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NOTE

A Class of Hypergraph Arrangements with Shellable Intersection Lattice

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For every hypergraph on n vertices there is an associated subspace arrangement in $\mathbb{R}^n$ called a hypergraph arrangement. We prove shellability for the intersection lattices of a large class of hypergraph arrangements. This class incorporates all the hypergraph arrangements which were previously shown to have shellable intersection lattices. © 1999 Academic Press

1. INTRODUCTION

With every hypergraph $\mathscr{H}$ one can associate a subspace arrangement $\mathscr{A}_{\mathscr{H}}$, see Definition 2.5, which is called a hypergraph arrangement. The following, characteristic for topological combinatorics, question arises immediately: *What is the connection between the combinatorial properties of* $\mathscr{H}$ *and topology of* $\mathscr{A}_{\mathscr{H}}$?

One of the most important examples of hypergraph arrangements considered up to now are the so called k-equal arrangements, $\mathscr{A}_{n,k}$. Given n and $2 \geqslant k \geqslant n$, such an arrangement consists of the $\binom{n}{k}$ subspaces obtained by setting some k of the coordinates equal to each other. The corresponding hypergraph is the k-regular hypergraph $\mathscr{H} = \{H \subseteq [n] \mid |H| = k\}$. The topology of k-equal arrangements was studied in [BWe]. It was proved later in [BWa94] that the intersection lattices of $\mathscr{A}_{n,k}$ are shellable, which of course immediately gives the topological implications, though in a more structural way. A few more hypergraph arrangements were considered in [B94], [B95], [We].

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In this paper we consider a quite broad class of hypergraphs. The class of the associated hypergraph arrangements includes k-equal arrangements and also all other hypergraph arrangements the intersection lattices of which were proved to be shellable (pure or non-pure) up to now.

In Section 2 we shortly define all basic notions used in this paper. In Section 3 we prove the shellability of the intersection lattices of the considered class of hypergraph arrangements, (Theorem 3.1). Then we specialize this result to the few known cases in Corollary 3.2 and Remark 3.3.

2. BASIC NOTIONS AND DEFINITIONS

In this section we give a short summary of the standard notions used throughout the text.

For a finite poset P we will denote its chain complex by $\Delta(P)$. We say that P is *pure* if $\Delta(P)$ is pure. Such posets are also often called graded.

DEFINITION 2.1. A simplicial complex Δ is called *shellable* if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$, in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. Such an ordering of facets is called a *shelling order*.

DEFINITION 2.2. A poset P is said to be *EL-shellable* if one can label its edges with elements from a poset Λ so that for every interval $[x, y]$ in P,

(i) there is a unique rising maximal chain c in $[x, y]$ (rising means that the associated labels form a strictly increasing sequence in Λ);

(ii) $c \prec c'$ for all other maximal chains c' in $[x, y]$.

Here the symbol "$\prec$" means "lexicographically preceding." We will often say "lexicographically less" or just "less."

The notion of EL-shellability was first introduced in [B80, Chapter 2]. It was proved there that if P is EL-shellable, then $\Delta(P)$ is shellable. See also [BWa83] for further investigations and [BWa94] for the non-pure version.

DEFINITION 2.3. A family of sets $\mathscr{H} \subseteq 2^{[n]}$ such that for any $H, H' \in \mathscr{H}$, H is not included in H', is called a *hypergraph*.

DEFINITION 2.4. A finite collection $\mathscr{A} = \{K_1, \ldots, K_t\}$ of linear proper subspaces in $\mathbb{R}^n$ is called a *subspace arrangement*. The *intersection lattice* $\mathscr{L}_{\mathscr{A}}$ of an arrangement $\mathscr{A} = \{K_1, \ldots, K_t\}$ is the collection of all intersections

$K_{i_1} \cap \cdots \cap K_{i_p}$, $1 \leqslant i_1 < \cdots < i_p \leqslant t$, ordered by reverse inclusion: $x \leqslant y \Leftrightarrow y \subseteq x$, and extended by a unique minimal element $\hat{0}$.

Definition 2.5. For each subset $S = \{i_1, \ldots, i_s\} \subseteq [n]$, such that $s \geqslant 2$, let $K_S = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{i_1} = \cdots = x_{i_s}\}$. Then a hypergraph $\mathscr{H} \in 2^{[n]}$ (without singletons) determines a subspace arrangement $\mathscr{A}_{\mathscr{H}} = \{K_S \mid S \in \mathscr{H}\}$. The arrangement $\mathscr{A}_{\mathscr{H}}$ is called a *hypergraph arrangement*.

Hypergraph arrangements were introduced in [B94, Section 3]. It was also suggested there to denote the intersection lattice of $\mathscr{A}_{\mathscr{H}}$ by $\Pi_{\mathscr{H}}$.

3. SHELLABILITY OF THE INTERSECTION LATTICES

Theorem 3.1. *Let us fix a partition of* $\{1, \ldots, n\} = E_1 \cup \cdots \cup E_r$, *such that* $\max E_i < \min E_{i+1}$ *for* $i = 1, \ldots, r-1$. *Let* $\mathscr{H}$ *be a hypergraph* $\{H_1, \ldots, H_l\}$ *without singletons, such that the following conditions are satisfied*:

(1) $|H_i \cap E_j| \leqslant 1$ *for any* $1 \leqslant i \leqslant l$ *and* $1 \leqslant j \leqslant r$;

(2) *for any* H_i *and* $x \notin H_i$ *there exists* j *such that* $H_i \cup H_j = H_i \cup \{x\}$, *i.e.* $x \in H_j$, $H_j \subseteq H_i \cup \{x\}$;

(3) *let* $C = H_{i_1} \cup \cdots \cup H_{i_d}$, *then there exists* j *and* s *such that*

$$H_j \cap E_m = \begin{cases} \min(C \cap E_m), & \textit{if} \quad C \cap E_m \neq \varnothing \textit{ and } 1 \leqslant m \leqslant s; \\ \varnothing, & \textit{otherwise}. \end{cases}$$

Then $\Pi_{\mathscr{H}}$ *is EL-shellable.*

Proof. Clearly $\Pi_{\mathscr{H}}$ is a subposet of the partition lattice Π_n. We will adapt the practice of talking about blocks and singletons from there.

We label edges of $\Pi_{\mathscr{H}}$ with 3 different types of labels. Labels within each type are ordered and for the labels of different types we will use the following order:

$$\text{type } 1 < \text{type } 2 < \text{type } 3.$$

It is easy to see that all the edges (covering relations) of $\Pi_{\mathscr{H}}$ are of one of the following types.

(1) Edges corresponding to a merging of two nonsingleton blocks $B_1, B_2 \to B_1 \cup B_2$. The label is $(\max(B_1 \cup B_2))_1$.

(2) Edges corresponding to an insertion of a singleton $x \in E_i$ into a nonsingleton block B, such that either

$$B \cap E_i \neq \varnothing \qquad \text{and} \qquad x < \min(B \cap E_i)$$

or

$$B \cap E_i = \varnothing \qquad \text{and} \qquad x < \max B.$$

The label is $(x)_2$.

(3)(a) Creating a block of size k out of k singletons:

$$s_1, \ldots, s_k \to B = \{s_1, \ldots, s_k\}.$$

The label is $(B)_3$.

(b) Insertion of a singleton $x \in E_i$ into a block B such that either

$$B \cap E_i \neq \varnothing \qquad \text{and} \qquad x > \min(B \cap E_i)$$

or

$$B \cap E_i = \varnothing \qquad \text{and} \qquad x > \max B.$$

The label is $(x)_3$.

Next we have to specify how to order labels internally within each type. For the labels of types 1 and 2 we just use the usual ordering of integers. We order labels of type 3 in the lexicographic order, considering integers as sets consisting of one element.

Any interval $[a, b]$ in $\Pi_{\mathscr{H}}$ is a direct product of intervals $[x, y]$ such that

(1) y consists of a single block, say B,

(2) x consists of blocks $B_1, \ldots, B_t$ and singletons $a_1, \ldots, a_p$ (t and p may be equal to 0 and the singletons are ordered $a_1 < \cdots < a_p$).

We claim that such an interval $[x, y]$ is isomorphic to $\Pi_{\mathscr{H}'}$ for some hypergraph $\mathscr{H}'$ satisfying conditions (1)–(3).

Assume first $t = 0$. The ground set of $\mathscr{H}'$ is $\{a_1, \ldots, a_p\} = S$ and $E'_i = E_i \cap S$. Furthermore $H \in \mathscr{H}'$ iff $H \in \mathscr{H}$ and $H \subseteq S$, in other words iff $H \in \mathscr{H} \cap 2^S$. It is not difficult to check that the conditions (1)–(3) are satisfied.

Assume now that $t > 0$. The ground set of $\mathscr{H}'$ is $\{1, \ldots, t + p\}$ and

$$E'_i = \begin{cases} \{i\}, & \text{for} \quad i = 1, \ldots, t; \\ \{t + j \mid a_j \in E_{i-t}\}, & \text{for} \quad i = t+1, \ldots, t+p. \end{cases}$$

Some of the $E'_{t+1}, \ldots, E'_{t+p}$ may be empty. Let us describe the hypergraph $\mathscr{H}'$, $H' \in \mathscr{H}'$ iff

$$\text{either} \quad H' = \{x, y\},\ 1 \leqslant x \leqslant t;$$
$$\text{or} \quad H' = \{t + j \mid a_j \in H\} \text{ for some fixed } H \in \mathscr{H},\ H \subseteq \{a_1, \ldots, a_p\}.$$

This construction is quite similar to the one of $\Pi_{n,k}(l)$, see [BWe].

It is clear that condition (1) is then true.

Let us now check condition (2). If $x \leqslant t$ then $H_j = \{x, y\}$, where y is any element of H_i. If there exists $y \in H_i$, such that $y \leqslant t$ then $H_j = \{x, y\}$. Finally, if min $H_i > t$ and $x > t$ then condition (2) follows from the fact that it is true for the hypergraph $\mathscr{H}$.

Finally we check condition (3). If min $C \leqslant t$ then

$$H_j = \{\min C, \min(C \backslash \{\min C\})\}.$$

If min $C > t$ then the statement follows from the fact that the condition (3) is true for $\mathscr{H}$.

So we have decomposed the interval $[a, b]$ into a direct product of simpler intervals $[x_i, y_i]$. Assume that in every such interval the rising chain is unique and that it coincides with the lexicographically least chain. What can we say about the total interval $[a, b]$?

Take a lexicographically least chain c in $[a, b]$, it projects to the lexicographically least chains in the intervals $[x_i, y_i]$ (if some of these chains can be replaced by a lexicographically preceding chain, then so can c). Take a rising chain d in $[a, b]$, it projects to rising chains in the intervals $[x_i, y_i]$, which in turn are also the lexicographically least chains. Hence both c and d consist of the same set of labels, just permuted. Since c is lexicographically least and d is rising we can conclude that $c = d$.

So it is enough to show only for the simpler intervals $[x, y]$ that there exists a unique rising chain which is also lexicographically least. We have to consider two cases.

Case 1. x consists of singletons only.

Let $E'_i = E_i \cap \{\text{the singletons in } x\}$. Let

$$\{m_1, \ldots, m_q\} = \{\min E'_i \mid E'_i \neq \emptyset,\ i = 1, \ldots, r\}.$$

We can assume $m_1 < \cdots < m_q$.

It is easy to exhibit a rising chain. In the first step, create a block $C = \{m_1, \ldots, m_s\}$ for some $s \leqslant q$. It exists according to the condition (3) of the theorem. Then insert the remaining singletons into the block C one by one in increasing order. This is possible according to condition (2) of the theorem. The obtained chain c is clearly rising.

Let us see that this chain is lexicographically least. The first step is always a creation of a block, clearly block C is lexicographically least possible. Further, one either creates a new block or inserts some element x into C. Clearly, both such edges are of type 3 and if $x = \min(\bigcup_{i=1}^{r} E_i' \backslash C)$ then such edge is lexicographically least. Continuing to argue in this way, we conclude that the chosen chain is lexicographically least.

Let us prove that there are no other rising chains. Consider another rising chain d. It starts with an edge of type 3a. If the next edge is of type 3a too then at some point we would have to merge two blocks and such an edge would have type 1. Hence all the subsequent edges must be of type 3b, which defines the rising chain d uniquely. So d must coincide with c.

Case 2. x consists of blocks $B_1, \ldots, B_t$ and singletons $a_1, \ldots, a_p$, such that, say $\max B_1 < \cdots < \max B_t$ and $a_1 < \cdots < a_p$.

It is clear that any rising chain should start with merging the blocks $B_1, \ldots, B_t$ with each other in increasing index order, i.e.

$$B_1, B_2 \rightarrow B_1 \cup B_2; \qquad B_1 \cup B_2, B_3 \rightarrow B_1 \cup B_2 \cup B_3; \quad \text{etc.}$$

What happens next is best explained via Fig. 1. The cross-painted areas denote the elements from $B_1 \cup \cdots \cup B_t$. The elements painted diagonally up&left–down&right are the ones, whose insertion at the first step would be of type 2 (we simply say elements of type 2), the up&right–down&left painted elements are the ones of type 3. Observe that if the element is of type 3, then it will have this type during the whole insertion process. Inserting an element e_i turns all of the elements above it (on the picture) to type 3.

We claim that the next step must be the insertion of the element e_1 (see the picture) into $B_1 \cup \cdots \cup B_t$. Because, if the insertion of e_1 will be done later, then it will still be of type 2 and since it has the smallest label the chain would not be rising.

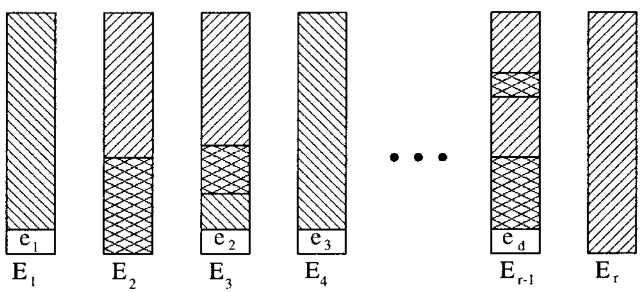

FIGURE 1

Arguing in the same manner we can conclude that the rising chain should continue by insertions of $e_2, e_3, \dots, e_d$. After that all other insertions will be of type 3, hence there is a unique way to insert the rest of the a_i's in increasing order (we can never create new blocks in the process, since otherwise we would have to have an edge of type 1 at some point and so the chain would not be rising).

It is not difficult to check that the obtained chain is lexicographically least in the interval $[x, y]$. Again, the mergings of blocks $B_1, \dots, B_t$ done in the described above order are the lexicographically least edges. Further insertions give the optimal steps as well, because of our choice of their order and the fact that it is (lexicographically) better to insert an element x, rather than create a new block with minimal element x.

So we have proved that in both cases there is a unique rising chain in the interval $[x, y]$ which is also lexicographically least. Taking in account the arguments we have mentioned above, this finishes the proof. ∎

We describe an important class of hypergraphs covered by Theorem 3.1.

Corollary 3.2. *Consider a partition of* $\{1, \dots, n\} = E_1 \cup \cdots \cup E_r$, *such that* $\max E_i < \min E_{i+1}$ *for* $i = 1, \dots, r-1$. *Let* $f\colon \{1, \dots, r\} \to \{2, 3, \dots\}$ *be a nondecreasing map. Consider a hypergraph* $\mathscr{H}$ *such that* $H \in \mathscr{H}$ *iff*

(1) $|H \cap E_i| \leqslant 1$ *for* $i = 1, \dots, r$;

(2) *if* $\min H \in E_i$ *then* $|H| = f(i)$.

Then $\Pi_{\mathscr{H}}$ *is EL-shellable.*

Proof. Follows from Theorem 3.1. ∎

Remark 3.3. Several special cases of Corollary 3.2 were studied before:

(1) $|E_1| = \cdots = |E_r| = 1$, $f(i) = k$ for all $i = 1, \dots, r$. The posets $\Pi_{\mathscr{H}}$ are usually denoted by $\Pi_{n,k}$ in this case. They are the intersection lattices of the so-called k-equal arrangements. The EL-shellability of $\Pi_{n,k}$ has been proved in [BWa94]. Other properties of the lattice $\Pi_{n,k}$ have been studied in [BWe, SW].

(2) $|E_1| = \cdots = |E_r| = 1$, $f(i) = 2$, for $i = 1, \dots, l$ and $f(i) = k$ for $i = l+1, \dots, r$. These posets are denoted by $\Pi_{n,k}(l)$. They were first defined in [BWe], the homology groups of $\Pi_{n,k}(l)$ were also computed there. The shellability of $\Pi_{n,k}(l)$ has been first established in [Koz, Proposition 8.3].

(3) $|E_1| = n_1, \dots, |E_r| = n_r$, $f(1) = \cdots = f(l) = 2$, $f(l+1) = \cdots = f(r) = k$. These posets are denoted by $\Pi_{n_1, \dots, n_r, k}(l)$, they are the intersection lattices of the generalized k-equal arrangements. The posets $\Pi_{n_1, \dots, n_r, k}(l)$ were introduced in [BWe], suggested by a question of Vassiliev. An attempt to

compute their homology groups was taken there and the formulated result was surely correct, unfortunately the presented proof was wrong. The shellability of the generalized k-equal arrangements was also proved independently by Volkmar Welker, [We].

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