Note

Inherited Arcs in Finite Affine Planes

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1. Introduction

Let $\Sigma = \Sigma(q)$, $q = p^h$, $p$ prime, be the family of all affine planes of order $q$. $\Sigma$ is not empty, as it contains the affine plane $AG(2, q)$ over the Galois field $GF(q)$. Let $\sigma$ be any plane belonging to $\Sigma$. We can identify the points of $\sigma$ with those of $AG(2, q)$, and regard the lines of $\sigma$ as point-sets in $AG(2, q)$. Moreover, if $\sigma$ is a translation plane, we may also assume that the translation groups of $AG(2, q)$ and $\sigma$ coincide and consist of all the maps with equations $x' = x + a$, $y' = y + b$, where $a, b \in GF(q)$.

In this paper we shall be concerned with questions of the following kind:

Suppose there are given an irreducible conic $K$ in $AG(2, q)$, an affine plane $\sigma$ belonging to $\Sigma$, and the information that $K$ is an inherited arc of $\sigma$, i.e., $K$ is still an arc of $\sigma$ which can be completed in a $(q + 1)$-arc in the projective closure $\bar{\sigma}$ of $\sigma$. How does the type (hyperbola, parabola, ellipse) of $K$ influence the structure of $\sigma$, in particular, under what conditions on $K$ and $\sigma$ can we conclude that $\sigma$ is isomorphic to $AG(2, q)$?

Examples of inherited arcs are known in various planes: in the associative André plane of odd order [7], in a nonassociative Moulton plane of order $q = p^h$, $h = 2g$, $p^g \equiv 1 \pmod{4}$ [5], in the Hughes plane [6] when $K$ is a hyperbola; in the associative Moulton plane of odd order [1] when $K$ is a parabola; in a nonassociative Moulton plane of order $q = p^h$, $h = 2g$, $p^g \equiv 3 \pmod{4}$ [5] when $K$ is an ellipse. It seems plausible that many other planes have inherited arcs. However, there are planes which cannot contain inherited arcs for each of the three types. In fact, we shall prove the following

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THEOREM 1. Let $K$ be a parabola in $AG(2, q)$ where $q$ is odd. If $K$ is still an arc in a translation plane $\sigma$ belonging to $\Sigma$, then $\sigma$ coincides with $AG(2, q)$.

Theorem 1 cannot be extended to the case where $q$ is even:

THEOREM 2. There is a parabola in $AG(2, 2^{2m})$ which is still an arc in the Hall plane $\mathcal{H}(2^{2m})$ of the same order.

We shall also prove

THEOREM 3. Let $K$ be an irreducible conic in $AG(2, q)$ which is still an arc in a plane $\sigma$ belonging to $\Sigma$. If any chord and any tangent of $K$ is also a line of $\sigma$, then $\sigma$ coincides with $AG(2, q)$.

2. Proof of Theorem 1

First, we prove the following

LEMMA. For any line $l$ of $\sigma$ there is a line $r$ of $AG(2, q)$ such that $|l \cap r| \geq 3$.

Take two distinct points $P$, $Q$ lying on $l$. Through these points there is also a line $r$ of $AG(2, q)$. The translation $\tau$ of $AG(2, q)$ mapping $P$ into $Q$ leaves $r$ invariant. $\tau$ is also a translation of $\sigma$. Thus $\tau$ leaves $l$ invariant. Therefore, both of the lines $l$ and $r$ contain $Q'$. Since by assumption $p \neq 2$, $Q' \neq P$ holds. Hence, $P$, $Q$, $Q'$ are three distinct common points of $l$ and $r$. This proves the lemma.

For any $m, d \in GF(q)$, we shall denote by $Q(m, d)$ the parabola of $AG(2, q)$ whose equation is $y = x^2 + mx + d$. Without loss of generality, we may assume that $K$ is one of such parabolas.

Consider the points of $AG(2, q)$ together with the vertical lines $x = c, c \in GF(q)$, and the parabolas $Q(m, d), m, d \in GF(q)$. This incidence structure is an affine plane $\mathcal{A}$ belonging to $\Sigma$.

The map

$$\Phi: x' = x, \quad y' = x^2 + y,$$

fixes the vertical lines and maps the line $y = mx + d$ into $Q(m, d)$. Hence $\Phi$ gives rise to an automorphism $\alpha \simeq AG(2, q)$.

Any non-vertical line of $AG(2, q)$ represent in $\alpha$ a parabola whose improper point is that of the vertical lines. In fact, the pre-image of the line
\[ y = gx + k, \quad g, \quad k \in \text{GF}(q), \quad \text{under } \Phi \text{ is the parabola with equation } y = -x^2 + gx + k. \]

Now let us consider the class \( S \) of parallel lines of \( \sigma \) whose improper point \( P^\infty \) is such that \( K \cup \{P^\infty\} \) is an arc in \( \tilde{\alpha} \). Then any line of \( S \) meets \( K \) in exactly one point. Let \( l \) be any line of \( S \). We prove that \( l \) coincides with a vertical line of \( \text{AG}(2, q) \). By way of contradiction, we assume that \( l \) contains two points \( A, B \) not lying on the same vertical line of \( \text{AG}(2, q) \). There is a parabola \( \Omega(m, d) \) which passes through \( A \) and \( B \). Let \( \tau \) be the translation of \( \text{AG}(2, q) \) mapping \( \Omega(m, d) \) into \( K \), \( \tau \) is again a translation of \( \sigma \). Thus \( l' = l^* \) is still a line of \( \sigma \). Clearly, \( l' \) meets \( K \) in two points: \( A' \) and \( B' \). Since \( l' \) belongs to \( S \), this is not possible.

We prove that any non-vertical line \( l \) of represent in \( \alpha \) a parabola whose improper point is that of the vertical lines. First, we prove that \( l \) is an arc in \( \alpha \) by showing that \( l \) meets any \( \Omega(m, d) \) in at most two points. By assumption this is true for that \( \Omega(m, d) \) which coincides with \( K \). Let \( \Omega(m', d') \neq \Omega(m, d) \). Let \( \tau \) be the translation of \( \text{AG}(2, q) \) mapping \( \Omega(m', d') \) into \( \Omega(m, d) \), \( \tau \) is again a translation of \( \sigma \). Thus \( l' = l^* \) is still a line of \( \sigma \). Since \( |l \cap \Omega(m', d')| = |l' \cap \Omega(m, d)| \), the assertion is proved.

In the projective closure \( \tilde{\alpha} \) of \( \alpha \), \( l \) can be completed in a \( (q + 1) \)-arc \( \tilde{l} \) by adjoining the improper point of the vertical lines. By Segre's theorem [9] (see also [4: 8.2.4]), \( \tilde{l} \) is an irreducible conic of \( \tilde{\alpha} \). This proves our assertion.

We are able to prove Theorem 1. We have to prove that any non-vertical line of \( \sigma \) coincides with a line of \( \text{AG}(2, q) \). Let \( l \) be any non-vertical line of \( \sigma \). By the lemma there is a line \( \varepsilon \) of \( \text{AG}(2, q) \) such that \( |l \cap \varepsilon| \geq 3 \). The same lines represent in \( \alpha \) two parabolas with the same improper point. Hence they coincide.

3. PROOF OF THEOREM 2

For any \( s \in \text{GF}(2^{2m}) - \text{GF}(2^m) \), let us consider the parabola \( C(s) \) whose equation is \( y = x^2 + sx \). We take the Hall plane \( \mathcal{H}(2^m) \) as the derived plane from \( \text{AG}(2, q) \) with respect to the derived set formed by all the improper points \((0, 1, 0), (1, n, 0) \ n \in \text{GF}(2^m) \) (see [3, p. 223–226]).

In order to prove that \( C(s) \) is again an arc in \( \mathcal{H}(2^m) \), we have to verify that there is no triangle inscribed in \( C(s) \) such that \( |l \cap \varepsilon| \geq 3 \). The same side belongs to the derived set.

Take three distinct points \( A_i = (x_i, x_i^2 + sx_i) \) \((i = 1, 2, 3)\) lying on \( C(s) \). Since the slope of the line \( A_iA_j \) \((1 \leq i < j \leq 3)\) is equal to \( x_i + x_j + s \), the improper point of \( A_iA_j \) belongs to the derived set if and only if \( x_i + x_j + s \in \text{GF}(2^m) \). As \((x_1 + x_2 + s) + (x_2 + x_3 + s) + (x_1 + x_3 + s) = s \) and by assumption \( s \notin \text{GF}(2^m) \), the assertion follows.
4. Proof of Theorem 3.

We have to prove that if a line \( l \) of \( \sigma \) is disjoint from \( K \), then \( l \) coincides with a line \( \nu \) of \( \text{AG}(2, q) \). Let \( \text{PG}(2, q) \) denote the projective closure of \( \text{AG}(2, q) \). Let \( \overline{K} \) be the conic of \( \text{PG}(2, q) \) which coincides with \( K \) in \( \text{AG}(2, q) \).

Let us consider the class \( \Pi \) of parallel lines of \( \sigma \) which includes \( l \). By assumption, those lines of \( \sigma \) which meet \( K \) are again lines of \( \text{AG}(2, q) \). Clearly, such lines of \( \text{AG}(2, q) \) belong to a class \( A \) of parallel lines. The improper point \( P^{\infty} \) of the lines of \( A \) does not lie on any line of \( \text{AG}(2, q) \) which meets \( K \) as well as \( l \). Therefore, any chord and any tangent of \( \overline{K} \) meets \( l = l \cup \{P^{\infty}\} \) in exactly one point. By [8] (see also [2] for \( q \) even) \( l \) coincides with an external line to \( \overline{K} \) of \( \text{AG}(2, q) \). Therefore, there is a line \( \nu \) of \( \text{AG}(2, q) \) which coincides with \( l \).

References