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Higher-Order Finite-Difference Methods for Nonlinear Second-Order Two-Point Boundary-Value Problems

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Abstract—Finite-difference methods of orders six and eight are presented for second-order, nonlinear, boundary-value problems. Both methods are economical in the sense that they use few function evaluations at interior grid points. The implementation of the methods is straightforward. The convergence of the methods is discussed. Numerical examples are considered to demonstrate computationally their order of convergence.

Keywords—Second-order boundary-value problems, Finite-difference methods, Numerical methods, Convergence.

1. INTRODUCTION

Consider the second-order nonlinear differential equation

$$y'' = f(x, y), \qquad a < x < b \tag{1a}$$

subject to the boundary conditions

$$y(a) = A, \qquad y(b) = B. \tag{1b}$$

For existence and uniqueness of a solution of (1), it is assumed that, for $x \in [a, b]$, $-\infty < y < \infty$, f is continuous, $\frac{\partial f}{\partial y}$ exists, and $\frac{\partial f}{\partial y} \geq 0$; see [1]. Special differential equations of the second order, and in particular systems of such equations, occur frequently, for instance in mechanical problems without dissipation (see [1, p. 289; 2, p. 252]). These special boundary-value problems also occur

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in other engineering contexts, for example in Troesh's problem relating to the confinement of a plasma column by radiation pressure, see references [3–7]. A finite-difference scheme for the second-order nonlinear boundary-value problem (1) is called an economical method of order $2p$ if

(i) the scheme is globally $O(h^{2p})$ convergent, and
(ii) each discretization of the differential equation at an interior grid point is based on $2p-1$ evaluations of f.

Economical methods of order two (classical second-order method) and four (Numerov's method) are well known (see [1]). Methods of orders six and eight are described for linear boundary-value problems by Jain [8], Chawla [9,10], and Usmani [11,12]. In the present paper, simple as well as economical methods of order six and eight are described for solving (1). The method of order six is a five-diagonal iterative method and uses three function evaluations at each mesh point not adjacent to the boundaries, as compared to Jain's method [8] which uses five function evaluations. The eighth-order method is a five-diagonal iterative method with five function evaluations at each mesh point not adjacent to the boundaries, as compared to Chawla's method [10] which requires seven function evaluations and contains interior grid points. At points adjacent to the boundaries extra function evaluations are called for to ensure that the principal part of the local truncation error at all interior mesh points is the same, thus ensuring that accuracy is not lost near the boundaries. In Section 2, the methods are described, in Section 3, the convergence of the sixth-order method is established and in Section 4 numerical examples are reported.

2. THE FINITE-DIFFERENCE METHODS

Let N be a positive integer, $h=(b-a)/(N+1)$, and let $x_k=a+kh$, $k=0,1,\dots,N+1$. At the grid points x_k, set $y_k=y(x_k)$, $f_k=f(x_k,y_k)$. Following the theory of multistep methods for special second-order differential equations (see, [1]) of the kind (1), the following scheme may be derived

$$Y=-\delta^2 y_m+\alpha\delta^4 y_m+(1-2\beta-2\gamma)h^2 f_m+\beta h^2(f_{m+1}+f_{m-1})+\gamma h^2(f_{m+2}+f_{m-2}). \quad (2)$$

An interesting feature of scheme (2) is that for different values of α, β, γ methods of different order are obtained. For example,

(i) for $\alpha=\beta=\gamma=0$ the classical second-order method is obtained;
(ii) for $\alpha=\gamma=0$, $\beta=1/9$, the second-order method of Twizell and Tirmizi [13] is obtained;
(iii) for $\alpha=\gamma=0$, $\beta=1/12$, Numerov's fourth-order method is derived;
(iv) for $\alpha=-1/20$, $\beta=2/15$, $\gamma=0$, a novel sixth-order method is obtained;
(v) for $\alpha=-31/252$, $\beta=172/945$, $\gamma=23/3780$, a new eighth-order method is obtained.

2.1. A Sixth-Order Method

As suggested above, scheme (2) with $\alpha=-1/20$, $\beta=2/15$, and $\gamma=0$ renders a sixth-order method, namely

$$\delta^2 y_m+\frac{1}{20}\delta^4 y_m=\frac{1}{15}h^2(2f_{m+1}+11f_m+2f_{m-1}), \qquad m=2,3,\dots,N-1, \quad (3)$$

$$\delta^2 y_1+\frac{1}{20}(-2y_0+5y_1-4y_2+y_3)=\frac{1}{15d_1}h^2\sum_{i=0}^{6}a_i f_i, \quad (4)$$

$$\delta^2 y_N+\frac{1}{20}(-2y_{N+1}+5y_N-4y_{N-1}+y_{N-2})=\frac{1}{15d_1}h^2\sum_{i=0}^{6}a_i f_{N+1-i}, \quad (5)$$

where

$$a_0=77081,\quad a_1=979962,\quad a_2=2307,\quad a_3=155852,$$
$$a_4=-92193,\quad a_5=30426,\quad a_6=-4315,\quad d_1=80640.$$

With these values of $a_i\,(i = 0, 1, \dots, 6)$ and d_1, the local truncation error, $T_m\,(m = 1, 2, \dots, N)$ at every mesh point is

$$T_m = \frac{23}{75600} h^8 y^{(viii)}(x_m) + O\left(h^{10}\right), \qquad \text{as } h \to 0.$$

2.2. An Eighth-Order Method

Scheme (2) with $\alpha = -31/252$, $\beta = 172/945$, $\gamma = 23/3780$, gives an eighth-order method, namely

$$\delta^2 y_m + \frac{31}{252}\delta^4 y_m = \frac{2358}{3780} h^2 f_m + \frac{172}{945}(f_{m+1} + f_{m-1}) + \frac{23}{3780} h^2 (f_{m+2} + f_{m-2}), \qquad (6)$$
$$m = 2, 3, N-1$$

for $m = 1$ and N, the schemes

$$\delta^2 y_1 + \frac{31}{252}(-2y_0 + 5y_1 - 4y_2 + y_3) = \frac{1}{3780 d_2} h^2 \sum_{i=0}^{8} a_i f_i, \qquad (7)$$

$$\delta^2 y_N + \frac{31}{252}(-2y_{N+1} + 5y_N - 4y_{N-1} + y_{N-2}) = \frac{1}{3780 d_2} h^2 \sum_{i=0}^{8} a_i f_{N+1-i} \qquad (8)$$

with

$$b_0 = 44867959, \quad b_1 = 655622428, \quad b_2 = -29280308, \quad b_3 = 275499076,$$
$$b_4 = -25478390, \quad b_5 = 162922276, \quad b_6 = -67722068, \quad b_7 = 16537468,$$
$$b_8 = -1803641, \quad d_2 = 4032$$

are suggested. With these values of $b_i\,(i = 0, 1, \dots, 8)$ and d_2 the local truncation error $T_m\,(m = 1, 2, \dots, N)$ at each mesh point is given by

$$T_m = \frac{79}{4762800} h^{10} y^{(x)}(x_m) + O\left(h^{12}\right), \qquad \text{as } h \to 0.$$

3. IMPLEMENTATION AND CONVERGENCE

3.1. The Sixth-Order Method

It may be shown by suitably arranging (3)–(5) that the sixth-order method may be written in matrix-vector form as

$$\left(J + \frac{1}{20}J^2\right)\mathbf{Y} + \left(\frac{1}{240}\right) h^2 M \mathbf{f}(x, \mathbf{Y}) = \mathbf{e}, \qquad (9)$$

where J is the familiar tridiagonal matrix of order N given by

$$J = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}, \qquad (10)$$

for which $\|J^{-1}\|_\infty = (N+1)^2/8$ (see [11]), M is the N-square matrix

$$M = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & & \\ 24 & 194 & 24 & -1 & & & & \\ -1 & 24 & 194 & 24 & -1 & & 0 & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & 0 & & -1 & 24 & 194 & 24 & -1 \\ & & & & -1 & 24 & 194 & 24 \\ & & m_6 & m_5 & m_4 & m_3 & m_2 & m_1 \end{bmatrix}, \qquad (11)$$

in which $m_i = a_i/d_1$ $(i = 1, 2, \dots, 6)$, $\mathbf{f}$ is the N-vector

$$\mathbf{f} = [f_1, f_2, \dots, f_N]^\top, \tag{12}$$

$\top$ denoting transpose, $\mathbf{e}$ is the N-vector

$$\mathbf{e} = \left(\frac{h^2}{240}\right)\left[-\frac{a_0}{d_1}, 1, 0, \dots, 0, 1, -\frac{a_0}{d_1}\right]^\top, \tag{13}$$

and $\mathbf{Y}$, which may be computed from (9) using a nonlinear algebraic system solver, is the N-vector

$$\mathbf{Y} = [y_1, y_2, \dots, y_N]^\top. \tag{14}$$

It follows from (11) that $\|M\|_\infty = 1149120/80640 \cong 14.25 = M^*$.

The theoretical-solution vector

$$\mathbf{y} = \mathbf{y}(x) = [y(x_1), y(x_2), \dots, y(x_N)]^\top \tag{15}$$

satisfies the equation

$$J\mathbf{y} + \left(\frac{h^2}{240}\right) M\mathbf{f}(x, \mathbf{y}) = \mathbf{e} + \mathbf{T}, \tag{16}$$

where

$$\mathbf{T} = [T_1, T_2, \dots, T_N]^\top \tag{17}$$

is the truncation error vector for which

$$\|\mathbf{T}\|_\infty = \left(\frac{23}{75600}\right) h^8 V_8$$

with

$$V_8 = \max_{a<x<b} \left|\frac{d^8 y(x)}{dx^8}\right|. \tag{18}$$

A standard convergence analysis then shows that method (3)–(5) is sixth-order convergent provided

$$\max_m \left|\frac{\partial f_m}{\partial y(x_m)}\right| < \frac{8}{(b-a)^2}. \tag{19}$$

3.2. The Eighth-Order Method

Following the approach of Section 3.1, it can be proved that method (6)–(8) is an eighth-order convergent process.

4. NUMERICAL ILLUSTRATIONS

To illustrate the six-order convergence of method (3)–(5), consider the following nonlinear problem:

$$y'' = \frac{1}{2}(x + y + 1)^3, \qquad 0 < x < 1$$

with boundary conditions

$$y(0) = y(1) = 0$$

for which the exact solution is

$$y(x) = \left(\frac{2}{2-x}\right) - x - 1.$$

Table 1.

	Sixth-Order Methods				Eighth-Order Methods			
	Method (3)–(5)		Jain [8]		Method (6)–(8)		Jain [8] Extrapolated	
h	$\|E\|_\infty$	% Error	$\|E\|_\infty$	% Error	$\|E\|_\infty$	% Error	$\|E\|_\infty$	% Error
$\frac{1}{8}$	0.632-06*	0.650-03	0.772-05	0.794-02	0.981-08	0.161-04	0.504-07	0.518-04
$\frac{1}{16}$	0.633-08	0.540-05	0.201-06	0.365-03	0.501-10	0.827-07	0.678-09	0.122-05
$\frac{1}{32}$	0.157-09	0.133-06	0.415-08	0.141-04	0.826-13	0.242-09	0.710-11	0.240-07
$\frac{1}{64}$	0.272-11	0.496-08	0.750-10	0.496-06	0.211-15	0.526-12	—	—

*0.632-06 = 0.632×10^{-6}.

The interval [0,1] was divided into $N+1$ equal subintervals each of width $h = 2^{-m}$ ($m = 3, \dots, 6$); the corresponding values of N are then given by $N = 2^m - 1$. The values of $\|E\|_\infty = |\mathbf{y}(x) - \mathbf{Y}|$ as well as the relative errors, expressed as percentages, for the sixth-order method (3)–(5) and Jain's sixth-order method are reported in Table 1. The numerical results verify the sixth-order convergence of the method. These results show an improvement in accuracy when compared to those of Jain [8]. The efficiency of the method lies in its implementation with less function evaluations at each mesh point, as compared to Jain's sixth-order method. Four iterations are required to obtain relative errors to three significant figures using the Newton-Raphson method for a nonlinear algebraic system. Furthermore, the CPU time was calculated for $h = 1/32$ ($N = 31$) for both of the methods .The sixth-order method (3)–(5) takes 12.5 ms while Jain's sixth-order method takes 14.2 ms when the program is run on a Pentium III personal computer of 800 mhz Intel with 256 mb RAM.

The above problem is again considered to show the eighth-order convergence of method (6)–(8). Table 1 also shows $\|\mathbf{E}\|_\infty$ and the relative errors expressed as percentages for the eighth-order method with $N = 7, 15, 31$. The results are compared with the eighth-order method of Jain [8] (obtained by extrapolating his sixth-order method). The results are more accurate than those of Jain. It is remarked that the proposed eighth-order method uses five function evaluations on a grid line as compared to eighth-order method of Chawla [10], which uses seven function evaluations. Four iterations are required to obtain $\|\mathbf{E}\|_\infty$ and relative errors to three significant figures.

5. SUMMARY

Sixth- and eighth-order, finite-difference methods have been developed for the numerical solution of a nonlinear, two-point, boundary-value problem. Numerical results confirm the orders of the methods, which are seen to be more accurate and economical than similar-order methods in the literature.

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