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Counting rooted near-triangulations on the sphere*

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Abstract

This paper provides the results on the enumerations of rooted simple outerplanar maps, rooted outerplanar near-triangulations, rooted 2-connected near-triangulations, rooted strict 2-connected near-triangulations and rooted simple 2-connected near-triangulations. The answer to an open problem proposed by one of the authors is also provided.

1. Introduction

A rooted planar near-triangulation is a rooted planar map in which each inner face is triangular. If the root-face of a rooted planar near-triangulation is also a triangle, then the map is called a triangulation. A rooted strict 2-connected near-triangulation is a rooted nonseparable one in which there do not exist two edges making up a circuit, with at least one vertex in the outer domain of the circuit. A rooted simple 2-connected near-triangulation is a rooted one in which there is no separating triangle. A separating triangle is defined by three edges of the near-triangulation making up a circuit, with at least one vertex in the inner domain of the circuit and at least one vertex in the outer domain.

Let $\mathscr{M}$ be a set of some maps. We define two functions as follows:

$$f_{\mathscr{M}}(x,y)=\sum_{M\in\mathscr{M}} x^{m(M)}y^{n(M)}$$

and

$$F_{\mathscr{M}}(x,y,z)=\sum_{M\in\mathscr{M}} x^{m(M)}y^{n(M)}z^{q(M)},$$

where $m(M)$, $n(M)$ and $q(M)$ are the valency of the root-face, the number of edges and the number of blocks in M, respectively. The coefficient of the term $x^m y^n$ (or $x^m y^n z^k$)

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in the power series of $f_{\mathscr{M}}(x, y)$ (or $F_{\mathscr{M}}(x, y, z)$) is the number of maps in $\mathscr{M}$ such that $m(M)=m$ and $n(M)=n$ (or $m(M)=m$, $n(M)=n$ and $q(M)=k$). Therefore, in order to enumerate $\mathscr{M}$ we only have to find the expressions of the power series of $f_{\mathscr{M}}(x, y)$ and $F_{\mathscr{M}}(x, y, z)$.

In this paper, we discuss rooted simple outerplanar maps, rooted outerplanar near-triangulations, rooted general 2-connected near-triangulations, rooted strict 2-connected near-triangulations and rooted simple 2-connected near-triangulations. We get a number of explicit expressions of the enumerating functions of these classes of maps with the valency of the root-face, the number of edges and the number of blocks as parameters.

2. For simple outerplanar maps and outerplanar near-triangulations

A rooted outerplanar map is a rooted planar map in which there is a face such that each vertex is on its boundary. This face is defined as the root-face of the map. A rooted outerplanar map is said to be simple if neither multiple edge nor loop exist on it. Let $\mathscr{M}^{\text{so}}$ and $\mathscr{M}^{\text{nso}}$ be the set of all rooted simple outerplanar maps and the set of all rooted nonseparable simple outerplanar maps, respectively. For convenience, we suppose the vertex-map ϑ is included in $\mathscr{M}^{\text{so}}$ but not in $\mathscr{M}^{\text{nso}}$.

First, we divide $\mathscr{M}^{\text{nso}}$ into two sets $\mathscr{M}_1^{\text{nso}}$ and $\mathscr{M}_2^{\text{nso}}$, $\mathscr{M}_1^{\text{nso}}=\{L \mid L \text{ is the link map}\}$ as shown in [3]. For a map $M \in \mathscr{M}_2^{\text{nso}}$, the valency of each inner face in M is at least 3. The structure of M is like the map in Fig. 1. In Fig. 1, r is the root-vertex of M, $k \geqslant 2$, $M_1, M_2, \ldots, M_k \in \mathscr{M}^{\text{nso}}$, and (r, r_k), (r, r_1), $(r_1, r_2), \ldots, (r_{k-1}, r_k)$ are the root-edges of M, $M_1, M_2, \ldots, M_k$, respectively. On the other hand, for any integer $k \geqslant 2$ and $M_1, M_2, \ldots, M_k \in \mathscr{M}^{\text{nso}}$, a map $M \in \mathscr{M}_2^{\text{nso}}$ can be constructed as in Fig. 1. Then we get an expression for $f_{\mathscr{M}_2^{\text{nso}}}(x, y)$ as follows:

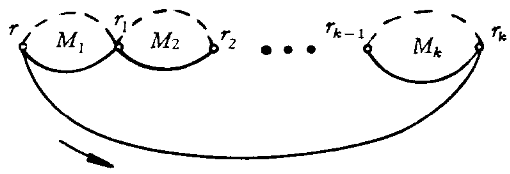
$$
\begin{aligned}
f_{\mathscr{M}_2^{\text{nso}}}(x, y) &= \sum_{k \geqslant 2} xy \prod_{i=1}^{k} \sum_{M_i \in \mathscr{M}^{\text{nso}}} x^{m(M_i)-1} y^{n(M_i)} \\
&= \frac{y(f_{\mathscr{M}^{\text{nso}}}(x, y))^2}{x - f_{\mathscr{M}^{\text{nso}}}(x, y)}.
\end{aligned}
$$

Further, we obtain an equation for $f_{\mathscr{M}^{\text{nso}}}(x, y)$:

$$
f_{\mathscr{M}^{\text{nso}}}(x, y) = x^2 y + \frac{y(f_{\mathscr{M}^{\text{nso}}}(x, y))^2}{x - f_{\mathscr{M}^{\text{nso}}}(x, y)}. \tag{2.1}
$$

This equation can be written as

$$
f = x^2 y - yf + \frac{xyf}{x - f}, \tag{2.2}
$$



$\boldsymbol{M}$

Fig. 1.

where $f = f_{\mathcal{M}^{\mathrm{nso}}}(x, y)$. Let $xyf/(x-f) = \xi$. From (2.2), we may deduce the following relations:

$$x = -\frac{\xi^2}{(xy)^2 + (xy-1)\xi}, \qquad f = \frac{x\xi}{xy+\xi}. \tag{2.3}$$

Let $xy + \xi = \varphi$, then

$$x = -\frac{\xi}{(\varphi-\xi)\dfrac{\varphi}{\xi} - 1}, \qquad xy = \varphi - \xi, \qquad f = \frac{x\xi}{\varphi}. \tag{2.4}$$

After some substitutions of parameters, the following relations with two parameters ψ and η can be found:

$$xy = \frac{\eta}{1+\psi}, \qquad y = \psi(1-\eta), \qquad \frac{f}{x} = \frac{1}{1+\psi}. \tag{2.5}$$

By using Lagrangian inversion theorem, we get the power series for $f_{\mathcal{M}^{\mathrm{nso}}}(x, y)$:

$$f_{\mathcal{M}^{\mathrm{nso}}}(x, y) = x^2 y + \sum_{\substack{m \geqslant 3 \\ 2m-3 \geqslant n \geqslant m}} \frac{1}{n}\binom{n}{m-1}\binom{m-3}{n-m} x^m y^n. \tag{2.6}$$

For a map $M \in \mathcal{M}^{\mathrm{nso}}$, it is easy to prove that $n(M) = 2m(M) - 3$ if and only if M is a rooted nonseparable outerplanar near-triangulation. Then we can find the power series of the generating function of $\mathcal{M}^{\mathrm{non}}$ which is the set of all rooted nonseparable outerplanar near-triangulations as

$$f_{\mathcal{M}^{\mathrm{non}}}(x, y) = \sum_{m \geqslant 2} \frac{1}{2m-3}\binom{2m-3}{m-1} x^m y^{2m-3}. \tag{2.7}$$

Here we suppose ϑ is not included in $\mathcal{M}^{\mathrm{non}}$.

Now we evaluate $F_{\mathcal{M}^{\mathrm{so}}}(x, y, z)$. Let $\mathcal{M}_1^{\mathrm{so}} = \{\vartheta\}$ and $\mathcal{M}_2^{\mathrm{so}} = \mathcal{M}^{\mathrm{so}} - \mathcal{M}_1^{\mathrm{so}}$. For any map $M \in \mathcal{M}_2^{\mathrm{so}}$ as shown in Fig. 2, we have $M_0 \in \mathcal{M}^{\mathrm{nso}}$, $k = m(M_0)$ and $M_1, M_2, \ldots, M_k \in \mathcal{M}^{\mathrm{so}}$. So, we may have

$$\begin{aligned} F_{\mathcal{M}_2^{\mathrm{so}}}(x, y, z) &= \sum_{M_0 \in \mathcal{M}^{\mathrm{nso}}} x^{m(M_0)} y^{n(M_0)} z \prod_{i=1}^{m(M_0)} \sum_{M_i \in \mathcal{M}^{\mathrm{so}}} x^{m(M_i)} y^{n(M_i)} z^{q(M_i)} \\ &= z f_{\mathcal{M}^{\mathrm{nso}}}(x F_{\mathcal{M}^{\mathrm{so}}}(x, y, z), y). \end{aligned} \tag{2.8}$$

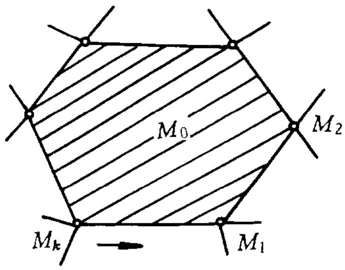


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Fig. 2.

From the known relations $F_{\mathcal{M}_1^{\mathrm{so}}}(x,y,z)=1$ and (2.5), we obtain the following expressions with two parameters ψ and η:

$$x=\frac{\eta(1+\psi)}{\eta z+(1-\eta)\psi(1+\psi)^2},\qquad y=\psi(1-\eta),$$

$$xF_{\mathcal{M}^{\mathrm{so}}}(x,y,z)=\frac{\eta}{(1-\eta)\psi(1+\psi)}. \tag{2.9}$$

Let $\eta/(1-\eta)=\xi$, then

$$x=\frac{\xi(1+\psi)}{\xi z+\psi(1+\psi)^2},\qquad y=\frac{\psi}{1+\xi},\qquad xF_{\mathcal{M}^{\mathrm{so}}}(x,y,z)=\frac{\xi}{\psi(1+\psi)}. \tag{2.10}$$

By using Lagrangian inversion theorem, an explicit expression of the power series of $F_{\mathcal{M}^{\mathrm{so}}}(x,y,z)$ is found as

$$F_{\mathcal{M}^{\mathrm{so}}}(x,y,z)=\sum_{k\geqslant 0} x^{2k}y^k z^k + \sum_{\substack{k\geqslant 1\\ m\geqslant 2k+1\\ 2m-3k\geqslant n\geqslant m-k+1}} \frac{1}{n}\binom{m}{k-1}\binom{n}{m-k}\binom{m-2k-1}{n-m-1+k}x^m y^n z^k. \tag{2.11}$$

For a map $M\in\mathcal{M}^{\mathrm{so}}$, if $q(M)=1$, it is easy to prove that $n(M)=2m(M)-3$ if and only if M is a rooted outerplanar nonseparable near-triangulation. Then, by induction we may see that for any map $M\in\mathcal{M}^{\mathrm{so}}$, M is a rooted outerplanar near-triangulation if and only if $n(M)=2m(M)-3q(M)$. Let $\mathcal{M}^{\mathrm{on}}$ be the set of all rooted outerplanar near-triangulations. By (2.11), we find that $F_{\mathcal{M}^{\mathrm{on}}}(x,y,z)$ has the following power-series form:

$$F_{\mathcal{M}^{\mathrm{on}}}(x,y,z)=1+\sum_{\substack{k\geqslant 1\\ m\geqslant 2k}} \frac{1}{2m-3k}\binom{2m-3k}{m-k}\binom{m}{k-1}x^m y^{2m-3k}z^k. \tag{2.12}$$

As a consequence, we may directly deduce the enumerating function of rooted trees as a special case of (2.12) due to the fact that M is a rooted tree if and only if $m(M)=2q(M)$ and $M\in\mathcal{M}^{\mathrm{on}}$. Let $\mathcal{M}^{\mathrm{t}}$ be the set of all rooted trees. From (2.12), we have

$$F_{\mathcal{M}^{\mathrm{t}}}(x,y,z)=1+\sum_{k\geq 1}\frac{1}{k}\binom{2k}{k-1}x^{2k}y^k z^k. \tag{2.13}$$

So there are

$$\frac{1}{k}\binom{2k}{k-1}$$

rooted trees with k edges.

3. Rooted 2-connected near-triangulations

A rooted 2-connected near-triangulation here is a rooted nonseparable map in which the boundary of each inner face consists of three edges. Let $\mathcal{M}^{\mathrm{g}}$ be the set of all rooted 2-connected near-triangulations. Here, $\mathcal{M}^{\mathrm{g}}$ does not contain the vertex map ϑ. We partition $\mathcal{M}^{\mathrm{g}}$ into three subsets:

$$\mathcal{M}^{\mathrm{g}}_1=\{L\},$$

$$\mathcal{M}^{\mathrm{g}}_2=\{M\mid M\in\mathcal{M}^{\mathrm{g}},\text{ and } M-R \text{ is separable}\},$$

$$\mathcal{M}^{\mathrm{g}}_3=\mathcal{M}^{\mathrm{g}}-\mathcal{M}^{\mathrm{g}}_1-\mathcal{M}^{\mathrm{g}}_2,$$

where, L is the link map. In this case, we may finally obtain:

$$f_{\mathcal{M}^{\mathrm{g}}_1}(x,y)=x^2y,\qquad f_{\mathcal{M}^{\mathrm{g}}_2}(x,y)=\frac{y}{x}(f_{\mathcal{M}^{\mathrm{g}}}(x,y))^2,$$

$$f_{\mathcal{M}^{\mathrm{g}}_3}(x,y)=\frac{y}{x}(f_{\mathcal{M}^{\mathrm{g}}}(x,y)-f_{\mathcal{M}^{\mathrm{g}}_{2,\cdot}}(x,y)), \tag{3.1}$$

with $\mathcal{M}^{\mathrm{g}}_{2,\cdot}$ being the set of M in $\mathcal{M}^{\mathrm{g}}$ while $m(M)=2$.

We can now obtain the relation between $f_{\mathcal{M}^{\mathrm{g}}}(x,y)$ and $f_{\mathcal{M}^{\mathrm{g}}_{2,\cdot}}$ as

$$f_{\mathcal{M}^{\mathrm{g}}}(x,y)=x^2y+\frac{y}{x}(f_{\mathcal{M}^{\mathrm{g}}}(x,y))^2+\frac{y}{x}(f_{\mathcal{M}^{\mathrm{g}}}(x,y)-f_{\mathcal{M}^{\mathrm{g}}_{2,\cdot}}(x,y)). \tag{3.2}$$

In brief, we write $f_{\mathcal{M}^{\mathrm{g}}_2}(x,y)=x^2f_2(y)$. From (3.2), we have

$$y(f_{\mathcal{M}^{\mathrm{g}}}(x,y))^2+(y-x)f_{\mathcal{M}^{\mathrm{g}}}(x,y)+x^2y(x-f_2(y))=0. \tag{3.3}$$

It can be verified that this equation has a power-series solution (i.e. a series in positive power only) for the only choice of function $f_2(y)$. From the known condition, the solution will have the form

$$f_{\mathcal{M}^{\mathrm{g}}}(x,y)=\frac{x-y-\sqrt{(y-x)^2-4x^2y^2(x-f_2(y))}}{2y}. \tag{3.4}$$

Let D denote the discriminant of equation (3.3), i.e.

$$D=y^2-2yx+(1+4y^2f_2(y))x^2-4y^2x^3. \tag{3.5}$$

It has been shown in [1] that D must have a repeated factor if $f_{\mathcal{M}^g}(x,y)$ is a power series. We may suppose

$$D=(y+ux)^2\,(1-vx)=y^2+(2yu-y^2v)x+(u^2-2yuv)x^2-u^2vx^3, \tag{3.6}$$

where u, v are functions of y. By identifying the coefficients of x^i in the forms of (3.5) and (3.6), the following parametric expressions are derived:

$$2yu-y^2v=-2y, \qquad u^2-2yuv=1+4y^2f_2(y), \qquad -u^2v=-4y^2. \tag{3.7}$$

From (3.7), we find

$$y^3=\frac{u^2(u+1)}{2}, \qquad v=\frac{2(u+1)}{y}. \tag{3.8}$$

Here u is treated as a parameter.

Now, from (3.4) by expanding the radical into a power series, we may have

$$\begin{aligned} f_{\mathcal{M}^g}(x,y)&=\left\{x-y+(y+xu)\left[1-\sum_{m\geq 1}\frac{(2m-2)!\,v^m x^m}{2^{2m-1}m!(m-1)!}\right]\right\}\Big/2y \\ &=-\sum_{m\geq 2}\left[\frac{(2m-2)!\,yv^m}{2^{2m-1}m!(m-1)!}+\frac{(2m-4)!\,uv^{m-1}}{2^{2m-3}(m-1)!(m-2)!}\right]\frac{x^m}{2y}. \end{aligned} \tag{3.9}$$

Let $\theta=u+1$, then

$$y^3=\frac{\theta(1-\theta)^2}{2}, \qquad v=\frac{2\theta}{y}. \tag{3.10}$$

From (3.10), (3.9) becomes

$$f_{\mathcal{M}^g}(x,y)=-\sum_{m\geq 2}\left[\frac{(2m-2)!\,yv^m}{2^{2m-1}m!\,(m-1)!}+\frac{(2m-4)!\,(\theta-1)\,v^{m-1}}{2^{2m-3}(m-1)!\,(m-2)!}\right]\frac{x^m}{2y}. \tag{3.11}$$

By using Lagrange's theorem, we have

$$v^m=\sum_{k\geq m}\frac{2^{m+k+1}\,m(3k-m-1)!}{(2k)!(k-m)!}\,y^{3k-m},$$

$$(\theta-1)v^{m-1}=-\sum_{k\geq m-1}\frac{2^{m+k-1}(2m-3)\,(3k-m-1)!}{(2k-1)!(k-m+1)!}\,y^{3k-m+1}.$$

Therefore, from (3.11), we may derive

$$f_{\mathcal{M}^g}(x,y)=\sum_{\substack{m\geq 2\\ k\geq m-1}}\frac{2^{k-m+2}(2m-3)!(3k-m-1)!}{(2k)!(k-m+1)!(m-2)!(m-2)!}\,x^m y^{3k-m}. \tag{3.12}$$

So, there are

$$\frac{2^{k-m+2}(2m-3)!(3k-m-1)!}{(2k)!(k-m+1)!(m-2)!(m-2)!}$$

distinct rooted 2-connected near-triangulations with root-face valency m and $3k-m$ edges.

4. Rooted strict 2-connected near-triangulations

It is obvious that a rooted near-triangulation M with $m(M)\geqslant 3$ is strict iff there do not exist two edges having the same pair of ends. We can also prove that a rooted near-triangulation M with $m(M)=2$ and $n(M)\geqslant 3$ is strict iff there do not exist two edges other than the two edges on the boundary of the root-face having the same pair of ends. Let $\mathscr{M}^s$ be the set of all rooted strict 2-connected near-triangulations with the root-face of valency at least 3. By the above definition, the following relation can be deduced:

$$f_{\mathscr{M}^g}(x,y)=f_{\mathscr{M}^g}(x,f_2(y))+x^2f_2(y). \tag{4.1}$$

We write $z=f_2(y)$, and from (3.7) the following parametric expressions are found:

$$z^3=\frac{-(u+1)(1+3u)^3}{16u^4}, \qquad y=\frac{-2u^2}{1+3u}z. \tag{4.2}$$

Here u is treated as a parameter. Now, we take y as a function of z.

From (4.1) and (3.9), we obtain

$$f_{\mathscr{M}^g}(x,z)=-\sum_{m\geqslant 2}\left[\frac{(2m-2)!v^m}{2^{2m}m!(m-1)!}+\frac{(2m-4)!uv^{m-1}/y}{2^{2m-2}(m-1)!(m-2)!}\right]x^m-x^2z, \tag{4.3}$$

where

$$v=\frac{2(1+u)}{y}.$$

Hence,

$$\begin{aligned} v^m z^m&=\frac{(-1)^m(1+u)^m(1+3u)^m}{u^2m},\\ \frac{uv^{m-1}}{y}z^m&=\frac{(-1)^m(1+u)^{m-1}(1+3u)^m}{2u^{2m-1}}. \end{aligned} \tag{4.4}$$

With $(1+u)/u=-w$, we get

$$z^3=\frac{w(2-w)^3}{16}, \qquad v^mz^m=w^m(2-w)^m, \qquad \frac{uv^{m-1}}{y}z^m=-\frac{(2-w)^m w^{m-1}}{2}. \tag{4.5}$$

By using the Lagrangian theorem, the power series of v^m and uv^{m-1}/y are as follows:

$$v^m = \sum_{t \geq m} \frac{2^{2m+1} m(4t-2m-1)!}{(3t-m)!(t-m)!} z^{3t-m},$$

$$uv^{m-1}/y = \sum_{t \geq m-1} \frac{2^{2m-2}(3-2m)(4t-2m)!}{(3t-m)!(t-m+1)!} z^{3t-m}. \tag{4.6}$$

Therefore, from (4.3), we have

$$f_{\mathscr{M}^s}(x,z) = \sum_{\substack{m \geq 3 \\ t \geq m-1}} \frac{2(4t-2m-1)!(2m-3)!}{(3t-m)!(t-m+1)!(m-1)!(m-3)!} z^{3t-m} x^m. \tag{4.7}$$

So, there are

$$\frac{2(4t-2m-1)!(2m-3)!}{(3t-m)!(t-m+1)!(m-1)!(m-3)!}$$

rooted strict 2-connected near-triangulations with $3t-m$ edges and the root-face of valency m, where $m \geq 3$.

Let $\mathscr{M}^s_{2,\cdot}$ be the set of all rooted strict 2-connected near-triangulations with the root-face of valency 2. For every $M \in \mathscr{M}^s_{2,\cdot}$, $n(M) > 1$, $M - R(M)$ is a rooted strict 2-connected triangulation. On the other hand, for every rooted strict 2-connected triangulation M', there exists only one map $M \in \mathscr{M}^s_{2,\cdot}$ such that $M - R(M) \cong M'$. So there are

$$\frac{2(4t-7)!}{(3t-4)!(t-1)!}$$

rooted strict 2-connected near-triangulations of $3t-2$ edges, $t \geq 2$, with the root-face of valency 2. It is obvious that there is only one map, the link map, of 1 edge with the root-face of valency 2.

5. Rooted simple 2-connected near-triangulations

A rooted simple 2-connected near-triangulation is a rooted strict one in which there is no separating triangle [5]. A separating triangle is defined by three edges of the near-triangulation which form a circuit with at least one vertex in each of the inner and outer domains of the circuit. Let $\mathscr{M}^p$ be the set of all rooted simple 2-connected near-triangulations with $m(M) \geq 3$.

We may now see the following relation between $f_{\mathscr{M}^s}(x,y)$and $f_{\mathscr{M}^s}(x,y)$:

$$f_{\mathscr{M}^s}(x,y) = x^2 y + x^3 y^3 + f_{\mathscr{M}^p}\left(\frac{xy}{f_3^{1/3}}, \frac{f_3^{2/3}}{y}\right) - x^3 f_3, \tag{5.1}$$

where $f_3 = [x^3] f_{\mathscr{M}^s}(x,y)$, the coefficient of x^3 in the power series of $f_{\mathscr{M}^s}(x,y)$.

From (4.2) and (4.3), we have

$$y^3=\frac{w(2-w)^3}{16}, \qquad f_3=\frac{w(1-w)}{2}, \tag{5.2}$$

and

$$\begin{aligned} &f_{\mathcal{M}^s}(x,y)-x^2y \\ &\quad=\sum_{m\geqslant 3}\left(\frac{(2m-4)!\,w^{m-1}(2-w)^m}{2^{2m-1}(m-1)!\,(m-2)!}-\frac{(2m-2)!\,w^m(2-w)^m}{2^{2m}m!\,(m-1)!}\right)\frac{x^m}{y^m}. \end{aligned} \tag{5.3}$$

Let

$$\tilde{x}=\frac{xy}{f_0^{1/3}}, \qquad \tilde{y}=\frac{f_0^{2/3}}{y}. \tag{5.4}$$

Then we get

$$\tilde{y}^3=\frac{4w(1-w)^2}{(2-w)^3} \tag{5.5}$$

and the following parametric expression of $f_{\mathcal{M}^p}(\tilde{x},\tilde{y})$ with w as a parameter:

$$\begin{aligned} f_{\mathcal{M}^p}(\tilde{x},\tilde{y})=&-\frac{w(1-w)}{2}\tilde{x}^3+\frac{4w(1-w)^2}{(2-w)^3}\tilde{x}^3 \\ &+\sum_{m\geqslant 3}\frac{2^{m+1}(2m-4)!\,w^{m-1}(1-w)^m(m-(2m-3)w)}{m!\,(m-2)!\,(2-w)^{2m}}\frac{\tilde{x}^m}{\tilde{y}^m}. \end{aligned} \tag{5.6}$$

Let $w/(2-w)=\theta$. We then obtain

$$\tilde{y}^3=\theta(1-\theta)^2, \tag{5.7}$$

and

$$\begin{aligned} f_{\mathcal{M}^p}(\tilde{x},\tilde{y})=&-\frac{\theta(1-\theta)}{(1+\theta)^2}\tilde{x}^3+\theta(1-\theta)^2\tilde{x}^3 \\ &+\sum_{m\geqslant 3}\frac{(2m-4)!\,\theta^{m-1}(1-\theta)^m(m-3(m-2)\theta)}{m!\,(m-2)!}\frac{\tilde{x}^m}{\tilde{y}^m}. \end{aligned} \tag{5.8}$$

Therefore, by using the Lagrangian inversion theorem, we find the explicit expression of the power series of $f_{\mathcal{M}^p}(x,y)$ as

$$\begin{aligned} f_{\mathcal{M}^p}(x,y)=x^3y^3+\sum_{j\geqslant 2}&\left[\frac{(3j-5)!}{(2j-3)!\,(j-1)!}\right. \\ &\left.-\sum_{1\leqslant i<i/2}\frac{(i+1)(4i-1)(3j-2i-5)!}{(2j-3)!\,(j-2i-1)!}\right]x^3y^{3j} \\ &+\sum_{\substack{m\geqslant 4\\ j\geqslant m-1}}\frac{(2m-4)!\,(3j-2m-1)!}{(m-1)!\,(m-4)!\,(j-m+1)!\,(2j-m)!}x^my^{3j-m}. \end{aligned} \tag{5.9}$$

From the above expression we know the number of rooted simple 2-connected near-triangulations with the root-face of valency at least 3 and at least 3 edges.

It is also easy to prove that the number of rooted simple 2-connected near-triangulations with the root-face of valency 2 and $3j+1$ edges is just the number of rooted simple 2-connected triangulations with $3j$ edges, where $j \geq 1$; the latter of course is found here.

6. Remarks

(a) The enumerating equation of rooted outerplanar near-triangulations with the valency of the root-face as a parameter has been obtained in [4]:

$$\phi = 1 + x^2\phi^2 - x^{-1}\frac{\phi - 1}{\phi} + (\phi - 1)x^{-1}, \tag{6.1}$$

where $\phi = \sum_{M \in \mathscr{M}^{\text{on}}} x^{m}(M)$, $m(M)$ is the valency of the root-face in M, and $\mathscr{M}^{\text{on}}$ is the set of all rooted outerplanar near-triangulations. But, in [4] the expressions of the coefficients in the power series of ϕ is very complicated. Here it is much simplified.

Let

$$f = x\phi. \tag{6.2}$$

From (6.1) and (6.2), we obtain an equation about f:

$$f^3 + \frac{1-x}{x^2}f^2 + \frac{x-2}{x}f + 1 = 0. \tag{6.3}$$

That is

$$x^2(1 + f + f^3) - (f^2 + 2f)x + f^2 = 0. \tag{6.4}$$

From (6.4), we know that

$$\frac{1}{x} = \frac{f + 2 \pm f\sqrt{1-4f}}{2f}. \tag{6.5}$$

According to $f(0) = 0$ and $f'(0) = 0$, we have

$$x = \frac{2f}{f + 2 - f\sqrt{1-4f}}. \tag{6.6}$$

Let $\sqrt{1-4f} = 1 - 2\theta$. Then

$$x = \frac{\theta - \theta^2}{1 + \theta^2(1-\theta)}, \qquad f = \theta - \theta^2. \tag{6.7}$$

We now construct a function $g(x, z)$ as follows:

$$x = \frac{\theta - \theta^2}{1 + z\theta^2(1-\theta)}, \qquad g = \theta - \theta^2. \tag{6.8}$$

It can be seen that $g(x,1)=f(x)$. From (6.8), we have

$$\theta=\frac{1+\psi}{1-\theta}x, \qquad \psi=\theta^2(1-\theta)z, \qquad g=\theta-\theta^2. \tag{6.9}$$

By using the Lagrangian inversion theorem, we obtain

$$g(x,z)=x+\sum_{\substack{m\geq 3\\ 1\leq k\leq (m-1)/2}} \frac{1}{2m-3k-2}\binom{m-1}{k-1}\binom{2m-3k-2}{m-k-1}x^m z^k. \tag{6.10}$$

So

$$\phi(x)=1+\sum_{\substack{m\geq 2\\ 1\leq k\leq m/2}} \frac{1}{2m-3k}\binom{m}{k-1}\binom{2m-3k}{m-k}x^m. \tag{6.11}$$

This is again formula (2.12).

(b) From the expression of $\phi(x)$ in [4] and (6.11), we find the following combinatorial identity:

$$2^{-(t+1)}+\sum_{k=0}^{t/3}\sum_{l=0}^{t-3k}\sum_{n\geq \max(1,k+1)} \frac{(-1)^l(2n-2k+t-l)!\,2^{-(2n-2k+t-l+1)}}{(n-k-l)!\,l!\,k!\,(n+k+1)!\,(t-3k-l)!}$$

$$=\sum_{1\leq k\leq t/2}\frac{1}{2t-3k}\binom{t}{k-1}\binom{2t-3k}{t-k}, \quad t\geq 1.$$

This is an answer for one of the open problems in [4].

References

[1] W.G. Brown, On the existence of square roots in certain rings of power series, Math. Annalen 158 (1965) 82–89.

[2] Y. Liu, Counting rooted planar maps, J. Math. Res. Expos. 4 (1984) 37–46.

[3] Y. Liu, Enumeration of rooted nonseparable outerplanar maps, Acta Math. Appl. Sinica, Eng. Series 5 (1989) 169–175.

[4] Y. Liu, On enumerating equations of planar maps, Chinese Adv. Math. 18(4) (1989) 446–460.

[5] W.T. Tutte, Counting rooted triangulations, Ann. Discrete Math. 12 (1982) 243–253.