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On root categories of finite-dimensional algebras

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ABSTRACT

For any finite-dimensional algebra A over a field k of finite global dimension, we investigate the root category $\mathcal{R}_A$ as the triangulated hull of the 2-periodic orbit category of A via the construction of B. Keller in "On triangulated orbit categories". This is motivated by Ringel–Hall Lie algebras associated to 2-periodic triangulated categories. As an application, we study the Ringel–Hall Lie algebras for a class of finite-dimensional k-algebras of global dimension 2, which turns out to give an alternative answer to a question of GIM-Lie algebras by Slodowy in "Beyond Kac–Moody algebra, and inside".

1. Introduction

Since Gabriel's work [11], the connection between representation theory of finite-dimensional algebras and Lie theory has been revealed by many mathematicians. In [11], Gabriel showed the existence of a bijection between the isomorphism classes of all indecomposable modules over a hereditary algebra of Dynkin type and the positive roots of the corresponding semisimple Lie algebra. In [12], Happel introduced the root categories for finite-dimensional hereditary algebras. Thus Gabriel's bijection has been extended to a bijection between indecomposable objects of the root category of a hereditary algebra of Dynkin type and all the roots of the corresponding semisimple Lie algebra.

Let A be a finite-dimensional hereditary algebra over a field k. Let $\mathcal{D}^b(\operatorname{mod} A)$ be the derived category of finitely generated right A-modules. Then the root category $\mathcal{R}_A$ of A is defined to be the 2-periodic orbit category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$, where Σ is the suspension functor. It was proved by Peng and Xiao [23] that the root category $\mathcal{R}_A$ is triangulated via the homotopy category of 2-periodic complexes of A-modules. With this triangle structure, Peng and Xiao [24] constructed a so-called

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Ringel–Hall Lie algebra associated to each root category and realized all the symmetrizable derived Kac–Moody Lie algebras. In particular, this provides a concrete and useful realization of Gabriel's correspondence. In fact, Peng–Xiao's construction is valid for any Hom-finite 2-periodic triangulated categories over finite fields. In [21], Lin and Peng realized the elliptic Lie algebras of type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$ via the 2-periodic orbit categories (which are triangulated) of corresponding tubular algebras. However, in general, for arbitrary finite-dimensional k-algebra A, the 2-periodic orbit category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is no longer triangulated with the inherited triangle structure from the one of $\mathcal{D}^b(\operatorname{mod} A)$ (*cf.* Section 2.7 or [17]). Up to now, there are no suitable Hom-finite 2-periodic triangulated categories to realize the other elliptic Lie algebras via the Ringel–Hall Lie algebras approach. The aim of this note is to enlarge the application of Ringel–Hall Lie algebra approach and try to establish more links between representation theory of algebras and Lie theory.

Our motivation also comes from the interaction between singularity theory and Lie theory. Inspired by the theory that the universal deformation and simultaneous resolution of a simple singularity are described by the corresponding simple Lie algebras [6], K. Saito associated in [27], a generalization of root system to any regular weight systems [28], and asked how to construct a suitable Lie theory in order to reconstruct the primitive forms for the singularities [29]. This has been well-done for simple singularities and simple elliptic singularities. But, in general, it is not clear how to construct a suitable Lie theory even for the 14 exceptional unimodular singularities. Based on the duality theory of weight systems and the homological mirror symmetry, Kajiura, Saito and Takahashi [14,15] (Takahashi [32]) associated a triangulated category $\mathcal{T}_W$ to each (simple or unimodular) singularity W. The triangulated category $\mathcal{T}_W$ is equivalent to the bounded derived category of certain finite-dimensional algebra A_W. If W is a simple singularity, then A_W is the path algebra of a Dynkin quiver Q of the corresponding type of W. Applying Peng–Xiao's theorem to the root category $\mathcal{R}_{A_W} = \mathcal{R}_{kQ}$ of A_W, one gets the desired simple Lie algebra to reconstruct the primitive forms. This fact suggests to study the Ringel–Hall Lie algebra of A_W for a unimodular singularity W. However, the algebra A_W is no longer hereditary and never derived equivalent to a hereditary category. It is not clear whether the 2-periodic orbit category $\mathcal{D}^b(\operatorname{mod} A_W)/\Sigma^2$ is triangulated or not.

In this paper, we propose to associate a 2-periodic triangulated category $\mathcal{R}_A$ to any finite-dimensional k-algebra A of finite global dimension using Keller's construction [17]. By the construction of $\mathcal{R}_A$, we have an embedding of categories $i : \mathcal{D}^b(\operatorname{mod} A)/\Sigma^2 \hookrightarrow \mathcal{R}_A$. Moreover, if the 2-periodic orbit category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ admits a canonical triangle structure, then the embedding i is an equivalence, where the canonical triangle structure means that the canonical projection functor $\pi : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is triangulated. This happens if A is a finite-dimensional hereditary k-algebra. In other words, the category $\mathcal{R}_A$ of a finite-dimensional hereditary algebra A coincides with the root category of A in the sense of Happel [12]. We also remark that, using $\mathcal{R}_A$, one can easily construct 2-periodic triangulated categories whose Grothendieck groups realize the root lattices for any homogeneous elliptic Lie algebras. It would be interesting to study the relation between the Ringel–Hall Lie algebras of these categories and the corresponding elliptic Lie algebras in the future. On the other hand, Keller's construction is valid for any algebraic triangulated categories satisfying some finiteness conditions. In [10], the authors have considered the construction for algebraic triangulated categories generated by spherical objects and have determined the structure of the corresponding Ringel–Hall Lie algebras.

This paper is organized as follows: in Section 2, for any finite-dimensional k-algebra A of finite global dimension, we introduce the root category $\mathcal{R}_A$ and study its basic properties. It is a Hom-finite 2-periodic triangulated category and admits AR-triangles. We prove that the Grothendieck group of $\mathcal{R}_A$ is isomorphic to the Grothendieck group of the derived category $\mathcal{D}^b(\operatorname{mod} A)$. A concrete example is also given to show that the 2-periodic orbit category is not triangulated with the triangle structure inherited from the one of $\mathcal{D}^b(\operatorname{mod} A)$. Section 3 is devoted to investigate the root categories of representation-finite hereditary algebras. Such root categories characterize the algebras up to derived equivalence. In Section 4, we study the Ringel–Hall Lie algebras associated to the root categories of a class of finite-dimensional k-algebras of global dimension 2. It turns out that we have a negative answer to a question on GIM-Lie algebra asked by Slodowy [31]. Let us mention that different counterexamples have been discovered in [1] by using different approach. In Appendix A, we discuss the universal property of the root category and study recollement associated to root categories. One can use this to construct inductively various algebras whose corresponding 2-periodic orbit categories are not triangulated with the inherited triangle structure from the bounded derived categories.

Throughout this paper, we fix a field k. All algebras are finite-dimensional k-algebras of finite global dimension. All modules are right modules. Let $\mathcal{C}$ be a k-category. For any $X, Y \in \mathcal{C}$, we write $\mathcal{C}(X, Y)$ for $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. A triangulated subcategory $\mathcal{C}$ of $\mathcal{T}$ is called *thick* if $\mathcal{C}$ is closed under direct summands. For a subcategory $\mathcal{M}$ in a triangulated category $\mathcal{T}$, we denote by $\mathrm{tria}(\mathcal{M})$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{M}$. For a Lie algebra $\mathfrak{g}$, let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$.

2. Root categories for finite-dimensional algebras

2.1. Reminder on differential graded categories

We follow [16,19]. Let k be a field. A differential graded (= dg) k-module V is a complex of k-modules. The tensor product of two dg k-modules V, W is the graded k-module $V \otimes_k W$ endowed with the differential $d_V \otimes 1_W + 1_V \otimes d_W$, where d_V and d_W are the differentials of V and W respectively. A *dg category* is a k-category $\mathcal{A}$ whose morphism spaces are dg k-modules and whose compositions

$$\mathcal{A}(Y, Z) \otimes_k \mathcal{A}(X, Y) \to \mathcal{A}(X, Y), \quad X, Y, Z \in \mathcal{A}$$

are morphisms of dg k-modules (complex of k-modules). We identify a dg algebra with a dg category with one object.

The dg category $\mathcal{C}_{dg}(k)$ has as objects all dg k-modules and its morphisms are defined by

$$\mathcal{C}_{dg}(k)(V, W) = \bigoplus_{p \in \mathbb{Z}} \mathcal{C}_{dg}(k)(V, W)^p,$$

where $\mathcal{C}_{dg}(k)(V, W)^p$ is the k-module formed by morphisms $f : V \to W$ of graded k-modules of degree p. The differential of $\mathcal{C}_{dg}(k)(V, W)$ is the commutator

$$d(f) = d_W \circ f - (-1)^p f \circ d_V, \quad f \in \mathcal{C}_{dg}(k)(V, W)^p.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is given by a map $F : \mathrm{obj}\,\mathcal{A} \to \mathrm{obj}\,\mathcal{B}$ and by morphisms of dg k-modules

$$F(X, Y) : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY), \quad X, Y \in \mathrm{obj}\,\mathcal{A},$$

compatible with the compositions and the units.

A *right dg $\mathcal{A}$-module* M is a dg functor $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k)$, where $\mathcal{A}^{op}$ is the opposite dg category of $\mathcal{A}$. Let $\mathrm{Dif}\,\mathcal{A}$ be the dg category of right dg $\mathcal{A}$-modules. A dg $\mathcal{A}$-module P is called *$\mathcal{K}$-projective* if $\mathrm{Dif}\,\mathcal{A}(P, ?)$ preserves acyclicity. For any dg category $\mathcal{B}$, let $\mathcal{Z}^0(\mathcal{B})$ be the category with the same objects of $\mathcal{B}$ whose Hom-space is given by

$$\mathcal{Z}^0(\mathcal{B})(X, Y) = Z^0\big(\mathcal{B}(X, Y)\big),$$

i.e. the 0th cocycle of dg k-module $\mathcal{B}(X, Y)$. Let $\mathcal{H}^0(\mathcal{B})$ be the category with the same objects of $\mathcal{B}$ whose Hom-space is given by

$$\mathcal{H}^0(\mathcal{B})(X, Y) = H^0\big(\mathcal{B}(X, Y)\big),$$

i.e. the 0th homology of dg k-module $\mathcal{B}(X, Y)$. For the dg category $\mathrm{Dif}\,\mathcal{A}$, we define $\mathcal{C}(\mathcal{A}) := \mathcal{Z}^0(\mathrm{Dif}\,\mathcal{A})$ and $\mathcal{H}(\mathcal{A}) := \mathcal{H}^0(\mathrm{Dif}\,\mathcal{A})$. A morphism $L \to N$ in $\mathcal{C}(\mathcal{A})$ is called a *quasi-isomorphism* if it induces an isomorphism in homology. Let $\mathcal{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$, *i.e.* the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. A dg $\mathcal{A}$-module L is called *compact* if $\mathcal{D}(\mathcal{A})(L, ?)$ commutes with arbitrary direct sums. For instance, the projective $\mathcal{A}$-modules $\mathcal{A}(?, A)$, $A \in \mathcal{A}$ are both $\mathcal{K}$-projective and compact. Let $\mathrm{per}(\mathcal{A})$ be the perfect derived category of $\mathcal{A}$, *i.e.* the smallest subcategory of $\mathcal{D}(\mathcal{A})$ containing $\mathcal{A}(?, A)$, $A \in \mathcal{A}$ and stable under shift, extensions and passage to direct factors.

Let X be a dg $\mathcal{A}^{op} \otimes_k \mathcal{B}$-module. It gives rise to a pair of adjoint dg functors

$$\mathrm{Dif}\,\mathcal{A} \underset{H_X}{\overset{T_X}{\rightleftarrows}} \mathrm{Dif}\,\mathcal{B}.$$

Assume that X is $\mathcal{K}$-projective as an $\mathcal{A}^{op} \otimes_k \mathcal{B}$-module, then (T_X, H_X) induces an adjoint pair of triangle functors $(\mathsf{L}T_X, \mathsf{R}H_X)$ over the derived categories, where $\mathsf{L}T_X$ is the left derived functor of T_X. If both $\mathcal{A}$ and $\mathcal{B}$ are dg k-algebras, we also write $? \overset{L}{\otimes}_{\mathcal{A}} X_{\mathcal{B}}$ for $\mathsf{L}T_X$.

2.2. 2-Periodic orbit categories

Let A be a finite-dimensional k-algebra of finite global dimension. Let $\mathcal{D}^b(\mathrm{mod}\,A)$ be the bounded derived category of finitely generated A-modules and Σ the suspension functor. Consider the left total derived functor of $A^{op} \otimes_k A$-module $\Sigma^2 A$

$$\Sigma^2 = ? \overset{L}{\otimes}_A \Sigma^2 A : \mathcal{D}^b(\mathrm{mod}\,A) \to \mathcal{D}^b(\mathrm{mod}\,A),$$

which is an equivalence. For all L, M in $\mathcal{D}^b(\mathrm{mod}\,A)$, the vector space

$$\mathcal{D}^b(\mathrm{mod}\,A)\big(L, \Sigma^{2n} M\big)$$

vanishes for all but finitely many $n \in \mathbb{Z}$. The 2-*periodic orbit category*

$$\mathcal{D}^b(\mathrm{mod}\,A)/\Sigma^2$$

of A is defined as follows:

- the objects are the same as those of $\mathcal{D}^b(\mathrm{mod}\,A)$;
- if L and M are in $\mathcal{D}^b(\mathrm{mod}\,A)$, the space of morphisms is isomorphic to the space

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{D}^b(\mathrm{mod}\,A)\big(L, \Sigma^{2n} M\big).$$

The composition of morphisms is obvious. If A is hereditary, the orbit category is called the *root category* of A which was first introduced by D. Happel in [12].

A Hom-finite k-additive triangulated category $\mathcal{R}$ is called 2-*periodic triangulated* if:

- $\Sigma^2 \cong \mathbf{1}$, where Σ is the suspension functor of $\mathcal{R}$;
- the endomorphism ring $\mathrm{End}_{\mathcal{R}}(X)$ for any indecomposable object X is a finite-dimensional local k-algebra.

For any finite-dimensional algebra A over a field k, the homotopy category $\mathcal{H}_2(\mathcal{P})$ of 2-periodic complexes of finitely generated projective A-modules is a 2-periodic triangulated category. Using $\mathcal{H}_2(\mathcal{P})$, Peng and Xiao [23] proved that the 2-periodic orbit category $\mathcal{D}^b(\mathrm{mod}\,A)/\Sigma^2$ of a hereditary algebra A

is 2-periodic triangulated with canonical triangle structure, where the canonical triangle structure means that the canonical projection functor $\pi : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2$ is triangulated. However, this is not true in general. The first non-triangulated example is due to A. Neeman who considers the algebra of dual numbers $k[x]/(x^2)$. The 2-periodic orbit category of $k[x]/(x^2)$ is not triangulated (*cf.* Section 3 of [17]). Note that the algebra $k[x]/(x^2)$ in this example is of infinite global dimension.

2.3. Root category via Keller's construction

When $\mathcal{D}^b(\mathsf{mod}\, A)$ is triangle equivalent to the bounded derived category of a hereditary category, the 2-periodic orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2$ is triangulated (*cf.* [23,17]). But in general, the 2-periodic orbit category is not triangulated. However, a *triangulated hull* was defined in [17] as the algebraic triangulated category $\mathcal{R}_A$ with the following universal properties:

- There exists an algebraic triangulated functor $\pi_A : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{R}_A$.
- Let $\mathcal{B}$ be a dg category and X an object of $\mathcal{D}(A^{op} \otimes \mathcal{B})$. If there exists an isomorphism in $\mathcal{D}(A^{op} \otimes \mathcal{B})$ between $\Sigma^2 A \overset{L}{\otimes}_A X$ and X, then the algebraic triangulated functor $? \overset{L}{\otimes}_A X : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}(\mathcal{B})$ factorizes through π into an algebraic triangulated functor.

Consider A as a dg algebra concentrated in degree 0. Let $\mathcal{S}$ be the dg algebra with underlying complex $A \oplus \Sigma A$, where the multiplication is that of the trivial extension:

$$(a, b)\big(a', b'\big) = \big(aa', ab' + ba'\big).$$

Let $\mathcal{D}(\mathcal{S})$ be the derived category of $\mathcal{S}$ and $\mathcal{D}^b(\mathcal{S})$ the bounded derived category, *i.e.* the full subcategory of $\mathcal{D}(\mathcal{S})$ formed by the dg modules whose homology has finite total dimension over k. Let $\mathsf{per}(\mathcal{S})$ be the perfect derived category of $\mathcal{S}$, *i.e.* the smallest thick subcategory of $\mathcal{D}(\mathcal{S})$ containing $\mathcal{S}$. Clearly, the perfect derived category $\mathsf{per}(\mathcal{S})$ is contained in $\mathcal{D}^b(\mathcal{S})$. Denote by $p : \mathcal{S} \to A$ the canonical projection. It induces a triangle functor $p_* : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathcal{S})$. By composition we obtain a functor

$$\pi_A : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathcal{S}) \to \mathcal{D}^b(\mathcal{S})/\mathsf{per}(\mathcal{S}),$$

where the functor $\mathcal{D}^b(\mathcal{S}) \to \mathcal{D}^b(\mathcal{S})/\mathsf{per}(\mathcal{S})$ is the canonical localization functor. Let $\mathsf{tria}(p_*A)$ be the thick subcategory of $\mathcal{D}^b(\mathcal{S})$ generated by the image p_*A. It is clear that $\mathcal{S}$ belongs to $\mathsf{tria}(p_*A)$. Hence, $\mathsf{per}(\mathcal{S})$ is a thick subcategory of $\mathsf{tria}(p_*A)$. By Theorem 2 of [17], the triangulated hull of the orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2$ is the category

$$\mathcal{R}_A := \mathsf{tria}(p_*A)/\mathsf{per}(\mathcal{S}).$$

Moreover, there are embeddings $i : \mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2 \hookrightarrow \mathcal{R}_A$ of categories and $\mathcal{R}_A \hookrightarrow \mathcal{H}_2(\mathcal{P})$ of triangulated categories. If i is dense, then we say that the 2-periodic orbit category $\mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2$ is triangulated with inherited triangle structure from the one of $\mathcal{D}^b(\mathsf{mod}\, A)$. If A is a finite-dimensional hereditary algebra over k, the embedding i is dense by the main theorem of Keller [17]. In this case, using the universal properties of $\mathcal{R}_A$, one implies that the triangle structure given by Peng and Xiao [23] is the same as the one described by Keller. Furthermore, we have equivalences of triangulated categories $\mathcal{D}^b(\mathsf{mod}\, A)/\Sigma^2 \cong \mathcal{R}_A \cong \mathcal{H}_2(\mathcal{P})$ in this case.

We have the following nice characterization of $\mathcal{R}_A$.

Lemma 2.1. *Let A be a finite-dimensional k-algebra of finite global dimension and $\mathcal{S}$ the dg algebra associated to A as above. Let $p_* : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathcal{S})$ be the triangle functor induced by the canonical homomorphism $p : \mathcal{S} \to A$ of dg algebras, where A is viewed as a dg algebra concentrated in degree* 0. *We have $\mathcal{R}_A = \mathcal{D}^b(\mathcal{S})/\mathsf{per}(\mathcal{S})$.*

Proof. We have inclusions $\mathrm{per}(\mathcal{S}) \subseteq \mathrm{tria}(p_*A) \subseteq \mathcal{D}^b(\mathcal{S})$. By the definition of $\mathcal{R}_A$, it suffices to show that $\mathrm{tria}(p_*A) = \mathcal{D}^b(\mathcal{S})$.

Since $\mathcal{S}$ is a negative dg algebra. It is well-known that there is a canonical t-structure $(\mathcal{D}^{\leqslant}, \mathcal{D}^{\geqslant})$ induced by homology over $\mathcal{D}(\mathcal{S})$. In particular, $\mathcal{D}^{\leqslant}$ is the full subcategory of $\mathcal{D}(\mathcal{S})$ whose objects are the dg modules X such that the homology groups $H^p(X)$ vanish for all $p > 0$. The t-structure restricts to the subcategory $\mathcal{D}^b(\mathcal{S})$ of $\mathcal{D}(\mathcal{S})$ with heart $\mathcal{H}$ which is equivalent to $\mathrm{mod}\, A$. Thus, each object $X \in \mathcal{D}^b(\mathcal{S})$ is a finite iterated extension of objects in $\mathcal{H}$. But every object in $\mathcal{H}$ belongs to $\mathrm{tria}(p_*A)$, since A is of finite global dimension. This implies that $\mathcal{D}^b(\mathcal{S}) \subseteq \mathrm{tria}(p_*A)$. Therefore we have $\mathcal{D}^b(\mathcal{S}) = \mathrm{tria}(p_*A)$. $\square$

Definition 2.2. We call the triangulated hull $\mathcal{R}_A := \mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S})$ the root category of A and $\pi_A : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S}) = \mathcal{R}_A$ the canonical functor.

Remark 2.3. One can also consider the construction for the orbit category $\mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^{-2}$ which is in fact the same as $\mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^2$. Then one replaces the dg algebra $\mathcal{S}$ by $\mathcal{S}' = A \oplus \Sigma^{-3}A$. The root category can be defined as $\mathcal{R}_A = \mathrm{tria}(p_*A)/\mathrm{per}(\mathcal{S}')$.

2.4. Alternative description of $\mathcal{R}_A$

There is another description of $\mathcal{R}_A$ in [17]. Let $\mathcal{A}$ be the dg category of bounded complexes of finitely generated projective A-modules. Naturally, the tensor product of $\Sigma^2 A$ defines a dg functor from $\mathcal{A}$ to $\mathcal{A}$. Then one can form the *dg orbit category* $\mathcal{B}$ as the dg category with the same objects of $\mathcal{A}$ and such that for any $X, Y \in \mathcal{B}$, we have

$$\mathcal{B}(X, Y) \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{A}\big(X, \Sigma^{2n} Y\big).$$

Now we have an equivalence of categories

$$\mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^2 \cong \mathcal{H}^0(\mathcal{B}).$$

Let $\mathcal{D}(\mathcal{B})$ be the derived category of the dg category $\mathcal{B}$. Let $\mathcal{M}$ be the triangulated subcategory of $\mathcal{D}(\mathcal{B})$ generated by the representable functors. The following is a special case of the remarkable theorem due to Keller (Theorem 2 in [17], *cf.* also [18]).

Theorem 2.4. *The category $\mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S})$ is triangle equivalent to the triangulated hull $\mathcal{M}$.*

In fact, Theorem 2 in [17] only implies that $\mathrm{tria}(p_*A)/\mathrm{per}(\mathcal{S}) \cong \mathcal{M}$. But by Lemma 2.1 we have $\mathrm{tria}(p_*A)/\mathrm{per}(\mathcal{S}) \cong \mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S})$ in the 2-periodic case. This equivalence was induced by the embedding $\mathcal{H}^0(\mathcal{B}) \cong \mathcal{D}^b(\mathrm{mod}\, A)/\Sigma^2 \xrightarrow{i} \mathcal{R}_A$.

Using this description, we have the following.

Proposition 2.5. *Let A be a finite-dimensional k-algebra of finite global dimension. Then the root category $\mathcal{R}_A$ is a* Hom*-finite* 2*-periodic triangulated category.*

Proof. The Hom-finiteness follows from the description of $\mathcal{M}$, since the homomorphisms between representable functors of $\mathcal{B}$ are finite-dimensional over k. Consider the $\mathcal{B}^{op} \otimes \mathcal{B}$-module X: $X(A, B) = \mathcal{B}(A, B)$ for any $A, B \in \mathcal{B}$, it induces the identity functor

$$\mathbf{1} : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}).$$

Let Y be the $\mathcal{B}^{op} \otimes \mathcal{B}$-module such that $Y(A, B) = \Sigma^2 \mathcal{B}(A, B)$ for any $A, B \in \mathcal{B}$. Clearly, the module Y induces the triangle functor

$$\Sigma^2 : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}).$$

By the definition of the dg orbit category $\mathcal{B}$, we deduce that X is isomorphic to Y as $\mathcal{B}^{op} \otimes \mathcal{B}$-modules, which induces an invertible morphism $\eta : \mathbf{1} \to \Sigma^2$ by Lemma 6.1 of [16]. Thus, to show that $\mathcal{R}_A$ is a 2-periodic triangulated category, we only need to show that $\mathcal{R}_A$ is a Krull–Schmidt category. It suffices to prove that each idempotent morphism of $\mathcal{R}_A$ is split, *i.e.* $\mathcal{R}_A$ is idempotent completed. In fact, $\mathcal{D}(\mathcal{B})$ admits arbitrary direct sums, which implies that $\mathcal{D}(\mathcal{B})$ is idempotent completed. Recall that $\mathcal{R}_A = \mathcal{M} \subset \mathcal{D}(\mathcal{B})$ is closed under direct summands in $\mathcal{D}(\mathcal{B})$, the result follows from the well-known fact that if an additive category $\mathcal{C}$ is idempotent completed, then a full subcategory $\mathcal{D}$ of $\mathcal{C}$ is idempotent completed if and only if $\mathcal{D}$ is closed under direct summands. □

2.5. Serre functor over $\mathcal{R}_A$

Keep the notations as above. Let $D = \mathrm{Hom}_k(?, k)$ be the usual duality over k. The $\mathcal{S}^{op} \otimes_k \mathcal{S}$-module $D\mathcal{S}$ induces a triangle functor

$$? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} : \mathcal{D}(\mathcal{S}) \to \mathcal{D}(\mathcal{S}).$$

We have the following well-known fact (see *e.g.* Lemma 1.2.1 of [2]).

Lemma 2.6. *There is a non-degenerate bilinear form*

$$\alpha_{X,Y} : \mathcal{D}(\mathcal{S})(X, Y) \times \mathcal{D}(\mathcal{S})(Y, X \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}) \to k,$$

which is bifunctorial for $X \in \mathrm{per}(\mathcal{S})$ and $Y \in \mathcal{D}^b(\mathcal{S})$.

Proposition 2.7. *The functor $? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$ restricts to auto-equivalences*

$$? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} : \mathcal{D}^b(\mathcal{S}) \to \mathcal{D}^b(\mathcal{S}) \quad \text{and} \quad ? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} : \mathrm{per}(\mathcal{S}) \to \mathrm{per}(\mathcal{S}).$$

Proof. Since A is of finite global dimension, we know that $DA \in \mathrm{per}\, A$ and hence $D\mathcal{S} \in \mathrm{per}\, \mathcal{S}$. Similarly, we have $\mathcal{S} \in \mathrm{tria}(D\mathcal{S}) \subseteq \mathcal{D}(\mathcal{S})$. This particularly implies that $D\mathcal{S}$ is a small generator of $\mathcal{D}(\mathcal{S})$. It is not hard to show that

$$\mathcal{D}(\mathcal{S})\big(\mathcal{S}, \Sigma^n \mathcal{S}\big) \cong \mathcal{D}(\mathcal{S})\big(D\mathcal{S}, \Sigma^n D\mathcal{S}\big), \quad n \in \mathbb{Z}.$$

Thus by Lemma 4.2 of [16], we know that $? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$ is an equivalence over $\mathcal{D}(\mathcal{S})$. Now the functor $? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$ restricts to $\mathrm{per}(\mathcal{S})$ follows from $\mathrm{tria}(D\mathcal{S}) = \mathrm{per}(\mathcal{S})$.

Consider the cofibrant resolution of A as a right $\mathcal{S}$-module, one computes directly that

$$H^1(A \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}) \cong DA \quad \text{and} \quad H^i(A \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}) = 0 \quad \text{for } i \neq 1.$$

By the existence of the canonical t-structure of $\mathcal{D}^b(\mathcal{S})$, we have $A \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} \cong \Sigma^{-1} DA$ in $\mathcal{D}^b(\mathcal{S})$. Now again by the finiteness of the global dimension of A, we have $p_* A \in \mathrm{tria}(p_*(DA)) \subseteq \mathcal{D}^b(\mathcal{S})$. In particular, we have $\mathrm{tria}(p_*(DA)) = \mathcal{D}^b(\mathcal{S})$ as $\mathrm{tria}(p_* A) = \mathcal{D}^b(\mathcal{S})$. Thus, $? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$ restricts to an equivalence $? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} : \mathcal{D}^b(\mathcal{S}) \to \mathcal{D}^b(\mathcal{S})$. □

Before stating the next result, we recall Amiot's construction [3] of bilinear form for quotient category. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N} \subset \mathcal{T}$ a thick subcategory of $\mathcal{T}$. Assume ν is an auto-equivalence of $\mathcal{T}$ such that $\nu(\mathcal{N}) \subset \mathcal{N}$. Moreover, we assume that there is a non-degenerate bilinear form:

$$\beta_{N,X} : \mathcal{T}(N, X) \times \mathcal{T}(X, \nu N) \to k,$$

which is bifunctorial in $N \in \mathcal{N}$ and $X \in \mathcal{T}$. Let $X, Y \in \mathcal{T}$. A morphism $p : N \to X$ is called a *local $\mathcal{N}$-cover of X relative to Y* if N is in $\mathcal{N}$ and it induces an exact sequence:

$$0 \to \mathcal{T}(X, Y) \xrightarrow{p^*} \mathcal{T}(N, Y).$$

The following theorem is due to Amiot (Lemma 1.1 and Theorem 1.3.1 of [3]).

Theorem 2.8.

1) *The bilinear form β naturally induces a bilinear form*

$$\beta'_{X,Y} : \mathcal{T}/\mathcal{N}(X, Y) \times \mathcal{T}/\mathcal{N}\big(Y, \nu\Sigma^{-1}X\big) \to k,$$

which is bifunctorial for $X, Y \in \mathcal{T}/\mathcal{N}$.

2) *Assume further $\mathcal{T}$ is* Hom*-finite. If there exists a local $\mathcal{N}$-cover of X relative to Y and a local $\mathcal{N}$-cover of νY relative to X, then the bilinear form $\beta'_{X,Y}$ is non-degenerate.*

Recall that $\mathcal{R}_A = \mathcal{D}^b(\mathcal{S})/\mathrm{per}(\mathcal{S})$. Now we have the following

Proposition 2.9.

1) *The bilinear form α induces a bifunctorial bilinear form α':*

$$\alpha'_{X,Y} : \mathcal{R}_A(X, Y) \times \mathcal{R}_A\big(Y, \Sigma^{-1}X \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}\big) \to k.$$

2) *The bilinear form α' is non-degenerate over $\mathcal{R}_A$.*

Proof. The first statement follows from Lemma 2.6, Proposition 2.7 and Theorem 2.8 directly.

The proof of part 2) is quite related to the proof of Theorem 4.3 of [2]. Let $P_A = \mathrm{Tot}(\cdots \to \Sigma^n\mathcal{S} \to \Sigma^{n-1}\mathcal{S} \to \cdots \to \Sigma^2\mathcal{S} \to \Sigma\mathcal{S} \to \mathcal{S} \to 0 \to \cdots)$, *i.e.* P_A is the cofibrant resolution of $\mathcal{S}$-module A. Then one can easily check that $\mathcal{D}^b(\mathcal{S})(A, \Sigma^m A)$ is finite-dimensional over k for any $m \in \mathbb{Z}$. In particular, we have

$$\mathcal{D}^b(\mathcal{S})\big(A, \Sigma^{2m}A\big) \cong A \quad \text{and} \quad \mathcal{D}^b(\mathcal{S})\big(A, \Sigma^{2m+1}A\big) = 0,$$

for $m \geqslant 0$ and $\mathcal{D}^b(\mathcal{S})(A, \Sigma^m A) = 0$ for $m < 0$. Since $p_*(A) = A \in \mathcal{D}^b(\mathcal{S})$ generates the category $\mathcal{D}^b(\mathcal{S})$, which implies that $\mathcal{D}^b(\mathcal{S})$ is Hom-finite, *i.e.* for any $X, Y \in \mathcal{D}^b(\mathcal{S})$, we have $\dim_k \mathcal{D}^b(\mathcal{S})(X, Y) < \infty$. Since the non-degeneracy is extension closed, it suffices to show that $\alpha'_{\Sigma^n A, \Sigma^m A}$ is non-degenerate. Equivalently, it suffices to show that $\alpha'_{A, \Sigma^n A}$ is non-degenerate for any $n \in \mathbb{Z}$. By 2) of Theorem 2.8, it suffices to show that there exists a local $\mathrm{per}(\mathcal{S})$-cover of A relative to $\Sigma^n A$ and a local $\mathrm{per}(\mathcal{S})$-cover of $\Sigma^n A$ relative to $A \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$. For $n < 0$, since $\mathcal{D}^b(\mathcal{S})(A, \Sigma^n A) = 0$, one can take $p : \mathcal{S} \to A$ as the local $\mathrm{per}(\mathcal{S})$-cover of A relative to $\Sigma^n A$. Now assume that $n \geqslant 0$. Let

$$P_{A,\Sigma^n A} := \mathrm{Tot}\big(\cdots \to 0 \to \Sigma^n\mathcal{S} \to \Sigma^{n-1}\mathcal{S} \to \cdots \to \Sigma\mathcal{S} \to \mathcal{S} \to 0 \to \cdots\big).$$

Clearly $P_{A,\Sigma^n A} \in \operatorname{per}(\mathcal{S})$. One can easily see that $p : P_{A,\Sigma^n A} \to A$ is a local $\operatorname{per}(\mathcal{S})$-cover of A relative to $\Sigma^n A$.

On the other hand, note that $A \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S} \cong \Sigma^{-1} DA$. A local $\operatorname{per}(\mathcal{S})$-cover of $\Sigma^n A$ relative to $\Sigma^{-1} DA$ is equivalent to a local $\operatorname{per}(\mathcal{S})$-cover of A relative to $\Sigma^{-n-1} DA$. If $n \geqslant 0$, we have $\mathcal{D}^b(\mathcal{S})(A, \Sigma^{-n-1} DA) = 0$ and $p : \mathcal{S} \to A$ is a local $\operatorname{per}(\mathcal{S})$-cover of A relative to $\Sigma^{-n-1} DA$. Suppose that $n < 0$. One can show that

$$P_{A,\Sigma^{-n-1} DA} := \operatorname{Tot}\bigl(\cdots \to 0 \to \Sigma^{-n-1}\mathcal{S} \to \Sigma^{-n-2}\mathcal{S} \to \cdots \to \Sigma\mathcal{S} \to \mathcal{S} \to 0 \to \cdots\bigr) \to A$$

is a local $\operatorname{per}(\mathcal{S})$-cover of A relative to $\Sigma^{-n-1} DA$. $\square$

The main result in this subsection is the following.

Theorem 2.10. *The root category $\mathcal{R}_A$ admits Auslander–Reiten triangles.*

Proof. Proposition 2.9 implies that $\Sigma^{-1}? \overset{L}{\otimes}_{\mathcal{S}} D\mathcal{S}$ is the Serre functor over $\mathcal{R}_A = \mathcal{D}^b(\mathcal{S})/\operatorname{per}\mathcal{S}$. By [25], one deduces that $\mathcal{R}_A$ admits Auslander–Reiten triangles. $\square$

2.6. The Grothendieck group of $\mathcal{R}_A$

We first recall the definition of the Euler bilinear form for a 2-periodic triangulated category. Let $\mathcal{R}$ be a 2-periodic triangulated category and $\mathsf{G}_0(\mathcal{R})$ the associated Grothendieck group. The Euler bilinear form $\chi_{\mathcal{R}}(-,-)$ on $\mathsf{G}_0(\mathcal{R})$ is defined to be

$$\chi_{\mathcal{R}}\bigl([X],[Y]\bigr) = \dim_k \mathcal{R}(X,Y) - \dim_k \mathcal{R}(X,\Sigma Y),$$

where $X, Y \in \mathcal{R}$. We claim that it is well-defined due to the 2-periodic property. Let

$$L \to X \to M \xrightarrow{t} \Sigma L$$

be any triangle in $\mathcal{R}$. By applying the functor $\mathcal{R}(-,Y)$, we obtain a long exact sequence

$$\begin{aligned} \cdots \to \mathcal{R}(\Sigma L, Y) \xrightarrow{t^*} \mathcal{R}(M,Y) \to \mathcal{R}(X,Y) \to \mathcal{R}(L,Y) \\ \to \mathcal{R}\bigl(\Sigma^{-1}M, Y\bigr) \to \mathcal{R}\bigl(\Sigma^{-1}X, Y\bigr) \to \mathcal{R}\bigl(\Sigma^{-1}L, Y\bigr) \xrightarrow{(\Sigma^{-2}t)^*} \mathcal{R}\bigl(\Sigma^{-2}M, Y\bigr) \to \cdots. \end{aligned}$$

Note that $\Sigma^{-2}t \cong t$ as $\Sigma^2 \cong \mathbf{1}$, which implies $\operatorname{im} t^* = \operatorname{im}(\Sigma^{-2}t)^*$. Hence we have

$$\chi_{\mathcal{R}}\bigl([X],[Y]\bigr) = \chi_{\mathcal{R}}\bigl([L],[Y]\bigr) + \chi_{\mathcal{R}}\bigl([M],[Y]\bigr).$$

Dually, one can show that if $K \to Y \to N \to \Sigma K$ is a triangle in $\mathcal{R}$, then we have

$$\chi_{\mathcal{R}}\bigl([X],[Y]\bigr) = \chi_{\mathcal{R}}\bigl([X],[K]\bigr) + \chi_{\mathcal{R}}\bigl([X],[N]\bigr).$$

Let A be a finite-dimensional k-algebra of finite global dimension and $\mathcal{R}_A$ the corresponding root category. Let $\mathsf{G}_0(\mathcal{R}_A)$ and $\mathsf{G}_0(\mathcal{D}^b(\operatorname{mod} A))$ be the Grothendieck groups of $\mathcal{R}_A$ and $\mathcal{D}^b(\operatorname{mod} A)$ respectively. Let $\chi_{\mathcal{R}_A}(-,-)$ be the Euler bilinear form over $\mathcal{R}_A$ defined as above and let $\chi_A(-,-)$ be the Euler bilinear form over $\mathsf{G}_0(\mathcal{D}^b(\operatorname{mod} A))$, *i.e.*

$$\chi_A\big([X],[Y]\big)=\sum_{i\in\mathbb{Z}}(-1)^i\dim_k\mathcal{D}^b(\mathsf{mod}\,A)\big(X,\Sigma^iY\big)$$

for any $X, Y\in\mathcal{D}^b(\mathsf{mod}\,A)$.

We are now in the position to state the main result of this subsection.

Proposition 2.11. *Let A be a finite-dimensional k-algebra of finite global dimension and $\mathcal{R}_A$ the corresponding root category. Let $\pi_A:\mathcal{D}^b(\mathsf{mod}\,A)\to\mathcal{R}_A$ be the canonical triangle functor. Then the functor π_A induces an isomorphism $\pi_A^*:\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))\to\mathsf{G}_0(\mathcal{R}_A)$ of Grothendieck groups which preserves the Euler bilinear form.*

Proof. Suppose that the algebra A has exactly n non-isomorphic simple modules, say $S_1,\ldots.S_n$ and let $P_1,\ldots,P_n$ be the corresponding projective covers. It is well-known that $\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))\cong\mathbb{Z}[S_1]\oplus\cdots\oplus\mathbb{Z}[S_n]$. Let $\mathcal{S}$ be the dg algebra defined in Section 2.3. Let $p_*:\mathcal{D}^b(\mathsf{mod}\,A)\to\mathcal{D}^b(\mathcal{S})$ be the triangle functor induced by the canonical homomorphism $p:\mathcal{S}\to A$ of dg algebras. Recall that the canonical functor $\pi_A:\mathcal{D}^b(\mathsf{mod}\,A)\to\mathcal{R}_A$ is the composition of the functor $p_*:\mathcal{D}^b(\mathsf{mod}\,A)\to\mathcal{D}^b(\mathcal{S})$ with the canonical localization $\mathcal{D}^b(\mathcal{S})\to\mathcal{D}^b(\mathcal{S})/\mathsf{per}(\mathcal{S})=\mathcal{R}_A$. We first show that the functor p_* induces an isomorphism of groups $p^*:\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))\to\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))$.

Since $\mathcal{S}$ is a negative dg algebra, there is a canonical t-structure over $\mathcal{D}^b(\mathcal{S})$ whose heart $\mathcal{H}$ is equivalent to $\mathsf{mod}\,A$. As in the proof of Lemma 2.1, each object of X in $\mathcal{D}^b(\mathcal{S})$ is a finite iterated extension of $\mathcal{H}$. Hence, the image $[X]$ of X in the Grothendieck group $\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))$ is a finite linear combination of the images of objects in $\mathcal{H}$. By the equivalence $\mathcal{H}\cong\mathsf{mod}\,A$ and the assumption that A is of finite global dimension, for any $N\in\mathcal{H}$, the image $[N]$ is a finite linear combination of $[p_*S_1],\ldots,[p_*S_n]$. Therefore, for any object X in $\mathcal{D}^b(\mathcal{S})$, the image $[X]$ of X is a finite linear combination of $[p_*S_1],\ldots,[p_*S_n]$. Let $i:A\to\mathcal{S}$ be the injective homomorphism of dg algebras, we have the induced triangle functor $i_*:\mathcal{D}^b(\mathcal{S})\to\mathcal{D}^b(\mathsf{mod}\,A)$ and the associated homomorphism of groups $i^*:\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))\to\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))$. Since $p\circ i=1_A$, we have $i^*\circ p^*=\mathbf{1}_{\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))}$, which implies that p^* is injective. Hence, $[p_*S_1],\ldots,[p_*S_n]$ are linearly independent over $\mathbb{Z}$ and form a $\mathbb{Z}$-basis of $\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))$. In particular, $p^*:\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))\to\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))$ is an isomorphism of groups.

We have the following exact sequence of triangulated categories

$$\mathsf{per}(\mathcal{S})\rightarrowtail\mathcal{D}^b(\mathcal{S})\twoheadrightarrow\mathcal{R}_A,$$

which induces an exact sequence of Grothendieck groups

$$\mathsf{G}_0\big(\mathsf{per}(\mathcal{S})\big)\xrightarrow{\psi}\mathsf{G}_0\big(\mathcal{D}^b(\mathcal{S})\big)\xrightarrow{\phi}\mathsf{G}_0(\mathcal{R}_A)\to 0.$$

In particular, we have $\mathsf{G}_0(\mathcal{R}_A)\cong\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))/\mathrm{im}\,\psi$. Let $e_1,\ldots,e_n$ be the orthogonal primitive idempotent elements of A. It is clear that $i(e_j)$, $j=1,\ldots,n$ are orthogonal idempotent elements of $\mathcal{S}$. Let $\widetilde{P}_j=i(e_j)\mathcal{S}$. By using the existence of canonical t-structure over $\mathcal{D}^b(\mathcal{S})$, it is not hard to see that $\widetilde{P}_j$, $j=1,\ldots,n$ are all the indecomposable direct summands of $\mathcal{S}$ in $\mathcal{D}^b(\mathcal{S})$. Since $\mathcal{S}$ is negative, each compact object is a finite extension of direct sum of $\Sigma^n\widetilde{P}_j$, $n\in\mathbb{Z}$ (*cf.* [16, Remark 5.3]). Thus for any $X\in\mathsf{per}(\mathcal{S})$, $[X]$ is a finite linear combination of $[\widetilde{P}_i]$, $i=1,\ldots,n$ in the Grothendieck group $\mathsf{G}_0(\mathsf{per}(\mathcal{S}))$. But $\psi([\widetilde{P}_j])=0$ in $\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))$ since $\psi([\widetilde{P}_j])=[P_j]+[\Sigma P_j]=0$. Therefore $\mathrm{im}\,\psi=0$ and $\phi:\mathsf{G}_0(\mathcal{D}^b(\mathcal{S}))\to\mathsf{G}_0(\mathcal{R}_A)$ is an isomorphism of groups.

Since the canonical functor $\pi_A:\mathcal{D}^b(\mathsf{mod}\,A)\to\mathcal{R}_A$ is the composition of p_* with the localization functor $\mathcal{D}^b(\mathcal{S})\to\mathcal{D}^b(\mathcal{S})/\mathsf{per}(\mathcal{S})=\mathcal{R}_A$, we infer that $\pi_A^*=p^*\circ\phi:\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))\to\mathsf{G}_0(\mathcal{R}_A)$ is an isomorphism of groups. The isomorphism π_A^* preserves the Euler bilinear form follows from a direct calculation and the definition of $\chi_{\mathcal{R}_A}$. $\square$

Remark 2.12. If A is a finite-dimensional hereditary k-algebra, then $\mathcal{R}_A\cong\mathcal{D}^b(\mathsf{mod}\,A)/\Sigma^2$. It follows that $\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A)/\Sigma^2)\cong\mathsf{G}_0(\mathcal{D}^b(\mathsf{mod}\,A))$.

2.7. A minimal example

Let Q be the following quiver

$$2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1.$$

Let A be the quotient of the path algebra kQ by the ideal generated by $\beta \circ \alpha$. Then A is representation-finite and has global dimension 2. Let $\mathcal{D}^b(\operatorname{mod} A)$ be the bounded derived category of finitely generated right A-modules. Let $\mathcal{A}$ be the dg enhancement of $\mathcal{D}^b(\operatorname{mod} A)$, *i.e.* the dg category of bounded complexes of finitely generated projective A-modules. Let $\Sigma^2 : \mathcal{A} \to \mathcal{A}$ be the dg enhancement of the square of suspension functor of $\mathcal{D}^b(\operatorname{mod} A)$. Let $\mathcal{B}$ be the dg orbit category of $\mathcal{A}$ respect to Σ^2 (*cf.* Section 2.4). The canonical dg functor $\pi : \mathcal{A} \to \mathcal{B}$ yields an $\mathcal{A}^{op} \otimes_k \mathcal{B}$-module

$$(B, A) \to \mathcal{B}(B, \pi A),$$

which induces the standard functors

$$\mathcal{D}(\mathcal{A}) \underset{\pi_\rho}{\overset{\pi_*}{\rightleftarrows}} \mathcal{D}(\mathcal{B}).$$

We also have a canonical triangle equivalence $F : \mathcal{D}(\operatorname{Mod} A) \to \mathcal{D}(\mathcal{A})$. Now the composition

$$\mathcal{D}^b(\operatorname{mod} A) \hookrightarrow \mathcal{D}(\operatorname{Mod} A) \xrightarrow{F} \mathcal{D}(\mathcal{A}) \xrightarrow{\pi_*} \mathcal{D}(\mathcal{B})$$

gives the canonical functor $\pi_A : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{R}_A$.

Proposition 2.13. *The canonical functor $\pi_A : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{R}_A$ is not dense.*

Proof. We will construct an object in $\mathcal{R}_A$ which is not in the image of π_A. Let S_i be the simple A-modules associated to the vertices i and P_i the corresponding indecomposable projective modules, $i = 1, 2$. Let $l : P_2 \to P_1$ be the embedding and $\gamma : P_1 \twoheadrightarrow S_1 \hookrightarrow P_2$ the composition of $P_1 \twoheadrightarrow S_1$ with $S_1 \hookrightarrow P_2$. Let X be the complex $\cdots \to 0 \to P_2 \xrightarrow{(l,0)} P_1 \oplus P_2 \xrightarrow{(0,l)^t} P_1 \to 0 \cdots$, where $P_1 \oplus P_2$ is in the 0th component. Let Y be the complex $\cdots \to 0 \to 0 \to P_2 \xrightarrow{0} P_2 \to 0 \cdots$, where the left P_2 is in the 0th component. Let f be the following morphism from X to Y in $\mathcal{D}^b(\operatorname{mod} A)$

$$f: \quad \begin{tikzcd} 0 \arrow[r] & P_2 \arrow[r, "{(l,0)}"] \arrow[d, "0"] & P_1 \oplus P_2 \arrow[r, "{(0,l)^t}"] \arrow[d, "{(\gamma,1)^t}"] & P_1 \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & P_2 \arrow[r, "0"] & P_2 \arrow[r] & 0, \end{tikzcd}$$

and g be the following morphism from X to $\Sigma^2 Y$ in $\mathcal{D}^b(\operatorname{mod} A)$

$$g: \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, "0"] & P_2 \arrow[r, "{(l,0)}"] \arrow[d, "1"] & P_1 \oplus P_2 \arrow[r, "{(0,l)^t}"] \arrow[d, "0"] & P_1 \arrow[r] & 0 \\ P_2 \arrow[r, "0"] & P_2 \arrow[r] & 0. \end{tikzcd}$$

We claim that the mapping cone of $\pi_A(f+g)$ is not in the image of π_A. Let

$$\pi_A(X) \xrightarrow{\pi_A(f+g)} \pi_A(Y) \to Z \to \Sigma\pi_A(X)$$

be the distinguished triangle. Applying the functor π_ρ, we get a triangle in $\mathcal{D}(\operatorname{Mod} A)$

$$\pi_\rho\pi_A(X) \xrightarrow{\pi_\rho\pi_A(f+g)} \pi_\rho\pi_A(Y) \to \pi_\rho Z \to \Sigma\pi_{\rho\pi_A(X)}.$$

Note that for any $X \in \mathcal{D}^b(\operatorname{mod} A)$, we have $\pi_\rho\pi_A(X) \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i}X$. Thus, $\pi_\rho Z$ is isomorphic to the mapping cone of the following chain map of complexes

$$\begin{tikzcd} \cdots \arrow[r] & P_1\oplus P_2 \arrow[r, "\left(\begin{smallmatrix}0&0\\l&0\end{smallmatrix}\right)"] \arrow[d, "(\gamma,1)^t"] & P_1\oplus P_2 \arrow[r, "\left(\begin{smallmatrix}0&0\\l&0\end{smallmatrix}\right)"] \arrow[d, "(\gamma,1)^t"] & P_1\oplus P_2 \arrow[r, "\left(\begin{smallmatrix}0&0\\l&0\end{smallmatrix}\right)"] \arrow[d, "(\gamma,1)^t"] & P_1\oplus P_2 \arrow[r] \arrow[d, "(\gamma,1)^t"] & \cdots \\ \cdots \arrow[r] & P_2 \arrow[r, "0"] & P_2 \arrow[r, "0"] & P_2 \arrow[r, "0"] & P_2 \arrow[r] & \cdots. \end{tikzcd}$$

In particular, the mapping cone is

$$\cdots \longrightarrow P_2\oplus P_1\oplus P_2 \xrightarrow{\begin{pmatrix}0&0&0\\-\gamma&0&0\\-1&-l&0\end{pmatrix}} P_2\oplus P_1\oplus P_2 \xrightarrow{\begin{pmatrix}0&0&0\\-\gamma&0&0\\-1&-l&0\end{pmatrix}} P_2\oplus P_1\oplus P_2 \longrightarrow \cdots.$$

Denote by h the composition $P_1 \twoheadrightarrow S_1 \hookrightarrow P_1$ and consider the complex $P : \cdots \to P_1 \xrightarrow{h} P_1 \xrightarrow{h} P_1 \to \cdots$. It is easy to check that the following is a quasi-isomorphism

$$\begin{tikzcd} \cdots \arrow[r] & P_2\oplus P_1\oplus P_2 \arrow[r, "\begin{pmatrix}0&0&0\\-\gamma&0&0\\-1&-l&0\end{pmatrix}"] \arrow[d, "(-l,1,0)^t"] & P_2\oplus P_1\oplus P_2 \arrow[r, "\begin{pmatrix}0&0&0\\-\gamma&0&0\\-1&-l&0\end{pmatrix}"] \arrow[d, "(-l,1,0)^t"] & P_2\oplus P_1\oplus P_2 \arrow[r] \arrow[d, "(-l,1,0)^t"] & \cdots \\ \cdots \arrow[r] & P_1 \arrow[r, "h"] & P_1 \arrow[r, "h"] & P_1 \arrow[r, "h"] & \cdots. \end{tikzcd}$$

In particular, $\pi_\rho Z$ is isomorphic to P in $\mathcal{D}(\operatorname{Mod} A)$. If there exists $U \in \mathcal{D}^b(\operatorname{mod} A)$ such that $\pi_A(U) = Z$, then $\pi_\rho Z \cong \bigoplus_{i\in Z} \Sigma^{2i}U$. But one can easily show that P is indecomposable in $\mathcal{D}(\operatorname{Mod} A)$. This completes the proof. □

This example implies that in general the orbit category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is not triangulated even if A is of small global dimension. In Appendix A, we propose a way to construct various examples from a known one by using recollement associated to root categories. It would be interesting to know whether there is an algebra A without oriented cycles such that the orbit category $\mathcal{D}^b(\operatorname{mod} A)/\Sigma^2$ is not triangulated with the inherited triangle structure.

3. The ADE root categories

In this section, we focus our attention on the root categories of finite-dimensional hereditary algebras of Dynkin type. We show that such root categories characterize these algebras up to derived equivalence.

3.1. Separation of AR-components

Let A be a finite-dimensional k-algebra of finite global dimension. Let $\pi_A : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{R}_A$ be the canonical triangle functor. By Theorem 2.10, we know that $\mathcal{R}_A$ has Auslander–Reiten triangles (AR-triangles). When π_A is dense, it is quite easy to show that π_A preserves the AR-triangles, *i.e.* each AR-triangle of $\mathcal{R}_A$ comes from an AR-triangle of $\mathcal{D}^b(\mathrm{mod}\, A)$ via the canonical functor π_A. In general, we have the following.

Theorem 3.1. *Let A be a finite-dimensional k-algebra of finite global dimension and $\pi_A : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{R}_A$ the canonical functor. Then the functor π_A maps AR-triangles of $\mathcal{D}^b(\mathrm{mod}\, A)$ to AR-triangles of $\mathcal{R}_A$. As a consequence, there is no irreducible morphism between $\mathrm{im}\,\pi_A$ and $\mathcal{R}_A \setminus \mathrm{im}\,\pi_A$.*

Proof. Recall that for arbitrary objects $X, Y \in \mathcal{D}^b(\mathrm{mod}\, A)$, we have the canonical isomorphism

$$\mathcal{R}_A\big(\pi_A(X), \pi_A(Y)\big) \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{D}^b(\mathrm{mod}\, A)\big(\Sigma^{2i}X, Y\big),$$

and $\mathcal{D}^b(\mathrm{mod}\, A)(\Sigma^{2i}X, Y)$ vanishes for all but finitely many i. Denote by S and $\widetilde{S}$ the Serre functors of $\mathcal{D}^b(\mathrm{mod}\, A)$ and $\mathcal{R}_A$ respectively. Firstly, we show that $\pi_A S(X) \cong \widetilde{S}\pi_A(X)$ for any indecomposable object $X \in \mathcal{D}^b(\mathrm{mod}\, A)$. Consider the functor $D\mathcal{R}_A(?, \pi_A S(X))$ over $\mathcal{R}_A$, where $D = \mathrm{Hom}_k(?, k)$ is the usual duality of k. We have the following canonical isomorphism

$$\begin{aligned} D\mathcal{R}_A\big(\pi_A X, \pi_A S(X)\big) &\cong D\Big(\bigoplus_{i\in\mathbb{Z}} \mathcal{D}^b(\mathrm{mod}\, A)\big(\Sigma^{2i}X, S(X)\big)\Big) \\ &\cong \bigoplus_{i\in\mathbb{Z}} D\mathcal{D}^b(\mathrm{mod}\, A)\big(\Sigma^{2i}X, S(X)\big) \\ &\cong \bigoplus_{i\in\mathbb{Z}} \mathcal{D}^b(\mathrm{mod}\, A)\big(X, \Sigma^{2i}X\big) \\ &\cong \mathcal{R}_A(\pi_A X, \pi_A X). \end{aligned}$$

The indecomposable property implies that $\mathcal{R}_A(\pi_A X, \pi_A X)$ is a local k-algebra. Let $\eta \in D\mathcal{R}_A(\pi_A X, \pi_A S(X))$ be the image of $1_{\pi_A X} \in \mathcal{R}_A(\pi_A X, \pi_A X)$ via the canonical isomorphism. Let $\eta^* : \mathcal{R}_A(\pi_A X, ?) \to D\mathcal{R}_A(?, \pi_A S(X))$ be the natural transformation corresponding to η. It is clear that $\eta^*|_{\mathrm{im}\,\pi_A}$ is an isomorphism. Furthermore, if $f : Y \to Z$ is a morphism in $\mathcal{R}_A$ such that η^*_Y and η^*_Z are isomorphisms, then $\eta^*_{Cone(f)}$ is an isomorphism. Since $\mathcal{R}_A$ is the triangulated hull of $\mathrm{im}\,\pi_A$, one deduces that η^* is an isomorphism over $\mathcal{R}_A$. In particular, $D\mathcal{R}_A(?, \pi_A S(X))$ is representable. On the other hand, the Serre functor $\widetilde{S}$ implies that $D\mathcal{R}_A(?, \widetilde{S}\pi_A X)$ is also represented by $\mathcal{R}_A(\pi_A X, ?)$. Thus, we have $\pi_A S(X) \cong \widetilde{S}\pi_A X$.

Let $\Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X)$ be an AR-triangle in $\mathcal{D}^b(\mathrm{mod}\, A)$. Let $\pi_A(\Sigma^{-1}SX) \xrightarrow{u} W \to \pi_A X \xrightarrow{v} \pi_A S(X)$ be the AR-triangle in $\mathcal{R}_A$. Clearly, $\pi_A(f)$ is not a split monomorphism. Hence there is a morphism $t : W \to \pi_A Y$ such that $\pi_A(f) = t \circ u$. Namely, we have the following commutative diagram of triangles

$$\begin{tikzcd} \pi_A(\Sigma^{-1}SX) \arrow[r, "u"] \arrow[d, equal] & W \arrow[r] \arrow[d, "t"] & \pi_A X \arrow[r, "v"] \arrow[d, "s"] & \pi_A S(X) \arrow[d, equal] \\ \pi_A(\Sigma^{-1}SX) \arrow[r, "\pi_A(f)"] & \pi_A(Y) \arrow[r, "\pi_A(g)"] & \pi_A X \arrow[r, "\pi_A(h)"] & \pi_A S(X). \end{tikzcd}$$

We claim that s is an isomorphism. Otherwise s is nilpotent by the indecomposability of X. Since $\Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X)$ is an AR-triangle, we have $\pi_A(h) \circ s = 0$, which implies that $v = 0$, a contradiction. Thus, t is also an isomorphism. In particular, the image of $\Sigma^{-1}SX \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} S(X)$ is indeed an AR-triangle of $\mathcal{R}_A$.

Now one can easily deduce that there is no irreducible morphism between $\operatorname{im}\pi_A$ and $\mathcal{R}_A \backslash \operatorname{im}\pi_A$, which completes the proof. $\square$

Remark 3.2. For generalized cluster categories [3], the separation property of AR-components has been proved in [4] by using the theory of graded algebra (Theorem 5.2 in [4]). Let $\mathcal{C}_A$ be the generalized cluster category associated to A and $\pi_A : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{C}_A$ the canonical functor. Let S and $\widetilde{S}$ be the Serre functors of $\mathcal{D}^b(\operatorname{mod} A)$ and $\mathcal{C}_A$ respectively. By the 2-Calabi–Yau property of $\mathcal{C}_A$, one deduces that $\widetilde{S} \circ \pi_A(X) \cong \pi_A \circ S(X)$ for any $X \in \mathcal{D}^b(\operatorname{mod} A)$. Then one shows that the functor π_A preserves AR-triangles as above. This gives an alternative proof for Theorem 5.2 in [4].

3.2. *The ADE root categories*

Let A and B be finite-dimensional k-algebras of finite global dimension. If A and B are derived equivalent, it is clear that $\mathcal{R}_A \cong \mathcal{R}_B$. But the converse is not known in general. In what follows we will characterize the algebras sharing the root category with the path algebra of a Dynkin quiver. Since the derived category of a Dynkin quiver is independent of the choice of the orientation, we assume Q to be one of the following quivers for simplicity.

$$A_n: \quad 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

$$D_n: \quad \begin{tikzcd}[row sep=small] 1 \arrow[dr] & & & & \\ & 3 \arrow[r] & 4 \arrow[r] & \cdots \arrow[r] & n \\ 2 \arrow[ur] & & & & \end{tikzcd}$$

$$E_6: \quad \begin{tikzcd} & & 3 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \end{tikzcd}$$

$$E_7: \quad \begin{tikzcd} & & 3 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

$$E_8: \quad \begin{tikzcd} & & 3 \arrow[d] & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8. \end{tikzcd}$$

Theorem 3.3. *Let A be a finite-dimensional k-algebra of finite global dimension. If the root category $\mathcal{R}_A$ is equivalent to $\mathcal{R}_{kQ}$ for some Dynkin quiver Q, then A is derived equivalent to kQ.*

Proof. Since Q is a connected Dynkin quiver, the AR-quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ is connected. The canonical functor $\pi_{kQ} : \mathcal{D}^b(\operatorname{mod} kQ) \to \mathcal{R}_{kQ}$ is dense, which implies that the AR-quiver of $\mathcal{R}_{kQ}$ is connected. By Theorem 3.1, we know that the functor $\pi_A : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{R}_A$ is dense. In particular, each

$X \in \mathcal{R}_A$ has at least one preimage in $\mathcal{D}^b(\mathrm{mod}\,A)$. Let P_i, $i = 1, \ldots, n$ be the indecomposable projective kQ-modules. It is clear that

$$\dim_k \mathcal{R}_{kQ}(\pi_{kQ} P_i, \pi_{kQ} P_j) \leqslant 1 \quad \text{for } i \leqslant j \quad \text{and} \quad \mathcal{R}_{kQ}(\pi_{kQ} P_i, \pi_{kQ} P_j) = 0 \quad \text{for } i > j.$$

Let $F : \mathcal{R}_{kQ} \to \mathcal{R}_A$ be the triangle equivalence. We claim that there is an object M in $\mathcal{D}^b(\mathrm{mod}\,A)$ such that

$$\pi_A(M) = F\big(\pi_{kQ}(kQ)\big) \quad \text{and} \quad \mathcal{D}^b(\mathrm{mod}\,A)\big(M, \Sigma^t M\big) = 0 \quad \text{for } t \neq 0.$$

Let $\{\Sigma^{2r} X_i \mid r \in \mathbb{Z}\}$ be the set of preimages of $F(\pi_{kQ}(P_i))$ in $\mathcal{D}^b(\mathrm{mod}\,A)$. The condition that $\pi_A(M) = F(\pi_{kQ}(kQ)) = F(\pi_{kQ}(P_1 \oplus \cdots \oplus P_n))$ implies that M has exactly n non-isomorphic indecomposable direct summands, say $M_1, \ldots, M_n$. We assume that $M_i \in \{\Sigma^{2r} X_i \mid r \in \mathbb{Z}\}$, $i = 1, \ldots, n$ and construct M as follows.

Step 1: Choosing $M_n \in \{\Sigma^{2r} X_n \mid r \in \mathbb{Z}\}$ arbitrarily, since n is the unique sink vertex;
Step 2: Suppose M_j, $1 < j \leqslant n$ have been chosen. Let I_j be the set of vertices i such that there is an arrow from vertex i to j. Consider the full subquiver of Q whose vertices are j and $i \in I_j$. Since

$$\dim_k \mathcal{R}_A\big(F\big(\pi_{kQ}(P_i)\big), F\big(\pi_{kQ}(P_j)\big)\big) = \dim_k \mathcal{R}_{kQ}\big(\pi_{kQ}(P_i), \pi_{kQ}(P_j)\big) = 1$$

for each $i \in I_j$, there exists a unique $r_i \in \mathbb{Z}$ such that

$$\dim_k \mathcal{D}^b(\mathrm{mod}\,A)\big(\Sigma^{2r_i} X_i, M_j\big) = 1 \quad \text{and} \quad \mathcal{D}^b(\mathrm{mod}\,A)\big(\Sigma^{2t} X_i, M_j\big) = 0 \quad \text{for } t \neq r_i.$$

Then set $M_i = \Sigma^{2r_i} X_i$ for each $i \in I_j$;
Step 3: Repeat step 2 until $j = 1$.

It is clear that $\pi_A(M) \cong F(\pi_{kQ}(kQ))$ and

$$\mathcal{D}^b(\mathrm{mod}\,A)\big(M, \Sigma^{2r} M\big) = 0 \quad \text{for } r \neq 0.$$

On the other hand, we have

$$\mathcal{R}_{kQ}\big(\pi_{kQ}(kQ), \Sigma\pi_{kQ}(kQ)\big) = \bigoplus_{r\in\mathbb{Z}} \mathcal{D}^b(\mathrm{mod}\,kQ)\big(kQ, \Sigma^{2r+1} kQ\big) = 0,$$

which implies $\mathcal{D}^b(\mathrm{mod}\,A)(M, \Sigma^{2r+1} M) = 0$ for any $r \in \mathbb{Z}$. In particular, M is a (partial) tilting complex of $\mathcal{D}^b(\mathrm{mod}\,A)$. We have $\mathcal{D}^b(\mathrm{mod}\,kQ) \cong \mathcal{D}^b(\mathrm{mod}\,\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\,A)}(M)) \cong \mathrm{tria}(M)$, where $\mathrm{tria}(M)$ is the smallest thick subcategory of $\mathcal{D}^b(\mathrm{mod}\,A)$ containing M. We now prove that $\mathrm{tria}(M) = \mathcal{D}^b(\mathrm{mod}\,A)$. Let $i : \mathcal{D}^b(\mathrm{mod}\,kQ) \xrightarrow{\sim} \mathrm{tria}(M) \hookrightarrow \mathcal{D}^b(\mathrm{mod}\,A)$ be the composition. By the universal property of the root category, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}\,kQ) \arrow[r, hook, "i"] \arrow[d, "\pi_{kQ}"] & \mathcal{D}^b(\mathrm{mod}\,A) \arrow[d, "\pi_A"] \\
\mathcal{R}_{kQ} \arrow[r, "\bar{i}"] & \mathcal{R}_A,
\end{tikzcd}$$

where $\bar{i}$ is induced by the full embedding i. By Lemma A.2, we know that $\bar{i}$ is also fully faithful, thus an equivalence. It follows that i is dense and an equivalence. We remark that there is no reason that the induced functor $\bar{i}$ coincides with $F : \mathcal{R}_{kQ} \to \mathcal{R}_A$. □

3.3. Tame quiver of type $\widetilde{D}$ and $\widetilde{E}$

Let Q be one of the following quivers

$$\widetilde{D}_n: \begin{tikzcd} & 2 \arrow[d] & & & n-1 \arrow[d] & \\ 1 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n-2 \arrow[r] & n \arrow[r] & n+1 \end{tikzcd}$$

$$\widetilde{E}_6: \begin{tikzcd} & & 3 \arrow[d] & & \\ & & 4 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

$$\widetilde{E}_7: \begin{tikzcd} & & & 4 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \end{tikzcd}$$

$$\widetilde{E}_8: \begin{tikzcd} & & 3 \arrow[d] & & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \arrow[r] & 9. \end{tikzcd}$$

Theorem 3.3 also holds for tame quiver of type $\widetilde{D}$ and $\widetilde{E}$. One can adapt a variant proof of Theorem 3.3.

Proposition 3.4. *Let A be a finite-dimensional k-algebra of finite global dimension. If the root category $\mathcal{R}_A$ is equivalent to $\mathcal{R}_{kQ}$ for some tame quiver Q of type $\widetilde{D}$ or $\widetilde{E}$, then A is derived equivalent to kQ.*

Proof. It suffices to prove this proposition for Q as one of the above quivers. Note that the canonical functor $\pi_{kQ} : \mathcal{D}^b(\operatorname{mod} kQ) \to \mathcal{R}_{kQ}$ is dense. Let $F : \mathcal{R}_{kQ} \to \mathcal{R}_A$ be the triangle equivalence. It is well-known that the AR-quiver of $\mathcal{D}^b(\operatorname{mod} kQ)$ and hence the AR-quiver of $\mathcal{R}_{kQ} \cong \mathcal{R}_A$ is the union of preprojective–preinjective component and tubes up to shifts. We claim that the intersection of $\operatorname{im} \pi_A$ with the preprojective–preinjective component is nonempty. Otherwise, we have $\operatorname{im} \pi_A \subseteq F(T) \cup \Sigma F(T)$, where T is the union of kQ-modules in the tubes. It is clear that T is a hereditary abelian subcategory of $\operatorname{mod} kQ$. By Theorem 9.1 of [17], we know that $\mathcal{D}^b(T)/\Sigma^2$ is triangulated and we have the following commutative diagram

$$\begin{tikzcd} \mathcal{D}^b(T) \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathcal{D}^b(\operatorname{mod} kQ) \arrow[d, "\pi_{kQ}"] \\ \mathcal{D}^b(T)/\Sigma^2 \arrow[r, "\bar{i}"] & \mathcal{R}_{kQ}, \end{tikzcd}$$

where $\bar{i}$ is induced by i. In particular, the functor $\bar{i}$ is a full embedding. Now $\operatorname{im} \pi_A \subseteq F(T) \cup \Sigma F(T)$ implies that $\operatorname{tria}(\operatorname{im} \pi_A) \subseteq \operatorname{im} F \circ \bar{i}$, which contradicts to $\operatorname{tria}(\operatorname{im} \pi_A) = \mathcal{R}_A$.

Therefore the intersection of the image $\operatorname{im}\pi_A$ with the preprojective–preinjective component is nonempty. By Theorem 3.1, every object in this component belongs to $\operatorname{im}\pi_A$. In particular, we have a preimage of $F(\pi_{kQ}(kQ))$ in $\mathcal{D}^b(\operatorname{mod} A)$. Note that in these cases, we have

$$\dim_k \mathcal{R}_{kQ}\big(\pi_{kQ}(P_i), \pi_{kQ}(P_j)\big) \leqslant 1 \quad \text{for } i \leqslant j \quad \text{and} \quad \mathcal{R}_{kQ}\big(\pi_{kQ}(P_i), \pi_{kQ}(P_j)\big) = 0 \quad \text{for } i > j.$$

Then one can adapt the proof of Theorem 3.3 to deduce the desired result. □

4. Ringel–Hall Lie algebras and GIM-Lie algebras

Throughout this section, let k be a field with $|k| = q$. We study the Ringel–Hall Lie algebras of the root categories for a class of finite-dimensional k-algebras of global dimension 2. Building on the representation theory of these algebras, we give a negative answer to a question on GIM-Lie algebras addressed by Slodowy in [31]. We mention here that different counterexamples have been discovered by Alpen [1] who considered fixed point subalgebras of certain Lie algebras.

4.1. Generalized intersection matrix Lie algebras

We recall the generalized intersection matrix Lie algebra (GIM-Lie algebra for short) following Slodowy [31]. A matrix $C = (c_{ij}) \in M_l(\mathbb{Z})$ is called a *generalized intersection matrix*, or GIM for short, if the following are satisfied

$$c_{ii} = 2,$$
$$c_{ij} < 0 \iff c_{ji} < 0,$$
$$c_{ij} > 0 \iff c_{ji} > 0.$$

If moreover C is symmetric, then C is called an *intersection matrix* (IM for short). Note that if the off-diagonal elements of a GIM C are non-positive, then C is a generalized Cartan matrix. A generalized intersection matrix C is called *symmetrizable*, if there exists an invertible diagonal matrix $D = \operatorname{diag}\{d_1, \ldots, d_l\}$ such that DC is symmetric.

Given a GIM $C \in M_l(\mathbb{Z})$, a *root basis* with structural matrix C is a triplet $(H, \triangledown, \triangle)$ consisting of

- a finite-dimensional $\mathbb{Q}$-vector space H;
- a family $\triangledown = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\}$, where $\alpha_i^\vee \in H$;
- a family $\triangle = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i \in H^* = \operatorname{Hom}_{\mathbb{Q}}(H, \mathbb{Q})$

satisfy the following

1) both sets $\triangle$ and $\triangledown$ are linearly independent;
2) $\alpha_j(\alpha_i^\vee) = c_{ij}$ for all $1 \leqslant i, j \leqslant l$;
3) $\dim_{\mathbb{Q}} H = 2l - \operatorname{rank} C$.

We call a root basis a *GIM-root basis* (resp. an IM-root basis, resp. a GCM-root basis) if its structural matrix is a GIM (resp. an IM, resp. a generalized Cartan matrix).

The *GIM-Lie algebra* $\operatorname{gim}(C)$ attached to the root basis $(H, \triangledown, \triangle)$ is given by the generators $\mathfrak{h} = H \otimes_{\mathbb{Q}} \mathbb{C}$ and $e_{\pm\alpha}$, $\alpha \in \triangle$ satisfying the following relations:

$$(1) \quad \big[h, h'\big] = 0, \quad h, h' \in \mathfrak{h},$$

$$(2) \quad [h, e_\alpha] = \alpha(h) e_\alpha, \quad h \in \mathfrak{h},\ \alpha \in \pm\triangle,$$

$$\begin{aligned}&(3)\quad [e_\alpha, e_{-\alpha}] = \alpha^\vee, \quad \alpha \in \triangle,\\ &(4)\quad ad(e_\alpha)^{\max(1,1-\beta(\alpha^\vee))} e_\beta = 0, \quad \alpha \in \triangle,\ \beta \in \pm\triangle,\\ &(5)\quad ad(e_{-\alpha})^{\max(1,1-\beta(-\alpha^\vee))} e_\beta = 0, \quad \alpha \in \triangle,\ \beta \in \pm\triangle.\end{aligned}$$

If C is a symmetrizable generalized Cartan matrix, then the $\mathrm{gim}(C)$ is the Kac–Moody algebra associated to $(H, \triangledown, \triangle)$.

Let $ad : \mathrm{gim}(C) \to \mathrm{End}(\mathrm{gim}(C))$ be the adjoint representation of $\mathrm{gim}(C)$. Consider the restriction of ad to $\mathfrak{h}$, the Lie algebra $\mathrm{gim}(C)$ decomposes into a direct sum

$$\mathrm{gim}(C) = \bigoplus_{\gamma \in \mathfrak{h}^*} \mathrm{gim}(C)_\gamma$$

of eigenspaces

$$\mathrm{gim}(C)_\gamma = \bigl\{x \in \mathrm{gim}(C) \bigm| [h, x] = \gamma(h)x \text{ for all } h \in \mathfrak{h}\bigr\}.$$

Let $R \subset \mathfrak{h}^*$ be the set of all $\gamma \in \mathfrak{h}^*$ such that $\mathrm{gim}(C)_\gamma \neq \{0\}$. A non-zero $\alpha \in R$ is called a *root* of $\mathrm{gim}(C)$. We have $R \subset \Gamma := \mathbb{Z}\triangle$, where Γ is called the *root lattice* of $\mathrm{gim}(C)$.

It is clear that $\mathfrak{h} \subseteq \mathrm{gim}(C)_0$. The following question has been addressed in [31] by Slodowy.

Question 4.1. Do we have $\mathfrak{h} = \mathrm{gim}(C)_0$?

Let $\mathrm{gim}(C)' := [\mathrm{gim}(C), \mathrm{gim}(C)]$ be the derived subalgebra of $\mathrm{gim}(C)$. It is known that $\mathrm{gim}(C) = \mathrm{gim}(C)' + \mathfrak{h}$. Set $\mathfrak{h}' := \sum_{i=1}^{l} \mathbb{C}\alpha_i^\vee \subset \mathfrak{h}$, then $\mathrm{gim}(C)' \cap \mathfrak{h} = \mathfrak{h}'$ and $\mathrm{gim}(C)' \cap \mathrm{gim}(C)_\gamma = \mathrm{gim}(C)_\gamma$ for $\gamma \neq 0$. We remark that the derived subalgebra $\mathrm{gim}(C)'$ can be presented by generators $\alpha_i^\vee$, $1 \leqslant i \leqslant l$ and e_α, $\alpha \in \pm\triangle$ with the same relations of $\mathrm{gim}(C)$.

Clearly, Question 4.1 is equivalent to the following: Do we have $\dim_{\mathbb{C}} \mathrm{gim}(C)'_0 = l$? In [1], Alpen has given a negative answer by using Lie theory. In the following, a totally different approach is given, using representation theory of finite-dimensional algebras.

4.2. IM-Lie algebras

Let C be a generalized intersection matrix and $(H, \triangledown, \triangle)$ a root basis associated to C. For any $\alpha \in \triangle$, let $s_\alpha : H \to H$ be the transformation

$$s_\alpha(h) = h - \alpha(h)\alpha^\vee, \quad h \in H.$$

The contragredient action of s_α on H^* is given by

$$s_\alpha(\gamma) = \gamma - \gamma\bigl(\alpha^\vee\bigr)\alpha, \quad \gamma \in H^*.$$

The *Weyl group* W of the root basis $(H, \triangledown, \triangle)$ is defined to be the subgroup of $\mathrm{Aut}(H)$ generated by the transformations s_α, $\alpha \in \triangle$.

Two GIM-root bases $(H, \triangledown, \triangle)$ and $(H, \triangledown', \triangle')$ are called *braid equivalent* [31] if they can be transformed into each other by a sequence

$$(H, \triangledown, \triangle) = (H, \triangledown_1, \triangle_1) \mapsto (H, \triangledown_2, \triangle_2) \mapsto \cdots \mapsto (H, \triangledown_m, \triangle_m) = \bigl(H, \triangledown', \triangle'\bigr)$$

of transformations of the form

$$\nabla_{k+1} = \left(\nabla_k \setminus \left\{\beta^\vee\right\}\right) \cup \left(s_\alpha\left(\beta^\vee\right)\right),$$
$$\triangle_{k+1} = \left(\triangle_k \setminus \{\beta\}\right) \cup \left(s_\alpha(\beta)\right)$$

for some $\alpha, \beta \in \triangle_k$, $k = 1, \ldots, m-1$.

It is known that there are braid equivalent GIM-root bases giving rise to non-isomorphic GIM-Lie algebras. To remedy this defect, Slodowy [31] introduced another class of Lie algebras for intersection matrix.

Let C be an intersection matrix and $(H, \nabla, \triangle)$ the associated IM-root basis. The symmetric structural matrix $C = (c_{\alpha\beta})_{\alpha,\beta\in\triangle}$ induces a symmetric bilinear form

$$(\,,\,) : \Gamma \times \Gamma \to \mathbb{Z}$$

over the root lattice $\Gamma = \mathbb{Z}\triangle$ by

$$(\alpha, \beta) := c_{\alpha\beta}, \quad \text{for } \alpha, \beta \in \triangle.$$

Let $\mathsf{gim}(C)$ be the GIM-Lie algebra associated to the root basis $(H, \nabla, \triangle)$. Let τ be the ideal of $\mathsf{gim}(C)$ generated by all the elements $x \in \mathsf{gim}(C)_\gamma$ with $(\gamma, \gamma) > 2$. The *IM-Lie algebra* associated to the root basis $(H, \nabla, \triangle)$ is defined to be the quotient algebra

$$\mathsf{im}(C) := \mathsf{gim}(C)/\tau.$$

If C is a symmetric generalized Cartan matrix, we have $\mathsf{gim}(C) = \mathsf{im}(C) = \mathsf{gcm}(C)$, where $\mathsf{gcm}(C)$ is the Kac–Moody algebra associated to C.

The following theorem has been proved by Slodowy [31].

Theorem 4.2. *Let $(H, \nabla, \triangle)$ and $(H, \nabla', \triangle')$ be braid equivalent IM-root bases. Let C_1 and C_2 be the corresponding structural matrices of $(H, \nabla, \triangle)$ and $(H, \nabla', \triangle')$ respectively. Then the IM-Lie algebra $\mathsf{im}(C_1)$ is isomorphic to $\mathsf{im}(C_2)$.*

4.3. *The Ringel–Hall Lie algebra*

We recall the definition of the Ringel–Hall Lie algebra of a 2-periodic triangulated category following [24] (*cf.* also [34,35,10]). Let $\mathcal{R}$ be a Hom-finite k-linear triangulated category with suspension functor Σ. By $\mathsf{ind}\,\mathcal{R}$ we denote a set of representatives of the isoclasses of all indecomposable objects in $\mathcal{R}$.

Given any objects X, Y, L in $\mathcal{R}$, we define

$$W(X, Y; L) = \left\{(f, g, h) \in \mathsf{Hom}_{\mathcal{R}}(X, L) \times \mathsf{Hom}_{\mathcal{R}}(L, Y) \times \mathsf{Hom}_{\mathcal{R}}(Y, \Sigma X) \;\middle|\; X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} \Sigma X \text{ is a triangle}\right\}.$$

The action of $\mathsf{Aut}(X) \times \mathsf{Aut}(Y)$ on $W(X, Y; L)$ induces the orbit space

$$V(X, Y; L) = \left\{(f, g, h)^\wedge \;\middle|\; (f, g, h) \in W(X, Y; L)\right\},$$

where

$$(f, g, h)^\wedge = \left\{\left(af, gc^{-1}, ch(\Sigma a)^{-1}\right) \;\middle|\; (a, c) \in \mathsf{Aut}(X) \times \mathsf{Aut}(Y)\right\}.$$

Let $\mathsf{Hom}_{\mathcal{R}}(X, L)_Y$ be the subset of $\mathsf{Hom}_{\mathcal{R}}(X, L)$ consisting of morphisms $l : X \to L$ whose mapping cone $Cone(l)$ is isomorphic to Y. Consider the action of the group $\mathsf{Aut}(X)$ on $\mathsf{Hom}_{\mathcal{R}}(X, L)_Y$ by $d \cdot l = dl$, the orbit is denoted by l^* and the orbit space is denoted by $\mathrm{Hom}_{\mathcal{R}}(X, L)^*_Y$. Dually one can consider the subset $\mathrm{Hom}_{\mathcal{R}}(L, Y)_{\Sigma X}$ of $\mathrm{Hom}_{\mathcal{R}}(L, Y)$ with the group action $\mathrm{Aut}(Y)$ and the orbit space $\mathrm{Hom}_{\mathcal{R}}(L, Y)^*_{\Sigma X}$. The following proposition is an observation of [33].

Lemma 4.3. $|V(X, Y; L)| = |\mathrm{Hom}_{\mathcal{R}}(X, L)^*_Y| = |\mathrm{Hom}_{\mathcal{R}}(L, Y)^*_{\Sigma X}|$.

In the following, we set $F^L_{YX} = |V(X, Y; L)|$.

We assume further that $\mathcal{R}$ is 2-*periodic*, *i.e.* $\mathcal{R}$ is Krull–Schmidt and $\Sigma^2 \cong 1$.

Let $\mathrm{G}_0(\mathcal{R})$ be the Grothendieck group of $\mathcal{R}$ and $I_{\mathcal{R}}(-,-)$ the symmetric Euler bilinear form of $\mathcal{R}$, where $I_{\mathcal{R}}(-,-)$ is defined to be

$$I_{\mathcal{R}}\big([X],[Y]\big) = \chi_{\mathcal{R}}\big([X],[Y]\big) + \chi_{\mathcal{R}}\big([Y],[X]\big)$$

for any $[X],[Y] \in \mathrm{G}_0(\mathcal{R})$. Let $\mathfrak{h}$ be the subgroup of $\mathrm{G}_0(\mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by $\frac{[M]}{d(M)}$, where $M \in \operatorname{ind}\mathcal{R}$ and $d(M) = \dim_k(\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M))$. One can naturally extend the symmetric Euler bilinear form to $\mathfrak{h} \times \mathfrak{h}$. Let $\mathfrak{n}$ be the free abelian group with basis $\{u_X \mid X \in \operatorname{ind}\mathcal{R}\}$. Let

$$\mathfrak{g}(\mathcal{R}) = \mathfrak{h} \oplus \mathfrak{n}$$

be a direct sum of $\mathbb{Z}$-modules. Consider the quotient group

$$\mathfrak{g}(\mathcal{R})_{(q-1)} = \mathfrak{g}(\mathcal{R})/(q-1)\mathfrak{g}(\mathcal{R}).$$

By abuse of notations, for any $M \in \mathcal{R}$ we still use u_M, $[M]$ to denote the corresponding residues in $\mathfrak{g}(\mathcal{R})_{(q-1)}$. Now define a bilinear operation $[-,-]$ on $\mathfrak{g}(\mathcal{R})_{(q-1)}$ as follows.

(a) For any indecomposable objects $X, Y \in \mathcal{R}$,

$$[u_X, u_Y] = \sum_{L \in \operatorname{ind}\mathcal{R}} \big(F^L_{YX} - F^L_{XY}\big)u_L - \delta_{X,\Sigma Y}\frac{[X]}{d(X)},$$

where $\delta_{X,\Sigma Y} = 1$ for $X \cong \Sigma Y$ and 0 else.

(b) $[\mathfrak{h}, \mathfrak{h}] = 0$.

(c) For any objects $X, Y \in \mathcal{R}$ with Y indecomposable,

$$\big[[X], u_Y\big] = I_{\mathcal{R}}\big([X],[Y]\big)u_Y, \qquad \big[u_Y, [X]\big] = -\big[[X], u_Y\big].$$

The following remarkable theorem is due to Peng and Xiao (Theorem 3.4 of [24], *cf.* also [34]).

Theorem 4.4. *Together with the operation* $[-,-]$, $\mathfrak{g}(\mathcal{R})_{(q-1)}$ *is a Lie algebra over* $\mathbb{Z}/(q-1)\mathbb{Z}$.

Let us mention that, for an arbitrary finite-dimensional algebra A over $\mathbb{C}$, Xiao, Xu and Zhang [34] have proposed to study the homotopy category $\mathcal{H}_2(\mathcal{P})$ of 2-periodic complexes of finitely generated projective A-modules and have given a geometric construction of a Lie algebra over $\mathbb{C}$ directly instead of over finite fields like in [24].

A triangulated category $\mathcal{T}$ is called *proper*, if for any non-zero indecomposable object $X \in \mathcal{T}$, $[X]$ is non-zero in the Grothendieck group $\mathrm{G}_0(\mathcal{T})$. If the 2-periodic triangulated category $\mathcal{R}$ is proper, then $[u_X, u_{\Sigma X}] = -\frac{[X]}{d(X)}$, which coincides the origin definition in [24]. Note that the proof in [24] is still valid for non-proper 2-periodic triangulated category for the Lie bracket defined above (*cf.* [34]).

We have the following functorial property of $\mathfrak{g}(\mathcal{R})_{(q-1)}$ (*cf.* Corollary 1.16 of [30]).

Lemma 4.5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be 2-periodic triangulated categories over k and $G : \mathcal{R}_1 \to \mathcal{R}_2$ a fully faithful triangle functor. Then G induces a homomorphism of Lie algebras $\widetilde{G} : \mathfrak{g}(\mathcal{R}_1)_{(q-1)} \to \mathfrak{g}(\mathcal{R}_2)_{(q-1)}$.*

Proof. Let $\mathfrak{h}_1$ (resp. $\mathfrak{h}_2$) be the subgroup of $\mathrm{G}_0(\mathcal{R}_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\mathrm{G}_0(\mathcal{R}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$) generated by $\frac{[M]}{d(M)}$ with $M \in \operatorname{ind} \mathcal{R}_1$ (resp. $M \in \operatorname{ind} \mathcal{R}_2$). Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the free abelian groups with bases $\{u_X \mid X \in \operatorname{ind} \mathcal{R}_1\}$ and $\{u_Y \mid Y \in \operatorname{ind} \mathcal{R}_2\}$ respectively. We have

$$\mathfrak{g}(\mathcal{R}_1)_{(q-1)} = \mathfrak{h}_1/(q-1)\mathfrak{h}_1 \oplus \mathfrak{n}_1/(q-1)\mathfrak{n}_1 \quad \text{and} \quad \mathfrak{g}(\mathcal{R}_2)_{(q-1)} = \mathfrak{h}_2/(q-1)\mathfrak{h}_2 \oplus \mathfrak{n}_2/(q-1)\mathfrak{n}_2.$$

Let $G^* : \mathrm{G}_0(\mathcal{R}_1) \to \mathrm{G}_0(\mathcal{R}_2)$ be the homomorphism of groups induced by $G : \mathcal{R}_1 \to \mathcal{R}_2$. Since G is fully faithful, it is not hard to see that G^* induces a homomorphism of abelian groups $\widetilde{G} : \mathfrak{h}_1/(q-1)\mathfrak{h}_1 \to \mathfrak{h}_2/(q-1)\mathfrak{h}_2$. We extend $\widetilde{G}$ linearly to the whole $\mathbb{Z}/(q-1)\mathbb{Z}$-space $\mathfrak{g}(\mathcal{R}_1)_{(q-1)}$ by setting $\widetilde{G}(u_X) = u_{GX}$ for $X \in \operatorname{ind} \mathcal{R}_1$. This is well-defined, since $\{u_X \mid X \in \operatorname{ind} \mathcal{R}_1\}$ form a $\mathbb{Z}/(q-1)\mathbb{Z}$-basis of $\mathfrak{n}_1/(q-1)\mathfrak{n}_1$ and G is fully faithful.

In order to show that $\widetilde{G}$ is a homomorphism of Lie algebras, it suffices to show that $\widetilde{G}$ preserves the Lie operations (a), (b), (c). This is obvious for (b). Since G is fully faithful, we infer that for any $X, Y \in \mathcal{R}_1$, we have $I_{\mathcal{R}_1}([X], [Y]) = I_{\mathcal{R}_2}([GX], [GY])$. This implies in particular that $\widetilde{G}$ preserves the operation (c). For (a), let $X, Y \in \operatorname{ind} \mathcal{R}_1$, by the definition of $\widetilde{G}$ we have

$$\widetilde{G}\big([u_X, u_Y]\big) = \sum_{L \in \operatorname{ind} \mathcal{R}_1} \big(F^L_{Y,X} - F^L_{X,Y}\big) u_{GL} - \delta_{X,\Sigma Y} \frac{[GX]}{d(X)},$$

$$\big[\widetilde{G}(u_X), \widetilde{G}(u_Y)\big] = \sum_{N \in \operatorname{ind} \mathcal{R}_2} \big(F^N_{GY,GX} - F^N_{GX,GY}\big) u_N - \delta_{GX,\Sigma GY} \frac{[GX]}{d(GX)}.$$

Again by G is fully faithful, we have $d(X) = d(GX)$. Therefore, to show $\widetilde{G}[u_X, u_Y] = [\widetilde{G}(u_X), \widetilde{G}(u_Y)]$, it suffices to prove

$$\sum_{L \in \operatorname{ind} \mathcal{R}_1} \big(F^L_{Y,X} - F^L_{X,Y}\big) u_{GL} = \sum_{N \in \operatorname{ind} \mathcal{R}_2} \big(F^N_{GY,GX} - F^N_{GX,GY}\big) u_N.$$

Since G is fully faithful, one can view $\mathcal{R}_1$ as a triangulated subcategory of $\mathcal{R}_2$. For any $X, Y, L \in \operatorname{ind} \mathcal{R}_1$, we have $F^L_{X,Y} = F^{GL}_{GX,GY}$. On the other hand, if $F^N_{GX,GY} \neq 0$, there exists $M \in \mathcal{R}_1$ such that $GM \cong N$ and $F^M_{X,Y} = F^N_{GX,GY}$. Hence G induces a bijection between $\{L \in \operatorname{ind} \mathcal{R}_1 \mid F^L_{Y,X} - F^L_{X,Y} \neq 0\}$ and $\{N \in \operatorname{ind} \mathcal{R}_2 \mid F^N_{GY,GX} - F^N_{GX,GY} \neq 0\}$. Moreover, we have $F^L_{Y,X} - F^L_{X,Y} = F^{GL}_{GY,GX} - F^{GL}_{GX,GY}$, which implies the desired result. □

We now turn to the 'integral' version of Ringel–Hall Lie algebras for finite-dimensional k-algebras. Let A be a finite-dimensional k-algebra of finite global dimension. Let E be a field extension of k and set $V^E = E \otimes_k V$ for any k-space V. Then A^E is a finite-dimensional E-algebra and, for $M \in \operatorname{mod} A$, M^E has a canonical A^E-module structure. The field E is called *conservative* [26] for an indecomposable A-module X if $(\operatorname{End}_A(X)/\operatorname{rad}\operatorname{End}_A(X))^E$ is a field again.

Let $\bar{k}$ be the algebraic closure of k and set

$$\Omega = \{E \mid k \subseteq E \subseteq \bar{k} \text{ is a finite field extension and conservative for all simple } A\text{-modules}\}.$$

For any $E \in \Omega$, one can show that A^E has finite global dimension (*cf. e.g.* Section 2 of [9]). Let $\mathcal{R}_{A^E}$ be the root category of A^E which is a 2-periodic triangulated category. By Theorem 4.4, we have a Lie algebra $\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$ over $\mathbb{Z}/(|E|-1)\mathbb{Z}$. Now consider the direct product $\prod_{E\in\Omega}\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$ of Lie algebras and let $\mathcal{LC}(\mathcal{R}_A)$ be the Lie subalgebra of $\prod_{E\in\Omega}\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$ generated by $u_{S_i} := (u_{S_i^E})_{E\in\Omega}$ and $u_{\Sigma S_i} := (u_{\Sigma S_i^E})_{E\in\Omega}$ for all simple A-modules S_i. We call $\mathcal{LC}(\mathcal{R}_A)$ the *integral Ringel–Hall Lie algebra* of A. It is clear that $\mathcal{LC}(\mathcal{R}_A)$ has a gradation given by the Grothendieck group $\mathsf{G}_0(\mathcal{R}_A)$, namely,

$$\mathcal{LC}(\mathcal{R}_A)=\bigoplus_{\alpha\in\mathsf{G}_0(\mathcal{R}_A)}\mathcal{LC}(\mathcal{R}_A)_\alpha,$$

such that $\deg u_{S_i}=[S_i]$ and $\deg u_{\Sigma S_i}=[\Sigma S_i]$. By the Lie operation (a), one gets that $h_i := [u_{S_i},u_{\Sigma S_i}]\in\mathcal{LC}(\mathcal{R}_A)_0$.

Now we can state the main result of Peng and Xiao (Theorem 4.7 in [24]).

Theorem 4.6. *Let A be a finite-dimensional hereditary k-algebra with symmetrizable generalized Cartan matrix C. Let $\mathcal{R}_A$ be the root category of A. Let $\mathsf{gcm}(C)_{\mathbb{C}}$ be the derived Kac–Moody algebra over $\mathbb{C}$ associated to the generalized Cartan matrix C. Then we have an isomorphism of Lie algebras $\mathsf{gcm}(C)_{\mathbb{C}}\cong\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{LC}(\mathcal{R}_A)$.*

We end up this subsection with the 'integral' version of Lemma 4.5.

Lemma 4.7. *Let A be a basic finite-dimensional k-algebra of finite global dimension. Let e be an idempotent element of A and $B=A/AeA$. Suppose that B has finite global dimension and for every finite field extension E of k, the derived functor $F_E := ?\overset{L}{\otimes}_{B^E} B^E_{A^E}:\mathcal{D}^b(\mathrm{mod}\,B^E)\to\mathcal{D}^b(\mathrm{mod}\,A^E)$ is an embedding. Then the functor $F := ?\overset{L}{\otimes}_B B_A$ induces a homomorphism of Lie algebras $\widetilde{F}:\mathcal{LC}(\mathcal{R}_B)\to\mathcal{LC}(\mathcal{R}_A)$.*

Proof. Since A is basic, we deduce that

$$\begin{aligned}\Omega &:= \{E\mid k\subseteq E\subseteq\bar{k}\text{ is a finite field extension and conservative for all simple }A\text{-modules}\}\\ &=\{E\mid k\subseteq E\subseteq\bar{k}\text{ is a finite field extension of }k\}\\ &=\{E\mid k\subseteq E\subseteq\bar{k}\text{ is a finite field extension and conservative for all simple }B\text{-modules}\}.\end{aligned}$$

By Lemma A.2, the functor F_E induces an embedding $\overline{F_E}:\mathcal{R}_{B^E}\to\mathcal{R}_{A^E}$ for each $E\in\Omega$. Hence we have the induced homomorphisms of Lie algebras

$$\widetilde{F_E}:\mathfrak{g}(\mathcal{R}_{B^E})_{(|E|-1)}\to\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$$

for any $E\in\Omega$ by Lemma 4.5. Consider the product of $(\widetilde{F_E})_{E\in\Omega}$,

$$\prod_{E\in\Omega}\widetilde{F_E}:\prod_{E\in\Omega}\mathfrak{g}(\mathcal{R}_{B^E})_{(|E|-1)}\to\prod_{E\in\Omega}\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}.$$

Let $\widetilde{F}$ be the restriction of $\prod_{E\in\Omega}\widetilde{F_E}$ to the subalgebra $\mathcal{LC}(\mathcal{R}_B)$. It suffices to show that the image of $\widetilde{F}$ is contained in the subalgebra $\mathcal{LC}(\mathcal{R}_A)$ of $\prod_{E\in\Omega}\mathfrak{g}(\mathcal{R}_{A^E})_{(|E|-1)}$. Let S_i, $i=1,\dots,m$ be the pairwise non-isomorphic simple B-modules. Since $\mathcal{LC}(\mathcal{R}_B)$ is generated by u_{S_i} and $u_{\Sigma S_i}$ for $1\leqslant i\leqslant m$, we only need to check that $\widetilde{F}(u_{S_i})$ and $\widetilde{F}(u_{\Sigma S_i})$ belong to $\mathcal{LC}(\mathcal{R}_A)$. It is clear that $F_E(S_i^E)=F(S_i)^E$ are simple A^E-modules. We have

$$\widetilde{F}(u_{S_i})=\big(\widetilde{F_E}(u_{S_i^E})\big)_{E\in\Omega}=(u_{F(S_i)^E})_{E\in\Omega}=u_{F(S_i)}\in\mathcal{LC}(\mathcal{R}_A)$$

and

$$\widetilde{F}(u_{\Sigma S_i}) = \big(\widetilde{F_E}(u_{\Sigma S_i^E})\big)_{E\in\Omega} = u_{\Sigma F(S_i)} \in \mathcal{LC}(\mathcal{R}_A), \quad 1 \leqslant i \leqslant m.$$

This completes the proof. $\square$

4.4. A class of finite-dimensional k-algebras

Let Q be the following quiver

$$\begin{tikzcd} & & & 0 \arrow[d, shift right, "\delta"'] \arrow[d, shift left, "\alpha"] & & & \\ n & n-1 \arrow[l] & 2 \arrow[l, dotted] & 1 \arrow[l, "\gamma"] \arrow[r, "\beta"] & n+1 \arrow[r, dotted] & \circ \arrow[r] & n+m. \end{tikzcd}$$

We assume $m \geqslant 1$, $n \geqslant 2$. Let A be the quotient of path algebra kQ by the ideal generated by $\beta \circ \alpha$, $\gamma \circ \alpha$. It has global dimension 2. Let S_i, $i = 0, 1, \dots, m+n$ be the simple A-modules corresponding to the vertices i and χ_A the Euler bilinear form of A. Let C_A be the Cartan matrix of A. Namely, $C_A = (c_{ij})_{(m+n+1)\times(m+n+1)}$, where

$$c_{i+1,j+1} = \chi_A\big([S_i],[S_j]\big) + \chi_A\big([S_j],[S_i]\big), \quad 0 \leqslant i, j \leqslant m+n.$$

Note that C_A is a generalized intersection matrix but not a generalized Cartan matrix.

Let $\mathcal{R}_A$ be the root category of A and $\mathcal{LC}(\mathcal{R}_A)$ the corresponding integral Ringel–Hall Lie algebra. In the rest of this paper, we show that $\dim_C(\mathcal{LC}(\mathcal{R}_A)\otimes_{\mathbb{Z}}\mathbb{C})_0 \geqslant m+n+2$ and there exists a surjective morphism $\mathsf{gim}(C_A)' \to \mathcal{LC}(\mathcal{R}_A)\otimes_{\mathbb{Z}}\mathbb{C}$. Consequently, the equality of Question 4.1 does not hold for the generalized intersection matrix algebra $\mathsf{gim}(C_A)$ (*cf.* Corollary 4.12). We remark that the surjective morphism $\mathsf{gim}(C_A)' \to \mathcal{LC}(\mathcal{R}_A)\otimes_{\mathbb{Z}}\mathbb{C}$ does not factor through the canonical projection $\pi_{C_A}: \mathsf{gim}(C_A)' \to \mathsf{im}(C_A)'$.

The following lemma gives a lower bound for the dimension of $(\mathcal{LC}(\mathcal{R}_A)\otimes_{\mathbb{Z}}\mathbb{C})_0$.

Lemma 4.8. *Let $\mathcal{R}_A$ be the root category of A and $\mathcal{LC}(\mathcal{R}_A)$ the integral Ringel–Hall Lie algebra of $\mathcal{R}_A$. Set $h_i = [u_{S_i}, u_{\Sigma S_i}]$, $0 \leqslant i \leqslant m+n$ and let $\widetilde{\mathfrak{h}}$ be the subspace of $\mathcal{LC}(\mathcal{R}_A)$ spanned by h_i, $0 \leqslant i \leqslant n+m$. Let M be the unique indecomposable A-module with composition series S_0, S_1, S_2, S_{n+1}. Then $u_M = (u_{M^E})_{E\in\Omega} \in \mathcal{LC}(\mathcal{R}_A)$ and $0 \neq [u_M, u_{\Sigma M}] \notin \widetilde{\mathfrak{h}}$. As a consequence, we have* $\dim_{\mathbb{C}}(\mathcal{LC}(\mathcal{R}_A)\otimes_{\mathbb{Z}}\mathbb{C})_0 \geqslant m+n+2$.

Proof. Since A is basic, we have $\Omega = \{k \subseteq E \subseteq \bar{k} \mid E \text{ is a finite field extension of } k\}$. A directly computation shows $h_i = [u_{S_i}, u_{\Sigma S_i}] = ([S_i^E])_{E\in\Omega}$. On the other hand, one can easily check that $u_{M^E} = [[[u_{S_0^E}, u_{S_1^E}], u_{S_2^E}], u_{S_{n+1}^E}]$ for any $E \in \Omega$ by using Lemma 4.3. Thus, both u_M, $u_{\Sigma M}$ belong to $\mathcal{LC}(\mathcal{R}_A)$.

Let P_i be the indecomposable projective A-modules corresponding to vertex i. Let

$$0 \to P_0 \xrightarrow{l} P_1 \to P_2 \oplus P_{n+1} \to M \to 0$$

be the projective resolution of M. We clearly have that

$$\mathcal{R}_A(M,M) = \mathcal{D}^b(\mathsf{mod}\,A)(M,M) \oplus \mathcal{D}^b(\mathsf{mod}\,A)\big(M,\Sigma^2 M\big).$$

Moreover, $\dim_k \mathcal{R}_A(M,M) = 2$ and $\dim_k \mathsf{rad}\,\mathcal{R}_A(M,M) = 1$.

In order to compute $[u_M, u_{\Sigma M}]$, it remains to compute $F^L_{\Sigma M,M}$ and $F^L_{M,\Sigma M}$ for any indecomposable $L \in \operatorname{ind}\mathcal{R}_A$. Let $M \to L \to \Sigma M \xrightarrow{f} \Sigma M$ be a triangle in $\mathcal{R}_A$, then we can write $f = f_0 + f_1$, where $f_0 \in \mathcal{D}^b(\operatorname{mod} A)(\Sigma M, \Sigma M)$ and $f_1 \in \mathcal{D}^b(\operatorname{mod} A)(\Sigma M, \Sigma^3 M)$. If $f_0 \neq 0$, then f is an isomorphism and $L \cong 0$. Hence, we assume that $f_0 = 0$, *i.e.* $0 \neq f_1 = f \in \operatorname{rad}\mathcal{R}_A(M, M)$, and then the triangle $M \to L \to \Sigma M \xrightarrow{f} \Sigma M$ is induced by a triangle $\Sigma^2 M \to L \to \Sigma M \xrightarrow{f_1} \Sigma^3 M$ in $\mathcal{D}^b(\operatorname{mod} A)$. By computing the mapping cone of f in $\mathcal{D}^b(\operatorname{mod} A)$, we deduce that L is isomorphic to the complex

$$\cdots \to 0 \to P_0 \xrightarrow{(f,l)} M \oplus P_1 \to P_2 \oplus P_{n+1} \to 0 \cdots,$$

where $P_2 \oplus P_{n+1}$ lies in the -1th component.

We claim that L is indecomposable in $\mathcal{D}^b(\operatorname{mod} A)$. Indeed, suppose $L \cong X \oplus Y$ in $\mathcal{D}^b(\operatorname{mod} A)$. We have $H^*(L) \cong H^*(X) \oplus H^*(Y)$, where $H^*(-)$ is the homology groups of corresponding complex. On the other hand, the only non-zero homology groups of L are $H^{-1}(L) \cong H^{-2}(L) \cong M$, which are indecomposable A-modules. Thus, we may assume $X \cong \Sigma^2 M$ and $Y \cong \Sigma M$. Note that in the root category $\mathcal{R}_A$, we have $\Sigma^2 M \cong M$. In particular, we can rewrite the triangle $M \to L \to \Sigma M \xrightarrow{f} \Sigma M$ as $M \to M \oplus \Sigma M \to \Sigma M \xrightarrow{f} \Sigma M$. By Lemma 3 of [24], we deduce that this triangle is split and $f = 0$, a contradiction. Thus L is indecomposable. Moreover, for any non-zero f, h in $\operatorname{rad}\mathcal{R}_A(M, M)$, the mapping cones of f and h are isomorphic to each other since $\dim_k \operatorname{rad}\mathcal{R}_A(M, M) = 1$. Therefore there is a unique indecomposable L such that $F^L_{\Sigma M,M}$ is non-zero.

Let $\Sigma M \to N \to M \xrightarrow{g} \Sigma^2 M$ be a triangle in $\mathcal{R}_A$. Similarly, one can show that N is indecomposable if and only if $0 \neq g \in \mathcal{D}^b(\operatorname{mod} A)(M, \Sigma^2 M)$. Moreover, we have $N \cong \Sigma^{-1} L$. A direct calculation implies that $\dim_k \mathcal{R}_A(M, L) = 1$. By Lemma 4.3, we know that $F^L_{\Sigma M,M} = F^{\Sigma^{-1}L}_{M,\Sigma M} = 1$. Now by the definition of the Lie bracket, in $\mathfrak{g}(\mathcal{R}_A)_{(|k|-1)}$, we have

$$\begin{aligned}
[u_M, u_{\Sigma M}] &= -[M] + \sum_{X \in \operatorname{ind}\mathcal{R}_A} \left(F^X_{\Sigma M,M} - F^X_{M,\Sigma M}\right) u_X \\
&= -[M] + F^L_{\Sigma M,M} u_L - F^{\Sigma^{-1}L}_{M,\Sigma M} u_{\Sigma^{-1}L} \\
&= -[M] + u_L - u_{\Sigma L}.
\end{aligned}$$

It is not hard to show that $L \ncong \Sigma L$ in the root category $\mathcal{R}_A$. Hence we have $u_L - u_{\Sigma L} \neq 0$. Note that the proof above is valid for any finite field extension of k. Thus, in the integral Ringel–Hall Lie algebra $\mathcal{LC}(\mathcal{R}_A)$, we still have $[u_M, u_{\Sigma M}] = -[M] + u_L - u_{\Sigma L}$. On the other hand, the degree of $[u_M, u_{\Sigma M}]$ is zero since $[M] + [\Sigma M] = 0 \in \mathrm{G}_0(\mathcal{R}_A)$. Therefore we have $[u_M, u_{\Sigma M}] \in \mathcal{LC}(\mathcal{R}_A)_0$ and it is not in the space spanned by h_i, $i = 0, \ldots, m+n$. □

Remark 4.9. Let M_{ij}, $i \geqslant 2$, $j \geqslant 1$ be the unique indecomposable A-module with composition series $S_0, S_1, S_2, \ldots, S_i, S_{n+1}, \ldots, S_{n+j}$. Similar to the proof of Lemma 4.8, one can show $u_{M_{ij}} \in \mathcal{LC}(\mathcal{R}_A)$ and $0 \neq [u_{M_{ij}}, u_{\Sigma M_{ij}}]$ is not in the space spanned by the h_i, $i = 0, \ldots, m+n$.

As a byproduct of the proof, we have the following

Corollary 4.10. *The root category $\mathcal{R}_A$ is not triangle equivalent to the root category of a finite-dimensional hereditary k-algebra.*

Proof. Let B be any finite-dimensional hereditary k-algebra. It is well-known that the root category $\mathcal{R}_B$ is proper (*cf. e.g.* [24]). Suppose that there is a triangle equivalence $\mathcal{R}_A \cong \mathcal{R}_B$, then the root category $\mathcal{R}_A$ is proper. That is for every non-zero indecomposable $X \in \mathcal{R}_A$, one has $0 \neq [X] \in \mathrm{G}_0(\mathcal{R}_A)$. But the proof above shows that there is an indecomposable object $L \in \mathcal{R}_A$ fitting into a triangle $M \to L \to \Sigma M \to \Sigma M$, which implies $[L] = [M] + [\Sigma M] = 0 \in \mathrm{G}_0(\mathcal{R}_A)$. □

Recall that $\mathrm{gim}(C_A)$ is the generalized intersection matrix algebra associated to the GIM-root basis $(H, \nabla, \triangle)$. Let $\mathrm{gim}(C_A)' := [\mathrm{gim}(C_A), \mathrm{gim}(C_A)]$ be the derived subalgebra of $\mathrm{gim}(C_A)$. As mentioned at the end of Section 4.1, the derived subalgebra $\mathrm{gim}(C_A)'$ can be presented by generators $\alpha_i^\vee$, $i = 0, \dots, m+n$, and e_α, $\alpha \in \triangle = \{\alpha_i \mid i = 0, \dots, m+n\}$ with the same relations of $\mathrm{gim}(C_A)$.

Theorem 4.11. *There is a surjective homomorphism $\phi : \mathrm{gim}(C_A)' \to \mathcal{LC}(\mathcal{R}_A) \otimes_{\mathbb{Z}} \mathbb{C}$ of Lie algebras defined by*

$$\begin{aligned} \alpha_i^\vee &\mapsto h_i = [u_{S_i}, u_{\Sigma S_i}], \\ e_{\alpha_i} &\mapsto u_{S_i}, \\ e_{-\alpha_i} &\mapsto -u_{\Sigma S_i}, \quad 0 \leqslant i \leqslant n+m. \end{aligned}$$

Moreover, ϕ keeps the gradations.

Proof. The relations (1), (2), (3) for GIM-Lie algebra follow from the definition of Lie bracket of Ringel–Hall Lie algebra. It suffices to show u_{S_i}, $u_{\Sigma S_j}$, $0 \leqslant i, j \leqslant m+n$ satisfy the Serre relations (4) and (5). We separate the proof into 4 cases.

Case 1: $i, j \in \{0, 1\}$. We consider the quotient algebra $B = A/A(e_2 + e_3 + \dots + e_{n+m})A$, where e_i is the idempotent associated to the vertex i. Note that B is projective as a right A-module. Then the algebras A and B satisfy all the assumptions of Lemma 4.7 by Theorem 3.1 in [8]. In particular, we have a homomorphism $\mathcal{LC}(\mathcal{R}_B) \to \mathcal{LC}(\mathcal{R}_A)$ of Lie algebras. Moreover, this homomorphism restricts to a surjective homomorphism $\mathcal{LC}(\mathcal{R}_B) \twoheadrightarrow \langle u_{S_i}, u_{\Sigma S_i} \mid i = 0, 1\rangle \subset \mathcal{LC}(\mathcal{R}_A)$. In order to show u_{S_i}, $u_{\Sigma S_i}$, $i = 0, 1$ satisfy the Serre relations (4), (5), it suffices to prove that the preimage of u_{S_i}, $u_{\Sigma S_i}$, $i = 0, 1$ satisfy the Serre relations.
Since the algebra B is hereditary of type $\widetilde{A_1}$, we infer that $\mathcal{LC}(\mathcal{R}_B) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the derived affine Kac–Moody algebra of type $\widetilde{A_1}$ by Theorem 4.6. Therefore the preimages of u_{S_i}, $u_{\Sigma S_i}$, $i = 0, 1$ satisfy the Serre relations.

Case 2: $i, j \in \{1, 2, \dots, n+m\}$. Let $B = A/Ae_0A$. It is easy to see that $\mathrm{Ext}^i_A(B_A, B_A) = 0$ for $i > 0$. Again by Theorem 3.1 of [8], we deduce that the algebras A, B satisfy all the assumptions of Lemma 4.7. We have a surjective homomorphism of Lie algebras

$$\mathcal{LC}(\mathcal{R}_B) \twoheadrightarrow \langle u_{S_i}, u_{\Sigma S_i} \mid i = 1, 2, \dots, m+n\rangle \subset \mathcal{LC}(\mathcal{R}_A).$$

Note that in this case B is of Dynkin type A_{m+n}. The Ringel–Hall Lie algebra $\mathcal{LC}(\mathcal{R}_B) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the simple Lie algebra of type A_{m+n}. Now the result follows similarly.

Case 3: $i = 0$, $j \neq 1, 2, n+1$. In particular, by the definition of Lie bracket we only need to show that $[u_{S_0}, u_{S_j}] = 0$ and $[u_{S_0}, u_{\Sigma S_j}] = 0$. This follows from the fact that S_j has projective dimension 2 and the projective resolution of S_j does not involve P_0.

Case 4: $i, j \in \{0, 2, n+1\}$. For the case $i = 0$, $j = 2$, we consider the quotient algebra $B_1 = A/A(e_3 + \dots + e_{m+n})A$ and $B_2 = A/A(e_2 + \dots + e_n + e_{n+2} + \dots + e_{n+m})A$ for the case $i = 0$, $j = n+1$. Clearly, the algebra B_2 is isomorphic to B_1. By Theorem 3.1 of [8], we deduce that A and B_1 satisfy the whole assumptions of Lemma 4.7. The algebra B_1 turns out to be a tilted algebra of tame hereditary algebra of type $\widetilde{A_2}$. Thus the integral Ringel–Hall algebra $\mathcal{LC}(\mathcal{R}_{C_1}) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the derived Kac–Moody algebra of type $\widetilde{A_2}$. The surjective homomorphism of Lie algebras $\mathcal{LC}(\mathcal{R}_{B_1}) \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow \langle u_{S_i}, u_{\Sigma S_i} \mid i = 0, 2, n+1\rangle \subset \mathcal{LC}(\mathcal{R}_A)$ implies the desired result.

Therefore ϕ is a homomorphism of Lie algebras. Note that $\mathcal{LC}(\mathcal{R}_A)$ is generated by u_{S_i} and $u_{\Sigma S_i}$, $i = 0, \dots, m+n$, we deduce that ϕ is surjective. Furthermore, since ϕ is homogeneous on the generators, we know that ϕ keeps the gradations. $\square$

Combine Theorem 4.11 with Lemma 4.8, we get the following.

Corollary 4.12.

1) *Let C_A be the Cartan matrix of the algebra A and $(H, \nabla, \triangle)$ the GIM-root basis associated to C_A. Then $\dim_{\mathbb{C}} \mathsf{gim}(C_A)_0 > \dim_{\mathbb{C}} H \otimes_{\mathbb{Q}} \mathbb{C}$. In particular, this gives a negative answer to Slodowy's Question 4.1.*
2) *Let τ be the ideal of $\mathsf{gim}(C_A)$ generated by all the elements in $\mathsf{gim}(C_A)_\gamma$ with $(\gamma, \gamma) > 2$. Then the ideal $\tau \neq \{0\}$ and hence $\mathsf{im}(C_A) \neq \mathsf{gim}(C_A)$.*

Proof. 1) Since $C_A \in M_{m+n+1}(\mathbb{Z})$ and $\mathsf{rank}(C_A) = m+n$, we obtain that $\dim_{\mathbb{C}} H \otimes_{\mathbb{Q}} \mathbb{C} = 2(m+n+1) - \mathsf{rank}(C_A) = m+n+2$. Therefore $\dim_{\mathbb{C}} \mathsf{gim}(C_A)_0 = \dim_{\mathbb{C}} \mathsf{gim}(C_A)'_0 + 1 \geqslant \dim_{\mathbb{C}}(\mathcal{LC}(\mathcal{R}_A) \otimes_Z \mathbb{C})_0 + 1 \geqslant m+n+3 > \dim_{\mathbb{C}} H \otimes_{\mathbb{Q}} \mathbb{C}$ by Theorem 4.11 and Lemma 4.8.

2) Let $\beta = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_{n+1}$. It is clear that $(\beta, \beta) = 4$, where $(-,-)$ is the symmetric bilinear form over the root lattice Γ of $\mathsf{gim}(C_A)$. We have

$$\mathsf{gim}(C_A)_\beta = \mathsf{gim}(C_A)'_\beta \xrightarrow{\phi|_{\mathsf{gim}(C_A)'_\beta}} \mathcal{LC}(\mathcal{R}_A)_{[M]}.$$

By Lemma 4.8, we have $0 \neq u_M \in \mathcal{LC}(\mathcal{R}_A)_{[M]}$, which implies $\mathsf{gim}(C_A)_\beta \neq \{0\}$. Hence $\tau \neq \{0\}$. $\square$

We end up this section with the following remark.

Remark 4.13. Let J be the ideal of kQ generated by $\beta \circ \alpha$, $\gamma \circ \delta$. The quotient algebra kQ/J is derived equivalent to the hereditary algebra $k\widetilde{A}_{m+n}$ (*cf.* [5]) and its Cartan matrix $C_{kQ/J}$ coincides with C_A. Let $(H, \nabla, \triangle)$ be a GIM-root basis associated to $C_{kQ/J} = C_A$. It is not hard to see that $(H, \nabla, \triangle)$ is braid equivalent to a GCM-root basis $(H, \nabla', \triangle')$ whose structural matrix is the generalized Cartan matrix $C_{\widetilde{A}_{m+n}}$ of affine type $\widetilde{A}_{m+n}$. Note that for the generalized Cartan matrix $C_{\widetilde{A}_{m+n}}$, we have $\mathsf{gim}(C_{\widetilde{A}_{m+n}}) = \mathsf{im}(C_{\widetilde{A}_{m_n}})$. Applying Theorem 4.2, Theorem 4.6 and Theorem 4.11, we have the following commutative diagram of Lie algebras

$$\begin{tikzcd}
\mathsf{gim}(C_A)' \arrow[r, equal] \arrow[d, "\pi_{C_A}"] \arrow[dd, bend right, "\phi"'] & \mathsf{gim}(C_{kQ/J})' \arrow[r, two heads, "\phi\circ\pi_{C_{kQ/J}}"] \arrow[d, "\pi_{C_{kQ/J}}"] & \mathsf{gim}(C_{\widetilde{A}_{m+n}})' \arrow[d, equal] \\
\mathsf{im}(C_A)' \arrow[r, equal] & \mathsf{im}(C_{kQ/J})' \arrow[r, "\cong", "\theta"'] \arrow[d, "\eta^{-1}\circ\tau\circ\theta", "\cong"'] & \mathsf{im}(C_{\widetilde{A}_{m+n}})' \arrow[d, "\cong", "\mu"'] \\
\mathcal{LC}(\mathcal{R}_A) \otimes_{\mathbb{Z}} \mathbb{C} & \mathcal{LC}(\mathcal{R}_{kQ/J}) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\eta", "\cong"'] & \mathcal{LC}(\mathcal{R}_{\widetilde{A}_{m+n}}) \otimes_{\mathbb{Z}} \mathbb{C},
\end{tikzcd}$$

where the isomorphism θ is a consequence of Theorem 4.2, μ follows from Theorem 4.6 and η follows from the fact that kQ/J is derived equivalent to $k\widetilde{A}_{m+n}$. However, the surjective morphism ϕ does not factor through the canonical morphism $\pi_{C_A} : \mathsf{gim}(C_A)' \to \mathsf{im}(C_A)$. There is nothing to surprise, since we have shown that $\mathcal{R}_A$ is not triangle equivalent to $\mathcal{R}_{kQ/J}$ and the integral Ringel–Hall Lie algebra $\mathcal{LC}(\mathcal{R}_A)$ is quite different to $\mathcal{LC}(\mathcal{R}_{kQ/J})$. By 2) of Corollary 4.12, we know that the canonical morphism $\pi_{C_A} = \pi_{C_{kQ/J}}$ is not injective and $\mathsf{gim}(C_A)' = \mathsf{gim}(C_{kQ/J})' \ncong \mathsf{gim}(C_{\widetilde{A}_{m+n}})'$. This also provides a class of examples that braid equivalent GIM-root bases can give rise to non-isomorphic GIM-Lie algebras.

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Appendix A. Recollement lives in root categories

In this appendix, we show that a recollement of bounded derived categories lives in the corresponding root categories under suitable assumption. This allows us to construct inductively various algebras whose 2-periodic orbit categories are not triangulated with the inherited triangle structure from the one of the bounded derived categories.

A.1. The induced functors

Let A and B be finite-dimensional k-algebras of finite global dimension. Let $F : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathrm{mod}\, B)$ be a standard functor, *i.e.* $F \cong ? \overset{L}{\otimes}_A X_B$ for some complex of $A^{op} \otimes_k B$-module. Note that for any triangle functor $L : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathrm{mod}\, B)$, we have $L \circ \Sigma_A^2 \cong \Sigma_B^2 \circ L$. By the universal property of dg orbit category (*cf.* Section 9.4 in [17]), F naturally induces a triangle functor $\overline{F} : \mathcal{R}_A \to \mathcal{R}_B$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}\, A) \arrow[r, "F"] \arrow[d, "\pi_A"] & \mathcal{D}^b(\mathrm{mod}\, B) \arrow[d, "\pi_B"] \\
\mathcal{R}_A \arrow[r, "\overline{F}"] & \mathcal{R}_B,
\end{tikzcd}$$

where π_A, π_B are the canonical functors. In the following, we will study the induced functor $\overline{F}$ explicitly.

Let ${}_A X_B$ be $\mathcal{K}$-projective as an $A^{op} \otimes_k B$-module. Clearly, X has finite total homology. Moreover, ${}_A X_B$ is compact as a left A-module and a right B-module respectively for the reason that A and B are of finite global dimension. Then we have the canonical isomorphism $\mathrm{RHom}_B({}_A X_B, ?) \cong ? \overset{L}{\otimes}_B \mathrm{RHom}_B({}_A X_B, B)_A$. Let ${}_B Y_A \to {}_B \mathrm{RHom}_B({}_A X_B, B)_A$ be a $\mathcal{K}$-projective resolution of ${}_B \mathrm{RHom}_B({}_A X_B, B)_A$ as a $B^{op} \otimes_k A$-module. Thus, the right adjoint G of F is naturally isomorphic to $? \overset{L}{\otimes}_B Y_A$.

Let $\mathcal{A}$ be the dg category of bounded complexes of finitely generated projective A-modules and $\mathcal{B}$ the dg category of bounded complexes of finitely generated projective B-modules. The tensor products by X and Y define dg functors $? \overset{L}{\otimes}_A X : \mathcal{A} \to \mathcal{B}$ and $? \overset{L}{\otimes}_B Y : \mathcal{B} \to \mathcal{A}$. By abuse of notation, we denote these dg functors by F and G as well. Similarly, one can lift the square of the shift functors $\Sigma_A^2 : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathrm{mod}\, A)$ and $\Sigma_B^2 : \mathcal{D}^b(\mathrm{mod}\, B) \to \mathcal{D}^b(\mathrm{mod}\, B)$ to dg functors $\Sigma_A^2 : \mathcal{A} \to \mathcal{A}$ and $\Sigma_B^2 : \mathcal{B} \to \mathcal{B}$.

Let $\mathcal{R}_{\mathcal{A}}$ be the dg orbit category (*cf.* Section 5 of [17]) of $\mathcal{A}$ respects to Σ_A^2. Let $\mathcal{R}_{\mathcal{B}}$ be the dg orbit category of $\mathcal{B}$ respects to Σ_B^2. We have canonical dg functors $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{R}_{\mathcal{A}}$, $\pi_{\mathcal{B}} : \mathcal{B} \to \mathcal{R}_{\mathcal{B}}$ and natural isomorphisms $\Sigma_B^2 \circ F \cong F \circ \Sigma_A^2$, $\Sigma_A^2 \circ G \cong G \circ \Sigma_B^2$ of dg functors. Thus, by the universal property of dg orbit categories, F and G induce dg functors $\overline{F} : \mathcal{R}_{\mathcal{A}} \to \mathcal{R}_{\mathcal{B}}$ and $\overline{G} : \mathcal{R}_{\mathcal{B}} \to \mathcal{R}_{\mathcal{A}}$. Clearly, $\overline{F}$ yields an $\mathcal{R}_{\mathcal{A}}^{op} \otimes_k \mathcal{R}_{\mathcal{B}}$-module $X_{\overline{F}}$

$$X_{\overline{F}}(B, A) \mapsto \mathcal{R}_{\mathcal{B}}\big(B, \overline{F}(A)\big).$$

Similarly, $\overline{G}$ induces an $\mathcal{R}_{\mathcal{B}}^{op} \otimes_k \mathcal{R}_{\mathcal{A}}$-module $Y_{\overline{G}}$

$$Y_{\overline{G}}(A, B) \mapsto \mathcal{R}_{\mathcal{A}}\big(A, \overline{G}(B)\big).$$

Let $\mathsf{L}T_{X_{\overline{F}}} : \mathcal{D}(\mathcal{R}_{\mathcal{A}}) \to \mathcal{D}(\mathcal{R}_{\mathcal{B}})$ be the derived tensor functor of $X_{\overline{F}}$ and $\mathsf{L}T_{Y_{\overline{G}}} : \mathcal{D}(\mathcal{R}_{\mathcal{B}}) \to \mathcal{D}(\mathcal{R}_{\mathcal{A}})$ the derived tensor functor of $Y_{\overline{G}}$. In the following, we identify the objects of $\mathcal{A}$ with the ones of $\mathcal{R}_{\mathcal{A}}$ and the objects of $\mathcal{B}$ with the ones of $\mathcal{R}_{\mathcal{B}}$ respectively.

Lemma A.1. *$\mathsf{L}T_{X_{\overline{F}}}$ is left adjoint to $\mathsf{L}T_{Y_{\overline{G}}}$.*

Proof. Clearly, $X_{\overline{F}}^{\widetilde{A}} = X_{\overline{F}}(?, \widetilde{A})$ is $\mathcal{K}$-projective for any $\widetilde{A} \in \mathcal{A}$ and $\mathsf{L}T_{X_{\overline{F}}}$ is left adjoint to $\mathsf{R}H_{X_{\overline{F}}}$. It suffices to show that $\mathsf{L}T_{Y_{\overline{G}}} \cong \mathsf{R}H_{X_{\overline{F}}}$. For any $\widetilde{A} \in \mathcal{A}$, $X_{\overline{F}}(?, \widetilde{A}) \cong \mathcal{R}_{\mathcal{B}}(?, \overline{F}(\widetilde{A}))$ which is compact in $\mathcal{D}(\mathcal{R}_{\mathcal{B}})$. By Lemma 6.2 (a) in [16], we have $\mathsf{L}T_{X_{\overline{F}}^T} \cong \mathsf{R}H_{X_{\overline{F}}}$, where $X_{\overline{F}}^T$ is defined by

$$X_{\overline{F}}^T(\widetilde{A}, \widetilde{B}) = \mathsf{Dif}\,\mathcal{R}_{\mathcal{B}}\big(X_{\overline{F}}(?, \widetilde{A}), \widetilde{B}^\wedge\big).$$

Thus, it suffices to show that we have a quasi-isomorphism $Y_{\overline{G}} \to X_{\overline{F}}^T$ of $\mathcal{R}_{\mathcal{B}}^{op} \otimes_k \mathcal{R}_{\mathcal{A}}$-modules. For any $\widetilde{A} \in \mathcal{A}$ and $\widetilde{B} \in \mathcal{B}$, we have

$$\begin{aligned}
X_{\overline{F}}^T(\widetilde{A}, \widetilde{B}) &= \mathsf{Dif}\,\mathcal{R}_{\mathcal{B}}\big(X_{\overline{F}}(?, \widetilde{A}), \widetilde{B}^\wedge\big) \\
&= \mathsf{Dif}\,\mathcal{R}_{\mathcal{B}}\big(\overline{F}(\widetilde{A})^\wedge, \widetilde{B}^\wedge\big) \\
&\cong \mathcal{R}_{\mathcal{B}}\big(\overline{F}(\widetilde{A}), \widetilde{B}\big) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathcal{B}\big(F(\widetilde{A}), \Sigma_B^{2n}\widetilde{B}\big) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_B\big(\widetilde{A} \otimes_A X_B, \Sigma_B^{2n}\widetilde{B}\big) \\
&\cong \bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \mathsf{RHom}_B\big(X, \Sigma_B^{2n}\widetilde{B}\big)\big).
\end{aligned}$$

Recall that we have a quasi-isomorphism $\Sigma_B^{2n}\widetilde{B} \overset{L}{\otimes}_B \mathsf{RHom}_B(X, B) \to \mathsf{RHom}_B({}_AX_B, \Sigma_B^{2n}\widetilde{B})$ and $\widetilde{A}$ is $\mathcal{K}$-projective as a right A-module. It follows that we have a quasi-isomorphism

$$\bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \Sigma_B^{2n}\widetilde{B} \overset{L}{\otimes}_B \mathsf{RHom}_B(X, B)\big) \xrightarrow{q.is} \bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \mathsf{RHom}_B\big({}_AX_B, \Sigma_B^{2n}\widetilde{B}\big)\big).$$

On the other hand, we also have a quasi-isomorphism $\Sigma_B^{2n}\widetilde{B} \otimes_B Y \to \Sigma_B^{2n}\widetilde{B} \overset{L}{\otimes}_B \mathsf{RHom}_B(X, B)$, which implies

$$\bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \Sigma_B^{2n}\widetilde{B} \otimes_B Y\big) \xrightarrow{q.is} \bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \Sigma_B^{2n}\widetilde{B} \overset{L}{\otimes}_B \mathsf{RHom}_B(X, B)\big).$$

Moreover, we have canonical isomorphisms

$$\begin{aligned}\bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \Sigma_B^{2n}\widetilde{B}\otimes_B Y\big) &\cong \bigoplus_{n\in\mathbb{Z}} \mathsf{RHom}_A\big(\widetilde{A}, \Sigma_A^{2n}(\widetilde{B}\otimes_B Y)\big)\\ &\cong \bigoplus_{n\in\mathbb{Z}} \mathcal{A}\big(\widetilde{A}, \Sigma_A^{2n}G(\widetilde{B})\big)\\ &\cong \mathcal{R}_{\mathcal{A}}\big(\widetilde{A}, \overline{G}(\widetilde{B})\big)\\ &= Y_{\overline{G}}(\widetilde{A}, \widetilde{B}).\end{aligned}$$

In other words, we get a quasi-isomorphism $Y_{\overline{G}}(\widetilde{A}, \widetilde{B}) \to X^{T}_{\overline{F}}(\widetilde{A}, \widetilde{B})$, which is natural in both $\widetilde{A}$ and $\widetilde{B}$. This completes the proof. □

We also have the following

Lemma A.2. *If $F : \mathcal{D}^b(\mathsf{mod}\, A) \to \mathcal{D}^b(\mathsf{mod}\, B)$ is fully faithful, then $\mathsf{L}T_{X_{\overline{F}}} : \mathcal{D}(\mathcal{R}_{\mathcal{A}}) \to \mathcal{D}(\mathcal{R}_{\mathcal{B}})$ is fully faithful.*

Proof. It follows from Lemma 4.2 (a) and (b) of [16] directly. □

Let $\mathcal{R}_A$ be the perfect derived category of $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_B$ the perfect derived category of $\mathcal{R}_{\mathcal{B}}$. In other words, $\mathcal{R}_A$ and $\mathcal{R}_B$ are the root categories of A and B respectively. Clearly, the triangle functors $\mathsf{L}T_{X_{\overline{F}}}$ and $\mathsf{L}T_{Y_{\overline{G}}}$ restrict to an adjoint pair of triangle functors

$$\begin{tikzcd} \mathcal{R}_A \arrow[r, shift left, "\mathsf{L}T_{X_{\overline{F}}}"] & \mathcal{R}_B \arrow[l, shift left, "\mathsf{L}T_{Y_{\overline{G}}}"] \end{tikzcd}.$$

For simplicity, we still denote by $\overline{F}$ the functor $\mathsf{L}T_{X_{\overline{F}}} : \mathcal{R}_A \to \mathcal{R}_B$ and by $\overline{G}$ the functor $\mathsf{L}T_{Y_{\overline{G}}} : \mathcal{R}_B \to \mathcal{R}_A$.

A.2. Recollement lives in root categories

Suppose we are given triangulated categories $\mathcal{D}'$, $\mathcal{D}$, $\mathcal{D}''$ with triangle functors

$$\begin{tikzcd} \mathcal{D}' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & \mathcal{D}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

- $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples;
- i_*, $j_!$, j_* are fully faithful;
- $j^* \circ i_* = 0$;
- for any X in $\mathcal{D}$, there are distinguished triangles

$$i_!i^!X \to X \to j_*j^*X \to \Sigma i_!i^!X \quad \text{and} \quad j_!j^!X \to X \to i_*i^*X \to \Sigma j_!j^!X,$$

  where the morphisms $i_!i^!X \to X$, $X \to j_*j^*X$, etc. are adjunction morphisms.

Then we say that $\mathcal{D}$ admits a *recollement* relative to $\mathcal{D}'$ and $\mathcal{D}''$. This notation was first introduced by Beilinson, Bernstein and Deligne [7] in geometric setting with the idea that $\mathcal{D}$ can be viewed as being glued together from $\mathcal{D}'$ and $\mathcal{D}''$. It is not hard to show that if both $\mathcal{D}'$ and $\mathcal{D}''$ are Krull–Schmidt categories, so is $\mathcal{D}$. Recollement in algebraic setting was studied extensively due to its close relation with tilting theory (see *e.g.* [13,20]).

Let A, B, C be finite-dimensional k-algebras of finite global dimension. Suppose that the bounded derived category $\mathcal{D}^b(\mathrm{mod}\,B)$ admits a recollement relative to $\mathcal{D}^b(\mathrm{mod}\,A)$ and $\mathcal{D}^b(\mathrm{mod}\,C)$. In particular, we have the following diagram of triangulated categories and triangle functors

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}^b(\mathrm{mod}\,A) \arrow[r, "i_*=i_!"] & \mathcal{D}^b(\mathrm{mod}\,B) \arrow[l, shift right=4ex, "i^*"'] \arrow[l, shift left=4ex, "i^!"] \arrow[r, "j^*=j^!"] & \mathcal{D}^b(\mathrm{mod}\,C). \arrow[l, shift right=4ex, "j_!"'] \arrow[l, shift left=4ex, "j_*"]
\end{tikzcd}$$

Assume further that both the functors i^* and $j_!$ are standard. Then we have the following

Theorem A.3. *Keep the notations above. Let A, B and C be finite-dimensional k-algebras of finite global dimension such that the derived category $\mathcal{D}^b(\mathrm{mod}\,B)$ admits a recollement relative to $\mathcal{D}^b(\mathrm{mod}\,A)$ and $\mathcal{D}^b(\mathrm{mod}\,C)$. Assume that the functors i^* and $j_!$ are standard. Then the root category $\mathcal{R}_B$ admits a recollement relative to $\mathcal{R}_A$ and $\mathcal{R}_C$. Moreover, we have the following commutative diagram of recollements*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{D}^b(\mathrm{mod}\,A) \arrow[r, "i_*=i_!"] \arrow[d, "\pi_A"] & \mathcal{D}^b(\mathrm{mod}\,B) \arrow[l, shift right=4ex, "i^*"'] \arrow[l, shift left=4ex, "i^!"] \arrow[r, "j^*=j^!"] \arrow[d, "\pi_B"] & \mathcal{D}^b(\mathrm{mod}\,C). \arrow[l, shift right=4ex, "j_!"'] \arrow[l, shift left=4ex, "j_*"] \arrow[d, "\pi_C"] \\
\mathcal{R}_A \arrow[r, "\overline{i_*}=\overline{i_!}"] & \mathcal{R}_B \arrow[l, shift right=4ex, "\overline{i^*}"'] \arrow[l, shift left=4ex, "\overline{i^!}"] \arrow[r, "\overline{j^*}=\overline{j^!}"] & \mathcal{R}_C. \arrow[l, shift right=4ex, "\overline{j_!}"'] \arrow[l, shift left=4ex, "\overline{j_*}"]
\end{tikzcd}$$

Proof. Since i^* and $j_!$ are standard, then all the functors i_*, $i^!$, j^*, j_* are standard for the reason that A, B, C are all of finite global dimension. Thus, we have the corresponding induced functors $\overline{i^*}$, $\overline{i_*}$, $\overline{i^!}$, $\overline{j_!}$, $\overline{j^*}$, $\overline{j_*}$. The commutativity of the above diagram follows from the universal property of the root categories. It remains to show that $\mathcal{R}_B$ admits a recollement relative to $\mathcal{R}_A$ and $\mathcal{R}_C$ together with the functors $\overline{i^*}$, $\overline{i_*}$, $\overline{i^!}$, $\overline{j_!}$, $\overline{j^*}$, $\overline{j_*}$. By Lemma A.1, we deduce that $(\overline{i^*}, \overline{i_*}, \overline{i^!})$ and $(\overline{j_!}, \overline{j^*}, \overline{j_*})$ are adjoint triples. On the other hand, Lemma A.2 implies that $\overline{i_*}$, $\overline{j_!}$, $\overline{j_*}$ are fully faithful. Since $\mathcal{R}_A$ is generated by $\pi_A(A)$, to show that $\overline{j^*} \circ \overline{i_*} = 0$, it suffices to show $\overline{j^*} \circ \overline{i_*}(\pi_A(A)) = 0$. By the commutativity of the above diagram, this result follows from $j^* \circ i_* = 0$. It remains to show that for any $X \in \mathcal{R}_B$ there are distinguished triangles

$$\overline{i_!}\,\overline{i^!}X \to X \to \overline{j_*}\,\overline{j^*}X \to \Sigma \overline{i_!}\,\overline{i^!}X, \qquad \overline{j_!}\,\overline{j^!}X \to X \to \overline{i_*}\,\overline{i^*}X \to \Sigma \overline{j_!}\,\overline{j^!}X.$$

We prove the existence of the first triangle, the second one is similar.

If $X \in \operatorname{im}\pi_B$, then there is an object $Y \in \mathcal{D}^b(\mathrm{mod}\,B)$ such that $X = \pi_B(Y)$. By the definition of recollement in $\mathcal{D}^b(\mathrm{mod}\,B)$, we have the following triangle

$$i_!i^!Y \to Y \to j_*j^*Y \to \Sigma i_!i^!Y.$$

Applying the triangle functor π_B, we get a triangle in $\mathcal{R}_B$

$$\pi_B\big(i_!i^!Y\big) \to \pi_B(Y) \to \pi_B\big(j_*j^*Y\big) \to \Sigma\pi_B\big(i_!i^!Y\big).$$

By the commutativity of the functors, we have

$$\overline{i_!}\,\overline{i^!}\pi_B(Y) \to \pi_B(Y) \to \overline{j_*}\,\overline{j^*}\pi_B(Y) \to \Sigma\overline{i_!}\,\overline{i^!}\pi_B(Y).$$

Clearly, this triangle is isomorphic to

$$\overline{i_!}\,\overline{i^!}\pi_B(Y) \xrightarrow{\eta_X} \pi_B(Y) \xrightarrow{\epsilon_X} \overline{j_*}\,\overline{j^*}\pi_B(Y) \to \Sigma\overline{i_!}\,\overline{i^!}\pi_B(Y),$$

where η_X, ϵ_X are adjunction morphisms, which implies that the later one is a distinguished triangle.

Let $f : X \to Y$ be the morphism fitting into the triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$, where $X, Y \in \operatorname{im}\pi_B$. We consider the following commutative square

$$\begin{tikzcd}
\overline{i_!}\,\overline{i^!}X \arrow[r, "\eta_X"] \arrow[d, "\overline{j_*}\,\overline{j^*}f"] & X \arrow[d, "f"] \\
\overline{j_*}\,\overline{j^*}Y \arrow[r, "\eta_Y"] & Y,
\end{tikzcd}$$

from which one gets the following commutative diagram of triangles

$$\begin{tikzcd}
\overline{i_!}\,\overline{i^!}X \arrow[r, "\eta_X"] \arrow[d, "\overline{j_*}\,\overline{j^*}f"] & X \arrow[r, "\epsilon_X"] \arrow[d, "f"] & \overline{j_*}\,\overline{j^*}X \arrow[d, "\overline{j_*}\,\overline{j^*}f"] \\
\overline{i_!}\,\overline{i^!}Y \arrow[r, "\eta_Y"] \arrow[d, "\overline{i_!}g_1"] & Y \arrow[r, "\epsilon_Y"] \arrow[d, "g"] & \overline{j_*}\,\overline{j^*}Y \arrow[d, "\overline{j_*}g_2"] \\
\overline{i_!}U_Z \arrow[r, "u"] & Z \arrow[r, "v"] & \overline{j_*}V_Z
\end{tikzcd}$$

by nine lemma.

Let $\phi(u) : U_Z \to \overline{i^!}Z$ be the morphism corresponding to u and $\phi(v) : \overline{j^*}Z \to V_Z$ the morphism corresponding to v under the natural isomorphisms. It is clear that $\phi(u)$ and $\phi(v)$ are isomorphisms. Thus, one gets the following commutative diagram

$$\begin{tikzcd}
\overline{i_!}U_Z \arrow[r, "u"] \arrow[d, "\overline{i_!}\phi(u)"] & Z \arrow[r, "v"] \arrow[d, equal] & \overline{j_*}V_Z \arrow[r, "w"] \arrow[d, "\overline{j_*}(\phi(v)^{-1})"] & \Sigma\overline{i_!}U_Z \arrow[d, "\Sigma\overline{i_!}\phi(u)"] \\
\overline{i_!}\,\overline{i^!}Z \arrow[r, "\eta_Z"] & Z \arrow[r, "\epsilon_Z"] & \overline{j_*}\,\overline{j^*}Z \arrow[r, "\delta"] & \Sigma\overline{i_!}\,\overline{i^!}Z,
\end{tikzcd}$$

where $\delta = \overline{j_*}\phi(v) \circ w \circ \Sigma\overline{i_!}\phi(u)$. Hence we deduce that

$$\overline{i_!}\,\overline{i^!}Z \xrightarrow{\eta_Z} Z \xrightarrow{\epsilon_Z} \overline{j_*}\,\overline{j^*}Z \to \Sigma\overline{i_!}\,\overline{i^!}Z$$

is a distinguished triangle. Now this holds true for any $Z \in \mathcal{R}_B$ by 'devissage'. $\square$

Corollary A.4. *Keep the assumptions in Theorem A.3. If the canonical functor π_B is dense, then both π_A and π_C are dense.*

Proof. Let X be any object in $\mathcal{R}_A$. By the density of π_B, there is an object $Y \in \mathcal{D}^b(\operatorname{mod} B)$ such that $\pi_B(Y) \cong \overline{i_*}X$. In $\mathcal{D}^b(\operatorname{mod} B)$, we have the distinguished triangle

$$i_!i^!Y \to Y \to j_*j^*Y \to \Sigma i_!i^!Y.$$

Applying the functor π_B, we get

$$\pi_B\big(i_!i^!Y\big) \to \pi_B(Y) \to \pi_B\big(j_*j^*Y\big) \to \Sigma\pi_B\big(i_!i^!Y\big),$$

which is isomorphic to the distinguished triangle

$$\overline{i_!}\,\overline{i^!}(\overline{i_!}X) \to \overline{i_!}X \to 0 \to \Sigma\overline{i_!}\,\overline{i^!}(\overline{i_!}X).$$

It follows that $X \cong \pi_A(i^!Y)$. In particular, π_A is dense. Similar proof implies that π_C is dense too. □

Remark A.5. If only one of i^* and $j_!$ is standard, say i^* is standard, then Lemmas A.1, A.2 and Theorem 2.4 (a) of [22] imply that there is a recollement

$$\begin{tikzcd}[column sep=large] \mathcal{R}_A \arrow[r, "\overline{i_*}=\overline{i_!}" description] & \mathcal{R}_B \arrow[l, shift right=6, "\overline{i^*}"'] \arrow[l, shift left=6, "\overline{i^!}"] \arrow[r, "\overline{j^*}=\overline{j^!}" description] & \mathcal{R}_B/\overline{i_*}\mathcal{R}_A. \arrow[l, shift right=6, "\overline{j_!}"'] \arrow[l, shift left=6, "\overline{j_*}"] \end{tikzcd}$$

Corollary A.4 also holds in this case (one should replace the functor π_C).

The following is quite obvious.

Corollary A.6. *Let A and B be finite-dimensional k-algebras of finite global dimension. Assume the 2-periodic orbit category of A is not triangulated with the inherited triangle structure. For any finite-dimensional $B^{op} \otimes_k A$-module M, the 2-periodic orbit category of the triangular extension of A and B by M is not triangulated with the inherited triangle structure.*

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