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Almost maximally almost-periodic group topologies determined by T-sequences ☆

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Abstract

A sequence $\{a_n\}$ in a group G is a *T-sequence* if there is a Hausdorff group topology τ on G such that $a_n \xrightarrow{\tau} 0$. In this paper, we provide several sufficient conditions for a sequence in an abelian group to be a T-sequence, and investigate special sequences in the Prüfer groups $\mathbb{Z}(p^\infty)$. We show that for $p \neq 2$, there is a Hausdorff group topology τ on $\mathbb{Z}(p^\infty)$ that is determined by a T-sequence, which is close to being maximally almost-periodic—in other words, the von Neumann radical $\mathbf{n}(\mathbb{Z}(p^\infty), \tau)$ is a non-trivial finite subgroup. In particular, $\mathbf{n}(\mathbf{n}(\mathbb{Z}(p^\infty), \tau)) \subsetneq \mathbf{n}(\mathbb{Z}(p^\infty), \tau)$. We also prove that the direct sum of any infinite family of finite abelian groups admits a group topology determined by a T-sequence with non-trivial finite von Neumann radical.
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1. Introduction

Given a set X, a point $x_0 \in X$, and a sequence $\{x_n\}$ of distinct elements in X, one can easily construct a Hausdorff topology τ on X such that $x_n \xrightarrow{\tau} x_0$. This is, however, not the case for groups and group topologies, as Example 1.2 below demonstrates.

Following Zelenyuk and Protasov [7,11], who were the first to investigate this type of question, we say that a sequence $\{a_n\}$ in a group G is a *T-sequence* if there is a Hausdorff group topology τ on G such that $a_n \xrightarrow{\tau} 0$. In this case, the group G equipped with the finest group topology with this property is denoted by $G\{a_n\}$. A similar notion exists for filters, in which case one speaks of a *T-filter*. Zelenyuk and Protasov characterized T-sequences and T-filters in abelian groups [7, 2.1.3, 2.1.4] and [11, Theorems 1, 2], and studied the topological properties of $G\{a_n\}$, where $\{a_n\}$ is a T-sequence. (They also present a characterization of T-filters in non-abelian groups in [7, 3.1.4].)

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Example 1.1. For a prime number p, let $A = \mathbb{Z}(p^\infty)$ be the Prüfer group. It can be seen as the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the elements of p-power order, or the group formed by all p^nth roots of unity in $\mathbb{C}$. For $e_n = \frac{1}{p^n}$, $\{e_n\}$ is clearly a T-sequence in A, because $e_n \longrightarrow 0$ in the subgroup topology that A inherits from $\mathbb{Q}/\mathbb{Z}$.

Example 1.2. Keeping the notations of Example 1.1, set $a_n = -\frac{1}{p} + \frac{1}{p^n} = -e_1 + e_n$. If $a_n \xrightarrow{\tau} 0$ for some group topology τ, then also $e_{n-1} = pa_n \xrightarrow{\tau} 0$, and therefore, $e_1 = pa_{n+1} - a_n \xrightarrow{\tau} 0$. Hence, τ cannot be Hausdorff, and so $\{a_n\}$ is not a T-sequence in A.

Every topological group G admits a "largest" compact Hausdorff group bG and a continuous homomorphism $\rho_G : G \to bG$ such that every continuous homomorphism $\varphi : G \to K$ into a compact Hausdorff group K factors uniquely through ρ_G:

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, "\rho_G"'] & & K \\ & bG \arrow[ur, dashed, "\exists!\tilde{\varphi}"'] & \end{tikzcd} \tag{1}$$

The group bG is called the *Bohr-compactification* of G. The image $\rho_G(G)$ is dense in bG. The kernel of ρ_G is called the *von Neumann radical* of G, and denoted by $\mathbf{n}(G)$. One says that G is *maximally almost-periodic* if $\mathbf{n}(G) = 1$, and *minimally almost-periodic* if $\mathbf{n}(G) = G$ (cf. [4]). For an abelian topological group A, let $\hat{A} = \mathcal{H}(A, \mathbb{T})$ be the Pontryagin dual of A—in other words, the group of *continuous characters* of A (i.e., continuous homomorphisms $\chi : A \to \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$), equipped with the compact-open topology. It follows from the famous Peter–Weyl Theorem [6, Theorem 33] that the Bohr-compactification of A can be quite easily computed: $bA = \widehat{\hat{A}_d}$, where $\hat{A}_d$ stands for the group $\hat{A}$ with the discrete topology. Thus, $\mathbf{n}(A) = \bigcap\limits_{\chi \in \hat{A}} \ker \chi$.

T-sequences turn out to be a very useful tool for constructing "pathological" examples. For example, Zelenyuk and Protasov used T-sequences to show (independently of Ajtai et al. [1]) that every infinite abelian group admits a non-maximally almost-periodic Hausdorff group topology (cf. [7, 2.6.4], [11, Theorem 16]). There are plenty of examples of minimally (or maximally) almost-periodic groups (cf. [4,5]). Nevertheless, it appears that no example is known for a Hausdorff topological group G whose von Neumann radical $\mathbf{n}(G)$ is non-trivial and finite. We call such groups *almost maximally almost-periodic*. This raises the following question:

Problem I. Which abelian groups A admit a T-sequence $\{a_n\}$ such that $A\{a_n\}$ is almost maximally almost-periodic?

If $\mathbf{n}(G)$ is non-trivial and finite, then $\mathbf{n}(\mathbf{n}(G)) = 1$, and thus $\mathbf{n}(\mathbf{n}(G)) \neq \mathbf{n}(G)$, which leads to a second problem:

Problem II. Which abelian groups A admit a T-sequence $\{a_n\}$ such that $\mathbf{n}(\mathbf{n}(A\{a_n\}))$ is strictly contained in $\mathbf{n}(A\{a_n\})$?

Note that since $\mathbf{n}$ is productive, one may wish to focus on *algebraically* (respectively, *topologically*) *directly indecomposable groups*, that is, groups that cannot be expressed as an algebraic (respectively, topological) direct product of two of its proper subgroups.

Our ultimate goal in this paper is to present ample non-isomorphic algebraically directly indecomposable almost maximally almost-periodic Hausdorff abelian groups (i.e., having non-trivial finite von Neumann radical), whose topology is determined by a T-sequence. It will also show that Problems I and II are meaningful.

This aim is carried out according to the following structure: In Section 2, several results that provide sufficient conditions for a sequence in an abelian group to be a T-sequence are presented (Theorem 2.2). In Section 3, a partial answer to Problem I is provided, namely, we prove that the direct sum of any infinite family of finite abelian groups admits an almost maximally almost-periodic group topology determined by a T-sequence (Theorem 3.1). Groups of this form certainly fail to be algebraically directly indecomposable, and they need not be topologically directly indecomposable either. Thus, they fall short of our ultimate goal. In Section 4, special sequences of Prüfer groups are investigated (Theorem 4.2), and we prove that for $p \neq 2$, $\mathbb{Z}(p^\infty)$ admits a neither maximally nor minimally almost-periodic Hausdorff group topology τ (Theorem 4.4). Thus, $\mathbf{n}(\mathbb{Z}(p^\infty), \tau)$ is finite, and $\mathbf{n}(\mathbf{n}(\mathbb{Z}(p^\infty), \tau)) \subsetneq \mathbf{n}(\mathbb{Z}(p^\infty), \tau)$. In particular, Problem II is meaningful.

2. T-sequences in abelian groups

In this section, A is an abelian group and $\underline{a} = \{a_k\} \subseteq A$ is a sequence in A. In what follows, we provide several sufficient conditions for $\{a_k\}$ to be a T-sequence in A. For $l, m \in \mathbb{N}$, one puts

$$A(l,m)_{\underline{a}} = \left\{ m_1 a_{k_1} + \cdots + m_h a_{k_h} \mid m \leqslant k_1 < \cdots < k_h,\ m_i \in \mathbb{Z}\backslash\{0\},\ \sum |m_i| \leqslant l \right\}. \tag{2}$$

The Zelenyuk–Protasov criterion for T-sequences states:

Theorem 2.1. ([7, 2.1.4], [11, Theorem 2].) *A sequence $\{a_k\}$ in an abelian group A is a T-sequence if and only if for every $l \in \mathbb{N}$ and $g \neq 0$, there exists $m \in \mathbb{N}$ such that $g \notin A(l,m)_{\underline{a}}$.*

Put $A[n] = \{a \in A\colon\ na = 0\}$ for every $n \in \mathbb{N}$. One says that A is *almost torsion-free* if $A[n]$ is finite for every $n \in \mathbb{N}$ (cf. [9]).

Theorem 2.2. *Let A be an abelian group, and let $\{a_k\} \subseteq A$ be a sequence such that $t_k := o(a_k)$ is finite for every $k \in \mathbb{N}$. Consider the following statements*:

(i) $$\lim_{k\to\infty} \frac{t_k}{\gcd(t_k, \operatorname{lcm}(t_1, \ldots, t_{k-1}))} = \infty. \tag{3}$$

(ii) *For every $l \in \mathbb{N}$,*

$$\lim_{m\to\infty} \inf\left\{ \max_{1\leqslant i\leqslant l} \frac{t_{k_i}}{\gcd(t_{k_i}, \operatorname{lcm}(t_{k_1}, \ldots, t_{k_{i-1}}, t_{k_{i+1}}, \ldots, t_{k_l}))} \;\middle|\; \underline{k} \in \mathbb{N}^l_{m<} \right\} = \infty, \tag{4}$$

where $\mathbb{N}^l_{m<} = \{\underline{k} = (k_1, \ldots, k_l) \in \mathbb{N}^l \mid m \leqslant k_1 < \cdots < k_l\}$.

(iii) *For every $l \in \mathbb{N}$,*

$$\lim_{m\to\infty} \inf\left\{ \max_{1\leqslant i\leqslant l} o(a_{k_i} + A_{\underline{k}^i}) \mid \underline{k} \in \mathbb{N}^l_{m<} \right\} = \infty, \tag{5}$$

where $A_{\underline{k}^i} = \langle a_{k_1}, \ldots, a_{k_{i-1}}, a_{k_{i+1}}, \ldots, a_{k_l} \rangle$.

(iv) *For every $l, n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that $A[n] \cap A(l,m)_{\underline{a}} = \{0\}$ for every $m \geqslant m_0$.*

(v) *$\{a_k\}$ is a T-sequence.*

One has (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v)*, and if A is almost torsion-free, then* (v) $\Rightarrow$ (iv).

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Clearly, the order of a_{k_i} in $(\langle a_{k_i} \rangle + A_{\underline{k}^i})/A_{\underline{k}^i}$ is equal to its order modulo $\langle a_{k_i} \rangle \cap A_{\underline{k}^i}$, and $|\langle a_{k_i} \rangle \cap A_{\underline{k}^i}|$ divides both t_{k_i} and $\exp(A_{\underline{k}^i})$. The exponent $\exp(A_{\underline{k}^i})$, in turn, divides $d = \operatorname{lcm}(t_{k_1}, \ldots, t_{k_{i-1}}, t_{k_{i+1}}, \ldots, t_{k_l})$, because $A_{\underline{k}^i}$ is generated by elements of orders $t_{k_1}, \ldots, t_{k_{i-1}}, t_{k_{i+1}}, \ldots, t_{k_l}$. Therefore, $|\langle a_{k_i} \rangle \cap A_{\underline{k}^i}|$ divides their greatest common divisor of t_k and d. Hence,

$$\frac{t_{k_i}}{\gcd(t_{k_i}, \operatorname{lcm}(t_{k_1}, \ldots, t_{k_{i-1}}, t_{k_{i+1}}, \ldots, t_{k_l}))} \;\Bigg|\; \frac{|\langle a_{k_i} \rangle|}{|\langle a_{k_i} \rangle \cap A_{\underline{k}^i}|} = o(a_{k_i} + A_{\underline{k}^i}). \tag{6}$$

(iii) $\Rightarrow$ (iv): Given $l, n \in \mathbb{N}$, let $m_0 \in \mathbb{N}$ be such that $nl < \max_{1\leqslant i\leqslant h} o(a_{k_i} + A_{\underline{k}^i})$ for every $1 \leqslant h \leqslant l$ and every $\underline{k} \in \mathbb{N}^h_{m_0<}$. (By (5), such m_0 exists.) Let $g = m_1 a_{k_1} + \cdots + m_h a_{k_h} \in A(l,m)_{\underline{a}}$ be a non-zero element, where $m_0 \leqslant m \leqslant k_1 < \cdots < k_h$, $m_i \in \mathbb{Z}\backslash\{0\}$, and $\sum |m_i| \leqslant l$. It follows from the last two conditions that $h \leqslant l$. So, there exists $1 \leqslant i \leqslant h$ such that $nl < o(a_{k_i} + A_{\underline{k}^i})$, and thus $n < o(m_i a_{k_i} + A_{\underline{k}^i})$. To complete the proof, note that $g \in m_i a_{k_i} + A_{\underline{k}^i}$, and therefore $o(m_i a_{k_i} + A_{\underline{k}^i}) \mid o(g)$. Hence, $n < o(g)$, and so $g \notin A[n]$, as desired.

(iv) $\Rightarrow$ (v): Let $g \in A$ be a non-zero element. If the order of g is infinite, then $g \notin A(l,1)_{\underline{a}}$ for every $l \in \mathbb{N}$, and so suppose that $n := o(g)$ is finite. By (iv), for every $l \in \mathbb{N}$ there exists $m_0(l)$ such that $A[n] \cap A(l, m_0(l))_{\underline{a}} = \{0\}$. In particular, $g \notin A(l, m_0(l))_{\underline{a}}$ for every l.

(v) $\Rightarrow$ (iv): Given $l, n \in \mathbb{N}$, and suppose that $A[n] = \{0, g_1, \ldots, g_j\}$ is finite. For each g_i, pick $m_i(l) \in \mathbb{N}$ such that $g_i \notin A(l, m_i(l))_{\underline{a}}$, and put $m_0(l) = \max m_i(l)$. Clearly, one has $A[n] \cap A(l,m)_{\underline{a}} = \{0\}$ for every $m \geqslant m_0(l)$, as desired. $\square$

Remark 2.3. In Theorem 2.2, (iv) does not imply (iii). Indeed, although (iii) fails for the sequence $\{e_n\}$ from Example 1.1, it is a T-sequence in $\mathbb{Z}(p^\infty)$.

Corollary 2.4. *Let A be an abelian group, and let $\{a_k\} \subseteq A$ be a sequence such that $t_k := o(a_k)$ is finite for every $k \in \mathbb{N}$.*

(a) *If the t_k are pairwise coprime, then $\{a_k\}$ is a T-sequence.*
(b) *If $t_k \mid t_{k+1}$ and $\lim\limits_{k\to\infty} \frac{t_{k+1}}{t_k} = \infty$, then $\{a_k\}$ is a T-sequence.*

Proof. If the t_k are pairwise coprime, then the expression in (3) is equal to t_k and $t_k \longrightarrow \infty$. If $t_k \mid t_{k+1}$, then the expression in (3) is precisely $\frac{t_{k+1}}{t_k}$. In both cases, the statement follows from Theorem 2.2(i). $\square$

3. Direct sums of finite abelian groups

In this section, we provide a partial answer to Problem I:

Theorem 3.1. *Let $A = \bigoplus\limits_{\alpha \in I} F_\alpha$ be the direct sum of an infinite family $\{F_\alpha\}$ of non-trivial finite abelian groups. There exists a T-sequence $\{d_k\}$ in A such that $A\{d_k\}$ is almost maximally-almost periodic.*

Remark 3.2. In the setting of Theorem 3.1, A is obviously not algebraically directly indecomposable. Furthermore, $A\{d_k\}$ need not be topologically directly indecomposable either: Consider the group $B = \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus\limits_{n=1}^{\infty} \mathbb{Z}/3\mathbb{Z}$, and let τ be a Hausdorff group topology on B. The subgroup $B_2 = \bigoplus\limits_{n=1}^{\infty} \mathbb{Z}/3\mathbb{Z}$ is closed in τ, because it is the kernel of the continuous group homomorphism $x \mapsto 3x$. Thus, (B,τ) decomposes into a topological direct product of $B_1 = \mathbb{Z}/2\mathbb{Z}$ and B_2 (where B_1 and B_2 are equipped with the subgroup topology). This also shows that $B_1 \cap \mathbf{n}(B,\tau) = \{0\}$, because $(B,\tau) \to B/B_2 \cong \mathbb{Z}/2\mathbb{Z}$ is continuous. In particular, not every finite subgroup of an abelian group A is of the form $\mathbf{n}(A,\tau)$, where τ is a Hausdorff group topology on A.

In order to prove Theorem 3.1, we need the following result:

Proposition 3.3. *Let $A = \bigoplus\limits_{i=1}^{\infty} C_i$ be a direct sum of cyclic groups of order $n_i = |C_i| > 1$, and suppose that*

(a) *$n_i = n_{i+1}$ for every i, or*
(b) *$n_i < n_{i+1}$ for every i.*

Then, for every $x \in A$, there is a T-sequence $\{d_k\}$ such that $\mathbf{n}(A\{d_k\})$ is finite and contains x.

Proof. The construction below is a modification of [11, Example 5] and [7, 2.6.2]. The sequence d_k is constructed identically in both (a) and (b), and the two are distinguished only in the proof of $\{d_k\}$ being a T-sequence.

For each i, pick a generator g_i in C_i. Each $y \in A$ can be written as $y = \sum \alpha_i g_i \in A$, and the α_i are unique modulo n_i. We set $\Lambda(y) = \{i \in \mathbb{N} \mid \alpha_i \not\equiv 0 \bmod n_i\}$ and $\lambda(y) = |\Lambda(y)|$. Put $i_0 = \max \Lambda(x)$. We define two sequences:

$$a_k: \quad g_{i_0+1}, 2g_{i_0+1}, \ldots, (n_{i_0+1}-1)g_{i_0+1}, g_{i_0+2}, 2g_{i_0+2}, \ldots, (n_{i_0+2}-1)g_{i_0+2}, \ldots, \tag{7}$$

$$b_k: \quad -x+g_{i_0+1}, -x+g_{i_0+2}+g_{i_0+3}, -x+g_{i_0+4}+g_{i_0+5}+g_{i_0+6}, \ldots. \tag{8}$$

Let $\chi : A \to \mathbb{T}$ be a character of A. If χ is zero on all but finitely many of the C_i and $\chi(x) = 0$, then $\chi(a_k) = 0$ and $\chi(b_k) = 0$ for k large enough, and so $\chi(a_k) \longrightarrow 0$ and $\chi(b_k) \longrightarrow 0$. Conversely, suppose that $\chi(a_k) \longrightarrow 0$ and $\chi(b_k) \longrightarrow 0$. Then there is $k_0 \in \mathbb{N}$ such that $\chi(a_k) \subseteq (-\frac{1}{3}, \frac{1}{3})$ for every $k > k_0$. Thus, there is $j_0 \in \mathbb{N}$ such that $\chi(C_j) \subseteq (-\frac{1}{3}, \frac{1}{3})$ for every $j > j_0$. Since the only subgroup contained in $(-\frac{1}{3}, \frac{1}{3})$ is $\{0\}$, this means that χ is zero on all but finitely many of the C_i. Therefore, $\chi(b_k) = -\chi(x)$ for k large enough, and hence $\chi(x) = 0$.

The foregoing argument shows that if $\{d_k\}$ is any combination of the sequences $\{a_k\}$ and $\{b_k\}$ without repetitions (such as $a_1, b_1, a_2, b_2, \ldots$) and if $\{d_k\}$ is a T-sequence, then χ is a continuous character of $A\{d_k\}$ if and only if χ is zero on all but finitely many of the C_i and $\chi(x) = 0$. Thus, $x \in \mathbf{n}(A\{d_k\})$, and the character $\chi_j : A \to \mathbb{T}$ defined by $\chi_j(\sum \alpha_i g_i) = \frac{1}{n_j}\alpha_j$ is continuous on $A\{d_k\}$ for every $j > i_0$. Therefore, $\mathbf{n}(A\{d_k\}) \subseteq C_1 \oplus \cdots \oplus C_{i_0}$, and hence $\mathbf{n}(A\{d_k\})$ is finite, as desired.

We show that d_k is a T-sequence. First, observe that for every $l \in \mathbb{N}$ and every $j > i_0$ there exists $m \in \mathbb{N}$ such that

$$A(l,m)_{\underline{d}} \subseteq \langle x \rangle \oplus \bigoplus_{i=j}^{\infty} C_i. \tag{9}$$

Thus, for every $l \in \mathbb{N}$,

$$\bigcap_{m=1}^{\infty} A(l,m)_{\underline{d}} \subseteq \bigcap_{j>i_0} \left(\langle x \rangle \oplus \bigoplus_{i=j}^{\infty} C_i \right) = \langle x \rangle. \tag{10}$$

Therefore, the condition of the Zelenyuk–Protasov criterion (Theorem 2.1) holds for every $y \notin \langle x \rangle$, and it remains to show it for non-zero elements of $\langle x \rangle$. Let $l \in \mathbb{N}$, and for the time being assume only that $m > i_0$. If $\alpha x \in A(l,m)_{\underline{d}}$, then

$$\alpha x = (m_1 d_{k_1} + \cdots + m_{h_1} d_{k_{h_1}}) + (m_{h_1+1} d_{k_{h_1+1}} + \cdots + m_h d_{k_h}), \tag{11}$$

where $\sum |m_i| \leqslant l$, $m_i \neq 0$, $k_i \geqslant m$, d_{k_i} is a member of $\{a_k\}$ for $1 \leqslant i \leqslant h_1$, and of $\{b_k\}$ for $h_1 + 1 \leqslant i \leqslant h$. (Here, we only assume that the k_i are distinct, but they need not be increasing.) Thus,

$$\alpha x = (m_1 d_{k_1} + \cdots + m_{h_1} d_{k_{h_1}}) + \big(m_{h_1+1}(d_{k_{h_1+1}} + x) + \cdots + m_h(d_{k_h} + x)\big) - \sum_{i=h_1+1}^{h} m_i x. \tag{12}$$

Since $k_i \geqslant m > i_0$, the first and the second expression on the right side belong to $\bigoplus\limits_{j=i_0+1}^{\infty} C_j$, while the left side and the third summand on the right belong to $\langle x \rangle$. Therefore,

$$(m_1 d_{k_1} + \cdots + m_{h_1} d_{k_{h_1}}) + \big(m_{h_1+1}(d_{k_{h_1+1}} + x) + \cdots + m_h(d_{k_h} + x)\big) = 0. \tag{13}$$

The sets $\Lambda(m_{h_1+j}(d_{k_{h_1+j}} + x))$ are disjoint, because the k_i are distinct, and so

$$\lambda\big(m_{h_1+1}(d_{k_{h_1+1}} + x) + \cdots + m_h(d_{k_h} + x)\big) = \sum_{j=1}^{h-h_1} \lambda\big(m_{h_1+j}(d_{k_{h_1+j}} + x)\big). \tag{14}$$

Since $\lambda(a_k) = 1$ for every k, one has $\lambda(m_1 d_{k_1} + \cdots + m_{h_1} d_{k_{h_1}}) \leqslant h_1 \leqslant l$, and hence, by (13) and (14),

$$\lambda\big(m_{h_1+j}(d_{k_{h_1+j}} + x)\big) \leqslant l \tag{15}$$

for every $1 \leqslant j \leqslant h - h_1$.

(a) Pick $m > i_0$ such that $\lambda(d_k + x) \geqslant l + 1$ for every $k \geqslant m$ such that d_k is a member of $\{b_k\}$. Then each $d_{k_{h_1+j}} + x$ is a sum of at least $l + 1$ distinct base elements g_i of order $n = n_i$, and so $\lambda(m_{h_1+j}(d_{k_{h_1+j}} + x)) \leqslant l$

implies $m_{h_1+j}(d_{k_{h_1+j}}+x)=0$. Therefore, $n \mid m_{h_1+j}$, and in particular, $n \mid \sum_{i=h_1+1}^{h} m_i$. Hence, by (12), $\alpha x=0$, as desired.

(b) Pick m as in (a), but with the additional condition that $n_i > l$ for every $i \in \Lambda(d_k+x)$ and for every $k \geqslant m$ such that d_k is a member of $\{b_k\}$. This is possible because the $\Lambda(b_k+x)$ are disjoint and $\{n_i\}$ is increasing. Each $d_{k_{h_1+j}}+x$ is a sum of at least $l+1$ distinct base elements, but of different orders. Thus, $m_{h_1+j}g_i \neq 0$ for every $i \in \Lambda(d_{k_{h_1+j}}+x)$ and $1 \leqslant j \leqslant h-h_1$, because $|m_{h_1+j}| \leqslant l < n_i$ (by our assumption), and $m_{h_1+j} \neq 0$. Therefore,

$$\lambda\big(m_{h_1+j}(d_{k_{h_1+j}}+x)\big)=\sum_{i\in\Lambda(d_{k_{h_1+j}}+x)}\lambda(m_{h_1+j}g_i)\geqslant l+1, \tag{16}$$

contrary to (15). Hence, $h=h_1$, and $\alpha x=0$, as desired. □

Proof of Theorem 3.1. Since each F_α decomposes into a direct sum of cyclic subgroups, we may assume that the F_α are cyclic from the outset. The set $\{|F_\alpha| \mid \alpha \in I\}$ is either bounded or contains an increasing sequence. Thus, there is a countable subset $J \subseteq I$ such that $A_1=\bigoplus_{\alpha\in J} F_\alpha$ satisfies one of the conditions of Proposition 3.3. Pick $\gamma \in J$, and let g be a generator of F_γ. Then, by Proposition 3.3, there is a T-sequence $\{d_k\}$ in A_1 such that $\mathbf{n}(A_1\{d_k\})$ is finite and contains g. For $A_2=\bigoplus_{\alpha\in I\setminus J} F_\alpha$, one has $A\{d_k\}=A_1\{d_k\}\times A_2$, where A_2 is equipped with the discrete topology. Therefore, $\mathbf{n}(A\{d_k\})=\mathbf{n}(A_1\{d_k\})\times\mathbf{n}(A_2)=\mathbf{n}(A_1\{d_k\})$ is non-trivial and finite, as desired. □

4. Special sequences in the Prüfer groups

In this section, we present method for constructing an algebraically directly indecomposable Hausdorff abelian topological groups A such that $\mathbf{n}(A)$ is non-trivial and finite. An implicit yet rather thick hint for the construction of a group with these properties appears in [3], in the proof of Corollary 4.9 and the paragraph following it. It was Dikranjan who pointed out to the author that considering a suitable T-sequence in a Prüfer group (and the maximal Hausdorff group topology thus obtained) would lead to the desired example. Prüfer groups are distinguished by the property of having only finite proper subgroups, which implies that these subgroups are closed in any Hausdorff group topology. This property makes Prüfer groups particularly suitable candidates for the aforesaid construction, because if $\mathbf{n}(\mathbb{Z}(p^\infty)\{d_n\})$ is a non-trivial proper subgroup, then it must be finite. Therefore, in this section, we study certain sequences in the Prüfer groups $\mathbb{Z}(p^\infty)$, and construct a T-sequence $\{d_n\}$ such that $\mathbb{Z}(p^\infty)\{d_n\}$ is neither maximally nor minimally almost-periodic. A second important property that Prüfer groups, being p-groups, have is that for every $a, b \in \mathbb{Z}(p^\infty)$,

$$o(a)\neq o(b) \implies o(a+b)=\max\big\{o(a),o(b)\big\}. \tag{17}$$

One says that a group G is *potentially compact* if for every ultrafilter $\mathcal{U}$ on G there is $x \in G$ such that $\mathcal{U}-x$ is a T-filter, that is, $\mathcal{U} \xrightarrow{\tau} x$ in some group topology τ (cf. [8,10]). A third noteworthy property is that Prüfer groups are *not* potentially compact, because they are divisible torsion groups (cf. [10, Theorem 6]).

Recall that if A is a subgroup of an abelian Hausdorff topological group S, and $\{a_k\} \subseteq A$ is so that $a_k \longrightarrow b$ in S, where $\langle b\rangle \cap A=\{0\}$, then $\{a_k\}$ is a T-sequence in A (cf. [7, 2.1.5], [11, Theorem 3]). The setting of this result is so that the sequence a_n converges to an *external* element (namely, $b \notin A$) in some group topology. In contrast, in this section, we investigate sequences in $\mathbb{Z}(p^\infty)$ that converge to a non-zero (internal) *element of* $\mathbb{Z}(p^\infty)$ in the "usual" topology, that is, the one inherited from $\mathbb{Q}/\mathbb{Z}$.

We start off with an immediate consequence of Corollary 2.4.

Lemma 4.1. *Let $\{a_k\}$ be a sequence in $\mathbb{Z}(p^\infty)$, and suppose that $o(a_k)=p^{n_k}$. If $n_{k+1}-n_k \longrightarrow \infty$, then $\{a_k\}$ is a T-sequence.*

Proof. One has $\frac{o(a_{k+1})}{o(a_k)}=\frac{p^{n_{k+1}}}{p^{n_k}}=p^{n_{k+1}-n_k}$. Therefore, the statement follows from Corollary 2.4(b). □

Example 1.1 shows that the converse of Lemma 4.1 does not hold in general. Nevertheless, for some special sequences, the condition of $n_{k+1} - n_k \longrightarrow \infty$ turns out to be necessary for being a T-sequence, as Theorem 4.2 below reveals.

We proceed by introducing some terminology. A topological group A is *precompact* if for every neighborhood U of zero there is a finite subset $F \subseteq A$ such that $A = F + U$. Following [2], we say that a sequence $\{a_n\}$ on a group G is a *TB-sequence* if there is a precompact Hausdorff group topology τ on G such that $a_n \xrightarrow{\tau} 0$.

It is easy to see that A is precompact if and only if it carries the initial topology induced by its group of continuous characters. Thus, if $\{a_k\}$ is a sequence in an abelian group A, then by the universal property of $A\{a_k\}$, a character $\chi : A \to \mathbb{T}$ is continuous on $A\{a_k\}$ if and only if $\chi(a_k) \longrightarrow 0$. Therefore, for $H = \{\chi \in \hom_{\mathbb{Z}}(A, \mathbb{T}) \mid \chi(a_k) \longrightarrow 0\}$, the closure of $\{0\}$ in the initial topology induced by H is $\mathbf{n}(A\{a_k\}) = \bigcap_{\chi \in H} \ker \chi$. Hence, $\{a_k\}$ is a TB-sequence if and only if H separates the points of A. (Observe that $H = \widehat{A\{a_k\}_d}$.)

Theorem 4.2. *Let $x \in \mathbb{Z}(p^\infty)$ be a non-zero element, $\{n_k\} \subseteq \mathbb{N}$ an increasing sequence of positive integers, and set*

$$a_k = -x + e_{n_k} = -x + \frac{1}{p^{n_k}} \in \mathbb{Z}\big(p^\infty\big). \tag{18}$$

(a) *$\{a_k\}$ is a T-sequence if and only if $n_{k+1} - n_k \longrightarrow \infty$.*
(b) *$\{a_k\}$ is a T-sequence if and only if it is a TB-sequence. In this case, $\mathbb{Z}(p^\infty)\{a_k\}$ is maximally almost-periodic, and it has $\mathfrak{c}$ many faithful characters (in particular, $|\widehat{\mathbb{Z}(p^\infty)\{a_k\}}| = \mathfrak{c}$).*

Since every TB-sequence is a T-sequence, but the converse is not true in general, (b) of Theorem 4.2 is a non-trivial result. Its proof, however, requires a technical lemma. Note that the group of all characters of $\mathbb{Z}(p^\infty)$ is isomorphic to the group $\mathbb{Z}_p$ of the p-adic integers. In other words, $\mathbb{Z}_p = \hom_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{T})$.

Lemma 4.3. *Suppose that $n_{k+1} - n_k \longrightarrow \infty$. For $\chi = \sum_{n=0}^{\infty} \alpha_n p^n \in \mathbb{Z}_p$ $(0 \leqslant \alpha_n \leqslant p-1)$ and $\gamma \in (0,1)$, $\chi(e_{n_k}) \longrightarrow \gamma$ if and only if*

$$r_k := \frac{\sum_{l=n_k}^{n_{k+1}-1} \alpha_l p^{l-n_k}}{p^{n_{k+1}-n_k}} \longrightarrow \gamma. \tag{19}$$

Proof. One has

$$\chi(e_{n_{k+1}}) = \frac{\sum_{l=0}^{n_{k+1}-1} \alpha_l p^l}{p^{n_{k+1}}} = \frac{\sum_{l=0}^{n_k-1} \alpha_l p^l}{p^{n_{k+1}}} + \frac{\sum_{l=n_k}^{n_{k+1}-1} \alpha_l p^l}{p^{n_{k+1}}} = \frac{\chi(e_{n_k})}{p^{n_{k+1}-n_k}} + r_k, \tag{20}$$

and thus $\lim_{k\to\infty} \chi(e_{n_k}) = \lim_{k\to\infty} r_k$ in $\mathbb{T}$ (by the equality of limits we mean that one exists if and only if the other does, and in that case they are equal), because $n_{k+1} - n_k \longrightarrow \infty$. Since $\gamma \neq 0$, small enough neighborhoods of γ in $\mathbb{T}$ and $(0,1)$ are the same, and therefore $\lim_{k\to\infty} \chi(e_{n_k}) = \lim_{k\to\infty} r_k$ in $(0,1)$. $\square$

Proof of Theorem 4.2. (a) Since $\{n_k\}$ is increasing, one has $n_k \longrightarrow \infty$. Thus, $p^{n_k} > o(x)$ for k large enough, and so $o(a_k) = p^{n_k}$ except for maybe a finite number of k (by (17)). Hence, a_k is a T-sequence by Lemma 4.1.

Conversely, let $p^{n_0} = o(x)$, and assume that $n_{k+1} - n_k \not\longrightarrow \infty$. Then $o(p^{n_0-1}x) = p$,

$$p^{n_0-1} a_k = -p^{n_0-1} x + e_{n_k - n_0 + 1}, \tag{21}$$

and the differences $(n_{k+1} - n_0 + 1) - (n_k - n_0 + 1) = n_{k+1} - n_k \not\longrightarrow \infty$. Thus, it suffices to show that $p^{n_0-1} a_k$ is not a T-sequence. Therefore, without loss of generality, we may assume that $o(x) = p$ from the outset. Since $n_{k+1} - n_k \not\longrightarrow \infty$, there exists a number d and a subsequence k_r of k such that $n_{k_r+1} - n_{k_r} \leqslant d$ for every r. If $a_k \longrightarrow 0$ in a group topology τ on $\mathbb{Z}(p^\infty)$, then in particular, $p a_{n_{k_r+1}} = e_{n_{k_r+1}-1} \longrightarrow 0$, and so for every $1 \leqslant i \leqslant d$, $e_{n_{k_r+1}-i} \longrightarrow 0$. Thus, the sequence b_n defined as

$$e_{n_{k_1+1}-d}, e_{n_{k_1+1}-d+1}, \ldots, e_{n_{k_1+1}-1}, e_{n_{k_2+1}-d}, e_{n_{k_2+1}-d+1}, \ldots, e_{n_{k_2+1}-1}, \ldots$$

also converges to 0 in τ. One has $n_{k_r+1} - d \leqslant n_{k_r} \leqslant n_{k_r+1} - 1$, and therefore $e_{n_{k_r}}$ is a subsequence of b_n, and hence $e_{n_{k_r}} \longrightarrow 0$ in τ. Since $a_{k_r} = -x + e_{n_{k_r}}$, this shows that τ is not Hausdorff.

(b) If $\{a_k\}$ is a TB-sequence, then clearly it is a T-sequence. Conversely, suppose that $\{a_k\}$ is a T-sequence. In order to show that $\{a_n\}$ is a TB-sequence, we find a faithful continuous character of $\mathbb{Z}(p^\infty)\{a_k\}$, in other words, $\chi \in \mathbb{Z}_p$ such that $\chi(a_k) \longrightarrow 0$ and $\ker \chi = \{0\}$. Let $p^{n_0} = o(x)$. For $\chi = \sum_{n=0}^{\infty} \alpha_n p^n \in \mathbb{Z}_p$, a character of $\mathbb{Z}(p^\infty)$, if

$$\alpha_0 = 1, \qquad \alpha_1 = \cdots = \alpha_{n_0-1} = 0, \tag{22}$$

then χ acts on the subgroup $\langle x \rangle$ as the identity, where $\mathbb{Z}(p^\infty)$ is viewed as a subgroup of $\mathbb{T}$. Thus, $\chi(e_1) = e_1 \neq 0$ and $\chi(x) = x \neq 0$, and in particular, χ is faithful. By Lemma 4.3, $\chi(a_k) \longrightarrow 0$ (i.e., χ is continuous on $\mathbb{Z}(p^\infty)\{a_k\}$) if and only if

$$r_k = \frac{\sum_{l=n_k}^{n_{k+1}-1} \alpha_l p^{l-n_k}}{p^{n_{k+1}-n_k}} \longrightarrow x. \tag{23}$$

Conditions (22) and (23) are satisfied (simultaneously) by continuum many elements in $\mathbb{Z}_p$, which completes the proof. □

We proceed by presenting the construction of a non-minimally almost-periodic non-maximally almost-periodic Hausdorff group topology on the group $\mathbb{Z}(p^\infty)$ for $p \neq 2$. Our technique makes substantial use of the assumption that $p \neq 2$; nevertheless, we conjecture that a similar construction is available for $p = 2$.

Theorem 4.4. *Let p be a prime number such that $p \neq 2$, $x \in \mathbb{Z}(p^\infty)$ be a non-zero element with $p^{n_0} = o(x)$, and put*

$$b_n = -x + e_{n^3-n^2} + \cdots + e_{n^3-2n} + e_{n^3-n} + e_{n^3} = -x + \frac{1}{p^{n^3-n^2}} + \cdots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}}. \tag{24}$$

Consider the sequence d_n defined as $b_1, e_1, b_2, e_2, b_3, e_3, \ldots$. Then:

(a) *$\{d_n\}$ is a T-sequence in $\mathbb{Z}(p^\infty)$;*
(b) *the underlying group of $\widehat{\mathbb{Z}(p^\infty)\{d_n\}}$ is $p^{n_0}\mathbb{Z} \subseteq \mathbb{Z}_p = \hom_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{T})$;*
(c) *$\mathbf{n}(\mathbb{Z}(p^\infty)\{d_n\}) = \langle x \rangle$.*

In particular, $\mathbb{Z}(p^\infty)\{d_n\}$ is neither maximally almost-periodic nor minimally almost-periodic, and $\mathbf{n}(\mathbb{Z}(p^\infty)\{d_n\})$ is finite.

Corollary 4.5. *Let p be a prime number such that $p \neq 2$, and put*

$$b_n = -e_1 + e_{n^3-n^2} + \cdots + e_{n^3-2n} + e_{n^3-n} + e_{n^3} = -\frac{1}{p} + \frac{1}{p^{n^3-n^2}} + \cdots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}}. \tag{25}$$

Consider the sequence d_n defined as $b_1, e_1, b_2, e_2, b_3, e_3, \ldots$. Then:

(a) *$\{d_n\}$ is a T-sequence in $\mathbb{Z}(p^\infty)$;*
(b) *the underlying group of $\widehat{\mathbb{Z}(p^\infty)\{d_n\}}$ is $p\mathbb{Z} \subseteq \mathbb{Z}_p = \hom_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{T})$;*
(c) *$\mathbf{n}(\mathbb{Z}(p^\infty)\{d_n\}) = \langle \frac{1}{p} \rangle$.*

In particular, $\mathbb{Z}(p^\infty)\{d_n\}$ is neither maximally almost-periodic nor minimally almost-periodic, and $\mathbf{n}(\mathbb{Z}(p^\infty)\{d_n\})$ is finite.

In order to prove Theorem 4.4, several auxiliary results of a technical nature are required. Until the end of this section, we assume that $p \neq 2$. Each element $y \in \mathbb{Z}(p^\infty)$ admits many representations of the form $y = \sum \sigma_n e_n$, where $\sigma_n \in \mathbb{Z}$ (only finitely many of the σ_n are non-zero), and so we say that it is the *canonical form* of y if $|\sigma_n| \leqslant \frac{p-1}{2}$ for every $n \in \mathbb{N}$; in this case, we put $\Lambda(y) = \{n \in \mathbb{N} \mid \sigma_n \neq 0\}$ and $\lambda(y) = |\Lambda(y)|$.

Lemma 4.6. *Let $y = \sum \sigma_n e_n \in \mathbb{Z}(p^\infty)$. Then*:

(a) *y admits a canonical form $y = \sum \sigma'_n e_n$, and $\sum |\sigma'_n| \leqslant \sum |\sigma_n|$;*
(b) *the canonical form is unique, and so Λ is well-defined.*

Furthermore,

(c) *$\lambda(z) \leqslant l$ for every $z \in \mathbb{Z}(p^\infty)(l,1)_{\underline{e}}$ and $l \in \mathbb{N}$.*

Proof. (a) Let N be the largest index such that $\sigma_N \neq 0$. We proceed by induction on N. If $N = 1$, then $y = \sigma_1 c_1$. Thus, if $\sigma_1 = \sigma'_1 + mp$ is a division with residue in $\mathbb{Z}$, and σ'_1 is chosen to have the smallest possible absolute value, then $|\sigma'_1| \leqslant \frac{p-1}{2}$, and

$$y = (\sigma'_1 + pm)e_1 = \sigma'_1 e_1 + mpe_1 = \sigma'_1 e_1. \tag{26}$$

In particular, $|\sigma'_1| \leqslant |\sigma_1|$. Suppose now that the statement holds for all elements with representation with maximal non-zero index less than N. If $\sigma_N = \sigma'_N + kp$ is a division with residue in $\mathbb{Z}$, and σ'_N is chosen to have the smallest possible absolute value, then $|\sigma'_N| \leqslant \frac{p-1}{2}$, and

$$y - \sum_{n=1}^{N-2} \sigma_n e_n - (\sigma_{N-1} + k)e_{N-1} = -ke_{N-1} + \sigma_N e_N = -ke_{N-1} + (\sigma'_N + kp)e_N = \sigma'_N e_N. \tag{27}$$

The element $z = \sum_{n=1}^{N-2} \sigma_n e_n + (\sigma_{N-1} + k)e_{N-1}$ satisfies the inductive hypothesis, so $z = \sum_{n=1}^{N-1} \sigma'_n e_n$, where $|\sigma'_n| \leqslant \frac{p-1}{2}$ and $\sum_{n=1}^{N-1} |\sigma'_n| \leqslant \sum_{n=1}^{N-1} |\sigma_n| + |k|$. Therefore, $y = \sum \sigma'_n c_n$, $|\sigma'_n| \leqslant \frac{p-1}{2}$, and $\sum |\sigma'_n| \leqslant \sum |\sigma_n|$, because $|\sigma'_N| + |k| \leqslant |\sigma_N|$.

(b) Suppose that $\sum \sigma_n e_n = \sum \upsilon_n e_n$ are two distinct canonical representations of the same element. Then $\sum(\sigma_n - \upsilon_n)e_n = 0$, and $|\sigma_n - \upsilon_n| \leqslant p - 1$. Let N be the largest index such that $\sigma_N \neq \upsilon_N$. (Since all coefficients are zero, except for a finite number of indices, such N exists.) This means that $0 < |\sigma_N - \upsilon_N| \leqslant p - 1$, and $o((\sigma_N - \upsilon_N)e_N) = p^N$. Therefore, by (17), one has $o(\sum(\sigma_n - \upsilon_n)e_n) = p^N$, because $o(\sum_{n<N} (\sigma_n - \upsilon_n)e_n) \leqslant p^{N-1}$. This is a contradiction, and therefore $\sigma_n = \upsilon_n$ for every $n \in \mathbb{N}$.

(c) Let $z = \mu_1 e_{n_1} + \cdots + \mu_h e_{n_h}$, where $\sum |\mu_i| \leqslant l$ and $n_1 < n_2 < \cdots < n_h$. By (a), z admits a canonical form $z = \sum \mu'_n e_n$, and $\sum |\mu'_n| \leqslant \sum |\mu_i| \leqslant l$. Therefore, $\mu'_n \neq 0$ only for at most l many indices. □

Lemma 4.7. *Let $m \in \mathbb{Z}\backslash\{0\}$, and put $l = \lceil \log_p |m| \rceil$. If $n > l$, then $\Lambda(me_n) \subseteq \{n-l, \ldots, n-1, n\}$ and $1 \leqslant \lambda(me_n)$.*

Proof. It follows from $n > l$ that $p^n > |m|$, and so $me_n \neq 0$. Thus, $1 \leqslant \lambda(me_n)$. To show the first statement, expand $m = \mu_0 + \mu_1 p + \cdots + \mu_l p^l$, where $\mu_i \in \mathbb{Z}$ and $|\mu_i| \leqslant \frac{p-1}{2}$. Then

$$me_n = \mu_0 e_n + \mu_1 e_{n-1} + \cdots + \mu_l e_{n-l} \tag{28}$$

is in canonical form, and therefore $\Lambda(me_n) \subseteq \{n-l, \ldots, n-1, n\}$, as desired. □

Lemma 4.8. *Let $y, z \in \mathbb{Z}(p^\infty)$ such that $\lambda(y) > \lambda(z)$, and suppose that $\Lambda(y) = \{k_1, \ldots, k_g\}$ where $k_1 < \cdots < k_g$ and $g = \lambda(y)$. Then $o(y+z) \geqslant p^{k_{g-\lambda(z)}}$.*

Proof. Let $y = \sum \nu_n e_n$ and $z = \sum \mu_n e_n$ in canonical form. Then $y + z = \sum(\nu_n + \mu_n)e_n$, and $|\nu_n + \mu_n| \leqslant p - 1$. Clearly, $o(y+z) = p^N$ for N the largest index such that $\nu_N + \mu_N \neq 0$. By the definition of N, $\mu_n = -\nu_n$ for every $n > N$. In particular, $\mu_{k_i} \neq 0$ for every i such that $k_i > N$. Thus, there are at most $\lambda(z)$ many i such that $k_i > N$, and therefore $N \geqslant k_{g-\lambda(z)}$. □

Remark 4.9. If $y_1, y_2 \in \mathbb{Z}(p^\infty)$ and $\Lambda(y_1) \cap \Lambda(y_2) = \emptyset$, then $\Lambda(y_1 + y_2) = \Lambda(y_1) \cup \Lambda(y_2)$ and $\lambda(y_1 + y_2) = \lambda(y_1) + \lambda(y_2)$.

Proposition 4.10. *Let $y = \nu_1 e_{n_1} + \cdots + \nu_f e_{n_f}$, where $n_1 < \cdots < n_f$ and $\nu_i \neq 0$. Put $l_i = \lceil \log_p |\nu_i| \rceil$, and suppose that $n_i < n_{i+1} - l_{i+1}$ for each $1 \leqslant i \leqslant f$. Then*:

(a) $f \leqslant \lambda(y)$;
(b) *if $z \in \mathbb{Z}(p^\infty)$ is such that $\lambda(z) < \lambda(y)$, then $o(y+z) \geqslant p^{n_{f-\lambda(z)} - l_{f-\lambda(z)}}$.*

Proof. (a) By Lemma 4.7, $\Lambda(\nu_i e_{n_i}) \subseteq \{n_i - l_i, \ldots, n_i\}$, and since $n_{i-1} < n_i - l_i$, the sets $\Lambda(\nu_i e_{n_i})$ are pairwise disjoint. Therefore, by Remark 4.9, $\lambda(y) = \lambda(\nu_1 e_{n_1}) + \cdots + \lambda(\nu_f e_{n_f}) \geqslant f$, and

$$\Lambda(y) \subseteq \bigcup_{i=1}^{f} \{n_i - l_i, \ldots, n_i\} = \{n_1 - l_1, \ldots, n_1, \ldots, n_i - l_i, \ldots, n_i, \ldots, n_f - l_f, \ldots, n_f\}. \tag{29}$$

(b) By Lemma 4.8, $o(y+z) \geqslant p^{k_{\lambda(y)-\lambda(z)}}$, where $\Lambda(y) = \{k_1, \ldots, k_g\}$ (increasingly ordered). Since $\Lambda(\nu_i e_{n_i})$ is non-empty for each i, it follows from (29) that $k_{\lambda(y)-\lambda(z)} \geqslant n_{f-\lambda(z)} - l_{f-\lambda(z)}$. □

Corollary 4.11. *Let $l \in \mathbb{N}$, $z \in \mathbb{Z}(p^\infty)(l,1)_{\underline{e}}$, and $y = e_{n_1} + \cdots + e_{n_f}$ such that $n_1 < \cdots < n_f$, $l < f$, and $n_i < n_{i+1} - l$. Then $o(\mu y + z) \geqslant p^{n_{f-l} - l} \geqslant p^{n_1 - l}$ for every $\mu \in \mathbb{Z}$ such that $0 < |\mu| \leqslant l$.*

Proof. Since $|\mu| \leqslant l$, $\mu y = \mu e_{n_1} + \cdots + \mu e_{n_f}$ satisfies the conditions of Proposition 4.10 (because $\log_p |\mu| \leqslant l$), and thus, $l < f \leqslant \lambda(\nu y)$. On the other hand, by Lemma 4.6(c), $\lambda(z) \leqslant l$, and therefore $o(\nu y + z) \geqslant p^{n_{f-\lambda(z)} - l} \geqslant p^{n_{f-l} - l}$ pursuant to Proposition 4.10(b). □

Proof of Theorem 4.4. To shorten notations, put $A = \mathbb{Z}(p^\infty)$.

(a) In order to prove that $\{d_n\}$ is a T-sequence, we show that (iv) of Theorem 2.2 holds. For n large enough, $o(b_n) = p^{n^3}$, and so by Lemma 4.1, $\{b_n\}$ is a T-sequence; $\{e_n\}$ is evidently a T-sequence (cf. Example 1.1). Thus, by Theorem 2.2, there exists m_0 such that

$$A[n] \cap A(l,m)_{\underline{b}} = A[n] \cap A(l,m)_{\underline{e}} = \{0\} \tag{30}$$

for every $m \geqslant m_0$ (because A is almost torsion-free). Without loss of generality, we may assume that $m_0 > l + n + n_0$. Observe that

$$A(l,2m)_{\underline{d}} \subseteq A(l,m)_{\underline{b}} \cup A(l,m)_{\underline{e}} \cup \big(A(l,m)_{\underline{b}} \backslash \{0\} + A(l,m)_{\underline{e}} \backslash \{0\}\big), \tag{31}$$

and therefore it suffices to show that $A(l,m)_{\underline{b}} \backslash \{0\} + A(l,m)_{\underline{e}} \backslash \{0\}$ contains no element of $A[n]$ for every $m \geqslant m_0$. Let $z \in A(l,m)_{\underline{e}} \backslash \{0\}$ and $w = m_1 b_{n_1} + \cdots + m_h b_{n_h} \in A(l,m)_{\underline{b}} \backslash \{0\}$ where $m \leqslant n_1 < \cdots < n_h$ and $0 < \sum |m_i| \leqslant l$. Put $y = e_{n_h^3 - n_h^2} + \cdots + e_{n_h^3 - n_h} + e_{n_h^3}$. The number of summands in y is $n_h + 1$, and the differences between the indices of the terms is n_h. By the construction, $n_h \geqslant m \geqslant m_0 > l$. Thus, the conditions of Corollary 4.11 are satisfied, and since $|m_h| \leqslant l$, we get $o(m_h y + z) \geqslant p^{n_h^3 - n_h^2 - l} > p^{(n_h - 1)^3}$. Therefore, $o(-m_h x) \neq o(m_h y + z)$ (because $o(-m_h x) \leqslant p^{n_0} \leqslant p^{m_0 - 1} \leqslant p^{(n_h - 1)^3}$), and

$$o(m_h b_{n_h} + z) = o(-m_h x + m_h y + z) \overset{(17)}{=} \max\big\{o(-m_h x), o(m_h y + z)\big\} > p^{(n_h - 1)^3}. \tag{32}$$

One has

$$o(w - m_h b_{n_h}) \leqslant o(b_{n_{h-1}}) = p^{n_{h-1}^3} \leqslant p^{(n_h - 1)^3} \overset{(32)}{<} o(m_h b_{n_h} + z), \tag{33}$$

and hence

$$o(w + z) = o\big((w - m_h b_{n_h}) + (z + m_h b_{n_h})\big) \tag{34}$$

$$\overset{(17)}{=} \max\big\{o(w - m_h b_{n_h}), o(z + m_h b_{n_h})\big\} > p^{(n_h - 1)^3} > p^{(m_0 - 1)^3} > n. \tag{35}$$

(b) As noted earlier, a character $\chi \in \hom_{\mathbb{Z}}(\mathbb{Z}(p^\infty), \mathbb{T})$ is continuous on $\mathbb{Z}(p^\infty)\{d_n\}$ if and only if $\chi(d_n) \longrightarrow 0$ (by the universal property)—in other words, $\chi(b_n) \longrightarrow 0$ and $\chi(e_n) \longrightarrow 0$. The latter is equivalent to χ having the form of $m\chi_1$, where χ_1 is the natural embedding of $\mathbb{Z}(p^\infty)$ into $\mathbb{T}$ and $m \in \mathbb{Z}$ (cf. [11, Example 6], [3, 3.3]). Since

$$0 \leqslant \frac{1}{p^{n^3-n^2}} + \cdots + \frac{1}{p^{n^3-2n}} + \frac{1}{p^{n^3-n}} + \frac{1}{p^{n^3}} \leqslant \frac{n+1}{p^{n^3-n^2}} \longrightarrow 0, \tag{36}$$

one has $\chi_1(b_n) \longrightarrow -x$, and consequently $\chi(b_n) = m\chi_1(b_n) \longrightarrow 0$ if and only if $-mx = 0$ (i.e., $x \in \ker\chi$). This means that $\chi = m\chi_1$ if and only if $o(x) = p^{n_0} \mid m$, as desired.

(c) We have already seen that $x \in \ker\chi$ for every continuous character of $\mathbb{Z}(p^\infty)\{d_k\}$. On the other hand, $\mathbf{n}(\mathbb{Z}(p^\infty)\{d_k\}) \subseteq \ker p^{n_0}\chi_1 = \langle x \rangle$. □

Remark 4.12. A careful examination of the construction in Theorem 4.4 reveals that the only following properties of the sequence $\{b_n\}$ are essential:

(1) Growing number of summands in b_n—in other words, $\lambda(b_n) \longrightarrow \infty$;
(2) Growing gaps between the orders of summands in b_n (in its canonical form);
(3) $b_n \longrightarrow -x$ in the topology of inherited from $\mathbb{T}$, where $x \in \mathbb{Z}(p^\infty)$ and $x \neq 0$.

Condition (1) and (2) are needed in order to apply Corollary 4.11, while (3) guarantees that $m\chi_1$ is continuous if and only if $o(x) \mid m$ (where χ_1 is the natural embedding of $\mathbb{Z}(p^\infty)$ into $\mathbb{T}$).

We conclude with a problem motivated by Theorem 4.2 and Remark 4.12:

Problem III. Is there a T-sequence $\{a_k\}$ in $\mathbb{Z}(p^\infty)$ with bounded $\lambda(a_k)$ that is not a TB-sequence?

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