \beta'_\varepsilon(s) \, ds \, dx.$$
We may assume \( \alpha' \) is very close to \( \alpha \), and for \( |u|^m u \in \mathcal{H}_1 \) it is easy to see that
\[
\lim_{\epsilon \to 0} H_\epsilon(u) = \int_{\Omega} \int_0^u h(s) \, ds \, dx \equiv H(u) \quad (12)
\]
and
\[
\lim_{\epsilon \to 0} H^{(1)}_\epsilon(u) = \frac{m + 1}{m + \alpha' + 2} \int_\Omega |u|^{m + \alpha' + 2} \, dx \quad (13)
\]
Now, by virtue of (2),
\[
\int_\Omega |G'(u)||\nabla u||\beta'_\epsilon(u)||u_i| \, dx \\
\leq \frac{1}{2} \int |\beta'_\epsilon(u)||u_i|^2 + C_0 \int |u|^{2r} |\nabla u|^2 |\beta'_\epsilon(u)| \, dx
\]
and by (11) we have
\[
\frac{1}{2} \int \beta'_\epsilon(u) u_i^2 \, dx + \frac{d}{dt} \{I_{0,\epsilon}(u(t)) + \epsilon H^{(1)}_\epsilon(u)\} \\
\leq C_0 \int |u|^{2r} |\nabla u|^2 |\beta'_\epsilon(u)| \, dx
\]
where \( C_i \), \( i = 0, 1, 2, \ldots \) denotes positive constants independent of \( u_0 \) and \( \epsilon \), and where
\[
I_{0,\epsilon}(u) = \frac{1}{2} ||\nabla \beta(u)||^2 - H_\epsilon(u).
\]
in order to treat the right-hand side of (14) we multiply the equation (10) by \( |u|^{2r} u \) to get
\[
\frac{1}{2(r + 1)} \frac{d}{dt} \int |u|^{2r + 2} \, dx + (2r + 1) \int \beta'_\epsilon(u)|\nabla u|^2 |u|^{2r} \, dx \\
+ \epsilon \int |u|^{2r + \alpha' + 2} \, dx + \int h(u)|u|^{2r} u \, dx = 0
\]
and hence, by (3),
\[
\frac{d}{dt} \|u(t)\|_{2r + 2}^{2r + 2} + 2(r + 1)(2r + 1) \int \beta'_\epsilon(u)|\nabla u|^2 |u|^{2r} \, dx \\
\leq 2k_1(r + 1) \int |u|^{2r + \alpha' + 2} \, dx. \quad (15)
\]
From (14) and (15) we obtain
\[
\frac{d}{dt} \{ \mathcal{I}_{0,c}(u(t)) + \varepsilon H_r^{(1)}(u) + C_1 \| u(t) \|_{2r+2}^2 \} + \frac{1}{2} \int \beta'_e(u) u_t^2 \, dx \\
+ \int \beta'_e(u) |\nabla u|^2 |u|^{2r} \, dx \leq C_2 \int |u|^{2r+\alpha+2} \, dx.
\] (16)

We shall prove
\[
\int |u|^{2r+\alpha+2} \, dx \leq C_3 \| \nabla (|u|^{r+m/2} u) \|_2^2 \| u \|_{(2r+m)/(2r+m+2)}^{m/2}.
\] (17)

Indeed, if \( n(\alpha - m)/(2r + m + 2) \geq 1 \) we see that
\[
\| U \|_{2(2r+\alpha+2)/(2r+m+2)} \leq C \| \nabla U \|_{(2r+m+2)}^{2r+m+2} \| U \|_{(2r+m+2)/(2r+m+2)}^{2r + \alpha + 2} \| U \|_{n(\alpha - m)/(2r+m+2)}^{n(\alpha - m)/(2r+m+2)}
\]
by Lemma 2. Taking \( U = |u|^{r+m/2} u \), (16) is derived immediately.

When \( n(\alpha - m)/(2r + m + 2) < 1 \) (16) does not follow directly from Lemma 2. First for \( U \)
\[
\| U \|_{2(2r+\alpha+2)/(2r+m+2)} \leq C \| \nabla U \|_{\theta_0}\| U \|_{\theta_1}^{\theta_0}
\] (18)
with
\[
\theta_0 = \{2(2r+\alpha+2)/(2r+m+2) - 1\} \cdot \frac{2n}{n+2}
\]
and
\[
\theta_1 = 2\{2(2r+\alpha+2) - (\alpha - m) n\}/(2r+m+2)(n+2).
\]

Hölder's inequality and Lemma 2 imply
\[
\| U \|_1 \leq \| u \|_{n(\alpha - m)/2}^{n(\alpha - m)/4} \| U \|_{1 - n(\alpha - m)/(2r+m+2)}^{1 - n(\alpha - m)/(2r+m+2)}
\leq C \| u \|_{n(\alpha - m)/2}^{n(\alpha - m)/4} \| \nabla U \|_1^{\theta_2(1 - n(\alpha - m)/(2r+m+2))}
\]
with
\[
\theta_2 = \frac{2r + m + 2 - n(\alpha - m)}{2(2r+m+2) - n(\alpha - m)} \cdot \frac{2n}{n+2},
\]
and hence that
\[
\| U \|_1 \leq C \{ \| u \|_{n(\alpha - m)/2}^{n(\alpha - m)/4} \| \nabla U \|_{\theta_2[1 - n(\alpha - m)/(2r+m+2)]}^{\theta_2} \}^{1/\theta_2}.
\] (19)
with
\[ \theta_3 = 1 - \left( 1 - \theta_2 \right) \left[ 1 - \frac{n(a-m)}{2(2r + m + 2)} \right] = \frac{2(2r + m + 2) - (n - 2)(a-m)}{2(2r + m + 2)} \cdot \frac{n}{n+2}. \]

From (18) and (19) we obtain (17).

Moreover we use the inequalities
\[ \int \beta'_e(u) |\nabla u|^2 |u|^{2r} \, dx \geq C \| \nabla (|u|^{r+m/2} u) \|^2 \]
and
\[ \| u \|^2_{m^2 m/2} \leq C \| \nabla (\beta(u)) \|^2_{(a-m)/(m+1)}. \]

With (16), (17), (20) and (21) we imply the inequality
\[ \frac{d}{dt} \{ I_{0,\epsilon}(u(t)) + \epsilon H^{(1)}_e(u(t)) \} + \frac{1}{2} \int \beta'_e(u) u^2 dx + C \| \nabla (|u|^{r+m/2} u) \|^2 \{ d'_1 - \| \nabla (\beta(u)) \|^2_{(a-m)/(m+1)} \} \leq 0 \]
for a certain constant \( d'_1 > 0 \). This is the basic inequality for our argument.

Now, we recall the definition of \( \tilde{W}^c \):
\[ \tilde{W}^c = \{ U \in \dot{H}_1 \mid J_0(U) < \tilde{d}_0 \text{ and } \tilde{J}_1(U) > 0 \} \]
where we set, for \( U = |u|^m u = \beta(u) \),
\[ J_0(U) \equiv I_0(u) = \frac{1}{2} \| \nabla \beta(u) \|^2 - \frac{m+1}{m+\alpha+2} \| u \|_{m+\alpha+2} \]
and
\[ \tilde{J}_1(U) \equiv \tilde{I}_1(u) = \| \nabla \beta(u) \|^2 - \| u \|_{m+\alpha+2} \]
Also recall that \( \tilde{d}_0 = \inf_{U \in \dot{H}_1, U \neq 0} \sup_{\lambda > 0} J_0(\lambda U) \) and
\[ J_0(U) \equiv I_0(u) = \frac{1}{2} \| \nabla \beta(u) \|^2 + H(u). \]

Assume that
\[ U_0 \equiv |u_0|^m u_0 \in \tilde{W}, \quad J_0(U_0) < \tilde{d}_0 \quad \text{and} \quad \| \nabla U_0 \| < d'_1. \]
Then, taking a sufficiently small \( \epsilon \), we may assume \( I_{0,\epsilon}(u_0) + \epsilon H^{(1)}_e(u_0) < \tilde{d}_0 \) and
\[ \| \nabla \beta(u_0(t)) \| < d'_1 \]
on some interval, say \([0, T_c]\).
From (22) we have
\[
\frac{d}{dt} \{ I_{0, \varepsilon}(u(t)) + \varepsilon H^{(1)}_\varepsilon(u(t)) \} + \frac{1}{2} \int \beta'_\varepsilon(u) u^2_i \, dx \leq 0
\]  \ 
(25)
as long as (24) holds, and, in particular,
\[
\bar{J}_0(U(t)) \leq I_{0, \varepsilon}(u(t)) + \varepsilon H^{(1)}_\varepsilon(u(t)) 
\leq I_{0, \varepsilon}(u_0) + \varepsilon H^{(1)}_\varepsilon(u_0) < \bar{a}_0.
\]  \ 
(26)
This inequality when combined with standard argument concerning the potential well (cf. [11, 12]) implies that \( \beta(u(t)) \equiv |u|^m u = U(t) \in \mathcal{W} \) for all time as long as (24) holds. Since, by Lemma 3,
\[
\bar{a}_0 > J_0(U_0) \geq J_0(U(t)) \geq \left( \frac{1}{2} - \frac{m+1}{m+\alpha+2} \right) \| \nabla U(t) \|^2 
= \frac{\alpha-m}{2(m+\alpha+2)} \| \nabla U(t) \|^2 
\]
we have
\[
\| \nabla U(t) \|^2 \leq \frac{2(m+\alpha+2)}{\alpha-m} J_0(U_0).
\]  \ 
(27)
Here we set \( d_i = (\alpha-m) d'_i / 2(m+\alpha+2) \) and assume that \( J_0(U_0) < d_i \). Then (27) implies that
\[
\| \nabla U(t) \|^2 \leq d'_i - \delta_0
\]  \ 
(28)
with \( \delta_0 = 2(m+\alpha+2)(d_i - J_0(U_0)) / (\alpha-m) > 0. \)
Thus we can conclude that (24) holds for all time \( t \) and consequently \( U(t) \in \mathcal{W} \) for all time \( t \).
Thus it has been proved that the approximate solutions \( u_\varepsilon(t) \) of the modified problem (9) exist globally and that the desired estimates
\[
\| \nabla \beta(u_\varepsilon(t)) \|^2 < \bar{a}_0 \quad \text{and} \quad \int_0^\infty \int_\Omega \beta'_\varepsilon(u) u^2_i \, dx \, dt < \bar{a}_0
\]  \ 
(29)
hold for \( \forall t. \)
Based on the estimates (29) standard compactness argument (cf. [5, 6]) shows that \( u_\varepsilon \) converges to a function \( u \) as \( \varepsilon \to 0 \) in such a way that
\[
u_\varepsilon(t) \rightarrow u(t) \text{ in } L^1_{loc}([0, \infty); L^1),
\]
\[
\beta_\varepsilon(u_\varepsilon(t)) \rightarrow \beta(u(t)) \text{ weakly* in } L^\infty_{loc}([0, \infty); \mathcal{H}_1),
\]
\[
\frac{\partial}{\partial t} (|u|^m u) \rightarrow \frac{\partial}{\partial t} (|u|^m u) \text{ weakly in } L^2_{loc}([0, \infty); L^2)
\]
and
\[ h(u_0(t)) \rightarrow h(u(t)) \text{ in } L^1_{\text{loc}}([0, \infty); L^1). \]

Thus the proof of Theorem 1 is completed if \( u_0 \in C^2_0(\Omega) \). For general \( u_0 \) satisfying the assumptions we have only to take a sequence \( u_{0,p} \in C^2_0(\Omega) \) such that \( \beta(u_{0,p}) \rightarrow \beta(u_0) \) in \( H_1 \) as \( p \rightarrow \infty \) and to take the limit of \( u_p \), corresponding solutions to \( u_{0,p} \), as in the above.

3. PROOF OF THEOREM 2

In this section we shall prove Theorem 2, which asserts that the solutions of Theorem 1 decay to 0 at a certain rate as \( t \rightarrow \infty \). We give a somewhat formal proof, which is justified through the approximate solutions \( u_\epsilon \) in the previous section.

By (22) and (28) we have
\[ \frac{d}{dt} \left\{ I_0(u(t)) + C \| u(t) \|_{L_{\text{loc}}^1}^{2r + \frac{3}{2}} \right\} + \frac{1}{2} \int |u|^m |u_t|^2 \, dx \]
\[ + \delta_0 \| \nabla(|u|^{m+2} u) \|_2^2 \leq 0 \]  
(30)

with \( \delta_0 = 2(m + \alpha + 2)(\alpha - m)^{-1} \left( d_1 - I_0(U_0) \right) > 0 \).

On the other hand, multiplying the equation by \( |u|^m u = U - \beta(u) \)
\[ \int u_t \beta(u) \, dx \leq C \left( \int |u_t|^2 |u|^m \right)^{1/2} \| \nabla \beta(u) \|^{(m+2)/2(m+1)}. \]

By Lemma 4 we know \( 2J_0(U) \geq J_1(U) \geq C(\epsilon_0) \| \nabla U \|^2 \) with \( \epsilon_0 \equiv d_0 - J_0(U_0) \geq d_0 - J_0(U_0) \). Hence we have from above
\[ [J_0(U(t))]^{(3m+2)/2(m+1)} \leq C \int |u_t|^2 |u|^m \, dx. \]  
(31)

Also we note that
\[ \| \nabla(|u|^{m+2} u) \|^2 \geq C \| u \|_{L_{\text{loc}}^1}^{2r + m + 2}. \]  
(32)

From (30)--(32) we obtain
\[ \frac{d}{dt} \left\{ J_0(U(t)) + C \| u(t) \|_{L_{\text{loc}}^1}^{2r + \frac{3}{2}} \right\} + C J_0(U(t))^{(3m+2)/2(m+1)} \]
\[ + C \delta_0 \| u(t) \|_{L_{\text{loc}}^1}^{2r + m + 2} \leq 0. \]  
(33)
Using the boundedness of $\|u(t)\|_{2r+2}$ and $\|\nabla \beta(u(t))\|$ we can show from (32) that

$$\frac{d}{dt} \left\{ J_0(U(t)) + C \| u(t) \|_{2r+2}^2 \right\} + C(u_0) \left\{ J_0(U(t)) + C \| U(t) \|_{2r+2}^2 + \lambda_0 \right\} \leq 0$$

(34)

with $\lambda_0 = 1 + m/2 \left( \min(m, r) + 1 \right)$. This inequality implies

$$J_0(U(t)) + C \| u(t) \|_{2r+2}^2 \leq C(u_0)(1 + t)^{-2(\min(m, r) + 1)/m} \quad \text{if } m > 0$$

$$\leq C(u_0) e^{-\lambda t}, \quad \lambda > 0, \text{ if } m = 0.$$ Integrating (31) from $t$ to $\infty$ we can obtain the estimate for

$$\int_0^\infty \| (\partial/\partial t)(|u|^{m/2} u) \|^2 \, ds.$$

4. PROOFS OF COROLLARIES 1, 2

Multiplying the equation by $|u|^p u$ ($p \geq 0$) we have

$$\frac{1}{p + 2} \frac{d}{dt} \| u(t) \|_{p+2}^2 + \frac{4(p+1)(m+1)}{(p+m+2)^2} \int \| \nabla(|u|^{p+m/2} u) \|^2 \, dx$$

$$\leq k_1 \int |u|^{p+x+2} \, dx.$$ (35)

This is the same inequality as (2.1) with $\varepsilon = 0$ in [9], and using the fact that $\| \nabla \beta(u(t))\|$ tends to 0 as $t \to \infty$ the same argument in [9] implies

$$\| u(t) \|_q \leq C_q(\varepsilon_0, \| u_0 \|_q)(1 + t)^{-1/m}$$

for any $q \geq 2$. Combining this with the Moser's technique (cf. Alikakos [1], Nakao [(3.4), 9]) we can obtain the estimate for $u(t)$ in Corollary 1.

Next, we shall prove Corollary 2. Letting $u(t)$ and $v(t)$ be two solutions we have

$$\frac{1}{2} \frac{d}{dt} \| u(t) - v(t) \|^2 + \| \nabla(u(t) - v(t)) \|^2$$

$$\leq C \int |G(u) - G(v)| |\nabla(u - v)| \, dx$$

$$+ C \int |f(u) - f(v)| |u - v| \, dx$$

$$\leq C(\| u(t) \|_\infty, \| v(t) \|_\infty) \int (|u - v| |\nabla(u - v)| + |u - v|^2) \, dx.$$
and hence
\[
\frac{d}{dt} \| u(t) - v(t) \|^2 \leq \text{Const.} \| u(t) - v(t) \|^2
\]
which implies \( u \equiv v \).

Remark. The above argument for the proof of Corollary 2 is valid only for the case \( m = 0 \), and the uniqueness problem for \( m > 0 \) remains open.

REFERENCES

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