Cohomological invariants of Jordan algebras with frames

Mark L. MacDonald

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2

ARTICLE INFO

Article history:
Received 15 July 2009
Available online 22 October 2009
Communicated by Eva Bayer-Fluckiger

Keywords:
Cohomological invariants
Jordan algebras
Essential dimension
Quadratic and hermitian forms

ABSTRACT

In a previous paper the author determined all possible cohomolog-ical invariants of Aut(J)-torsors in Galois cohomology with mod 2 coefficients, where J is a split central simple Jordan algebra of odd degree \( n \geq 3 \). In the present paper we extend these results to a larger class of groups \( G_{n_1,...,n_d}^r \) indexed by \( r = 0, 1, 2 \) or 3, and \( n_i \) positive integers that add to \( n \geq 3 \) such that \( n_1 \) is odd. We also give precise values for the essential dimension of these groups.

© 2009 Elsevier Inc. All rights reserved.

The Stiefel–Whitney invariants for quadratic forms over a field \( k \) send an isometry class to an element in the Galois cohomology ring with mod 2 coefficients, \( H^*(k) := H^*(\text{Gal}(k_s/k), \mathbb{Z}/2\mathbb{Z}) \). Classical examples of Stiefel–Whitney invariants include the determinant and Hasse invariant. It is natural to ask whether there are any other cohomological invariants of isometry classes of quadratic forms. The answer is no, aside from linear combinations of the Stiefel–Whitney invariants, as shown in [GMS03, Chapter IV].

By Galois cohomology, one may identify isomorphism classes of \( n \)-dimensional quadratic forms over \( k \) with isomorphism classes of \( O_n \)-torsors over \( k \). In this sense, one is said to have classified all cohomological invariants of \( O_n \)-torsors. In [Mac08a] the author has classified all cohomological invariants of \( G \)-torsors, where \( G \) is the algebraic group of automorphisms of an odd degree simple Jordan algebra.

In this paper we will classify all cohomological invariants (with \( \mathbb{Z}/2\mathbb{Z} \) coefficients) of \( G \)-torsors for a larger class of algebraic groups \( G \), thus extending the results of [Mac08a].

The algebraic groups \( G \) that we cover will be denoted \( G_{n_1,...,n_d}^r \), where the indices are as follows. The upper index \( r \) is either 0, 1, 2 or 3, and the lower index, \( n_\bullet = (n_1, \ldots, n_d) \), is an ordered \( d \)-tuple of positive integers. The conditions we will impose on \( n_\bullet \) throughout are that \( n := n_1 + \cdots + n_d \geq 3 \), the first number \( n_1 \) is odd, and if \( r = 3 \) then \( n = 3 \). Explicitly, all of the groups \( G_{n_1,...,n_d}^r \) that we will consider are as follows:

E-mail address: mlm@math.ubc.ca.
Two idempotents are orthogonal the sum of two orthogonal idempotents in $n$-frame. An $n$-frame is a $d$-tuple of idempotents, each of rank $n_i$, defined below. Then we can use Galois cohomology to get a concrete description of the $G^{r}_{n_r}$-torsors.

The main result of this paper is that the cohomological invariants of $G^{r}_{n_r}$-torsors have an $H^*(k_0)$-basis, which is the simplest possible situation. See Theorem 4.7 for a different version of this theorem.

**Theorem 0.1.** Let $(n_1, \ldots, n_d)$ be any positive integers with $n_1$ odd, $r = 0, 1, 2$ or $3$, and $n := \sum_i n_i \geq 3$. If $r = 3$ then assume $n = 3$. Consider the polynomial in $\mathbb{Z}[t]$ given by

$$p(t) = 1 + t^r \prod_{i=1}^{d} (1 + t + \cdots + t^{n_i}),$$

unless $r = 0$, in which case we omit the leading 1. Then $\text{Inv}(G^{r}_{n_1, \ldots, n_d}, \mathbb{Z}/2\mathbb{Z})$ has an $H^*(k_0)$-basis such that the number of generators in $H^i(k_0)$ is the coefficient of $t^i$ in the polynomial $p(t)$.

Notice that the expression $p(t)$ is a polynomial if and only if there is an $n_i$ which is odd; this justifies the assumption that $n_1$ is odd.

This theorem contains several known results. When $d = 1$ this is the main theorem of [Mac08a]. The proof of Theorem 0.1 uses the case $d = 1$ and $r = 0$ from [GMS03]. The remaining $r = 0$ cases can easily be deduced from [GMS03, 16.5]. The $r = 3$ cases are known ([GMS03, 22.5] and [Ga09, 18.1 and 18.9]), and we give a new proof. The cases when $r = 1, 2$ and $d > 1$ are new.

Throughout we will assume that $k$ is a field extension of a fixed base field $k_0$ of characteristic not 2, and $k_s$ is a separable closure of $k$.

### 1. Preliminaries

(See also [KMRT98] and [Mc04].) A **Jordan algebra** over $k$ is a commutative, unital (not necessarily associative) $k$-algebra $J$ whose elements obey the identity

$$x^2(xy) = x(x^2 y) \quad \text{for all } x, y \in J.$$

A simple Jordan algebra is one with no proper ideals. An **idempotent** in $J$ is an element $u^2 = u \neq 0 \in J$. Two idempotents are **orthogonal** if they multiply to zero, and an idempotent is **primitive** if it is not the sum of two orthogonal idempotents in $J$. For any field extension $l/k$, we can **extend scalars** to $l$ by taking $J_l = J \otimes_k l$, for example $J_l = J \otimes_k k$. A Jordan algebra has **degree** $n$ if the identity in $J$ decomposes into $n$ pairwise orthogonal primitive idempotents over $k$. A degree $n$ Jordan algebra is **reduced** if the identity decomposes into $n$ orthogonal primitive idempotents over $k$. 

$$C_{n_1, \ldots, n_d}^3 \cong \text{SO}_{n_1} \times O_{n_2} \times \cdots \times O_{n_d},$$

$$C_{n_1, \ldots, n_d}^1 \cong \mathbb{Z}/2\mathbb{Z} \times \frac{\text{GL}_{n_1} \times \cdots \times \text{GL}_{n_d}}{\text{GL}_1},$$

$$C_{n_1, \ldots, n_d}^2 \cong \frac{\text{Sp}_{2n_1} \times \cdots \times \text{Sp}_{2n_d}}{\langle -1, \ldots, -1 \rangle},$$

$$G_3^3 \cong \text{Spin}(9),$$

$$G_3^{1,2} \cong \text{Spin}(8).$$

In the $G_{n_r}^r$ case, the subgroup $\text{GL}_1$ is embedded diagonally, and $\mathbb{Z}/2\mathbb{Z}$ acts by inverse transpose on each factor. For a more detailed explanation of these groups, see Section 2. $G_r^{n_r}$ is defined to be the subgroup of the automorphism of a simple Jordan algebra of degree $n$ and type $r$, which fixes an $n_r$-frame. An $n_r$-frame is a $d$-tuple of idempotents, each of rank $n_i$, defined below. Then we can use Galois cohomology to get a concrete description of the $G_r^{n_r}$-torsors.
The classification of reduced simple Jordan algebras of degree $\geq 3$ is closely related to the classification of composition algebras. A composition algebra over $k$ is a unital $k$-algebra $C$ together with a non-degenerate quadratic form $\phi$ on $C$ (called the norm form) such that for any $c_1, c_2 \in C$ we have that $\phi(c_1c_2) = \phi(c_1)\phi(c_2)$. Two composition algebras are isomorphic as $k$-algebras iff their norm forms are isometric. Every norm form is an $r$-fold Pfister form, which is to say

$$\phi = \langle \langle a_1, \ldots, a_r \rangle \rangle := \langle \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_r \rangle \rangle.$$

Furthermore, $r$ must be 0, 1, 2 or 3, and for any such $r$-fold Pfister form $\phi$, there is a composition algebra with $\phi$ as its norm form and a natural anti-homomorphism conjugation map $- : C \to C$. The reader might find it helpful to notice that these four cases over the reals correspond to: the real numbers, complex numbers, quaternions and octonions respectively.

Let $C$ be a composition algebra with norm form $\phi = \langle \langle a_1, \ldots, a_r \rangle \rangle$, and let $q = \langle b_1, \ldots, b_n \rangle$ be a non-degenerate quadratic form. Then we can construct a reduced Jordan algebra in the following way. Let $\Gamma = \text{diag}(b_1, \ldots, b_n)$, and let $\sigma_q(x) := \Gamma^{-1}x^3\Gamma$ define a map from $M_n(C)$ to $M_n(C)$. Then $\sigma_q$ is an involution (i.e. an anti-homomorphism such that $\sigma_q^2$ is the identity), so we can define $\text{Sym}(M_n(C), \sigma_q)$ to be the commutative algebra of symmetric elements (i.e. elements $x$ such that $\sigma_q(x) = x$). The product structure is defined by $x \circ y = \frac{1}{2}(xy + yx)$, using the multiplication in $C$. When $C$ is associative (i.e. $r = 0$, 1 or 2) we know $\text{Sym}(M_n(C), \sigma_q)$ is Jordan. For $r = 3$, it is only Jordan when $n \leq 3$, so in what follows we will always impose this condition in the $r = 3$ case.

We will also use the notation $\mathcal{H}(\phi, b) := \text{Sym}(M_n(C), \sigma_q)$ for this Jordan algebra, and we will recall that its isomorphism class only depends on the isomorphism classes of $q$ and $\phi$, and not on the diagonalization we have chosen for $q$. The following theorem states that in degrees $\geq 3$ these make up all of the reduced Jordan algebras up to isomorphism.

**Theorem 1.** (See coordinatization [Mc04, 17], [Ja68, p. 137].) Let $J$ be a reduced simple Jordan algebra of degree $n \geq 3$. Then there exists a composition algebra $C$ and an $n$-dimensional quadratic form $b$ such that $J \cong \text{Sym}(M_n(C), \sigma_q)$.

We will denote the split Jordan algebra $J^r_n := \mathcal{H}(\phi, q)$ for $\phi$ split and $q = \langle 1, \ldots, 1 \rangle$. The split composition algebra of dimension $2^r$ will be denoted $C^r$.

### 2. Automorphism groups

In this section we will explain the descriptions given in the Introduction of the groups $G^r_n$, for $n \geq 3$, as automorphism groups of Jordan algebras. We will use boldface to denote algebraic groups.

We will start by describing the groups when $d = 1$, in other words when the only idempotent that is preserved is the identity. In the $r = 3$ case, it is well known that the automorphism group of $J^3_n$ (also known as the split Albert algebra) is the split group of type $F_4$ [SV00, 72].

For $r = 0$, 1 or 2, the algebraic groups $G^r_n \cong \text{Aut}(M_n(C^r), \ast)$ are described in [KMRT98, Chapter III or pp. 403–405] as $\text{PGO}_n$, $\mathbb{Z}/2\mathbb{Z} \ltimes \text{PGL}_n$ and $\text{PSp}_{2n}$, respectively. The group $\text{PGO}_n$ will be explained in the proof of the next theorem; when $n$ is odd, it is isomorphic to $\text{SO}_n$.

We will use throughout the fact that the automorphism group of any simple Jordan algebra of degree $\geq 3$ is isomorphic to the automorphism group of its associated algebra with involution by [Ja68, p. 210]. In particular, for $r = 0$, 1, 2 and $n \geq 3$, we have $\text{Aut}(M_n(C^r), \ast) \cong \text{Aut}(J^r_n)$.

The remaining groups, $G^r_{n, \ast}$, are subgroups of the above groups.

**Definition 2.1.** Given a sequence of positive integers $n := (n_1, \ldots, n_d)$, and a Jordan algebra $J$ of degree $n = \sum n_i$ an $(n_1, \ldots, n_d)$-frame (or $n$-frame) in $J$ is a sequence of pairwise orthogonal idempotents $(u_1, \ldots, u_d)$ such that $1 = \sum u_i$ and $\text{rank}(u_i) = n_i$ for all $1 \leq i \leq d$. Then a Jordan algebra is reduced (as defined above) if it has a $(1, \ldots, 1)$-frame.

Given a diagonalization of $q$, there is a canonical $(n_1, \ldots, n_d)$-frame in the Jordan algebra $\mathcal{H}(\phi, q)$ given by the diagonal elements.
In [Mac08a] the author viewed isomorphism classes of simple Jordan algebras of degree \( n \) and type \( r \) as \( \text{Aut}(J^n_r) \)-torsors by using Galois cohomology. We can similarly view isomorphism classes of Jordan algebras with a frame \( (J; u_1, \ldots, u_d) \) as \( G_n^r \)-torsors, where \( G_n^r = \text{Aut}(J^n_r; e_1, \ldots, e_d) \subset \text{Aut}(J^n_r) \) is the subgroup of automorphisms that fix the canonical idempotents \( e_i \) for all \( i \). These idempotents are fixed from our definition of \( J^n_r \).

**Theorem 2.2.** For \( n_* = (n_1, \ldots, n_d) \) a partition of \( n \geq 3 \) with \( n_1 \) odd, we have the following isomorphisms of algebraic groups

\[
\begin{align*}
C_{n_*}^0 & := \text{Aut}(J^{0|n_*}) \cong \text{SO}^{n_1} \times \text{O}^{n_2} \times \cdots \times \text{O}^{n_d}, \\
C_{n_*}^1 & := \text{Aut}(J^{1|n_*}) \cong \mathbb{Z}/2\mathbb{Z} \times \frac{\text{GL}^{n_1} \times \cdots \times \text{GL}^{n_d}}{\text{GL}_1}, \\
C_{n_*}^2 & := \text{Aut}(J^{2|n_*}) \cong \text{Sp}^{2n_1} \times \cdots \times \text{Sp}^{2n_d} \langle -1, \ldots, -1 \rangle.
\end{align*}
\]

We are taking the quotients here by normal algebraic subgroups.

**Proof.** Case \( r = 0 \): We have \( C^0 = k \). By the Skolem–Noether theorem, the automorphisms of \( (M_n(C^0), \star) = (\text{End}_k(V_0), \sigma) \) are the inner automorphisms that commute with \( \sigma \), which is the adjoint involution of the bilinear form \( (1, \ldots, 1) \) with respect to the given basis on \( V_0 \). It is easy to check that the inner automorphism corresponding to an element \( g \in \text{End}_k(V_0) \) commutes with \( \sigma \) if and only if \( \sigma(g)g \) is a non-zero scalar. Such an element \( g \) is called a similitude [KMRT98, Chapter III]. The group of all similitudes of any algebra with orthogonal involution is denoted \( GO(A, \sigma) \).

So we see that the elements of \( \text{Aut}(M_n(k), \star) \) are inner automorphisms, \( \text{Int}(B) \), for matrices \( B \in M_n(k) \) such that \( B^tB = \lambda I_n \) for some \( \lambda \in k^* \). Moreover, two such matrices give the same automorphism if and only if they are non-zero scalar multiples of each other, so the group \( \text{Aut}(M_n(k), \star) \) is isomorphic to the group of projective similitudes in the terminology of [KMRT98], where it is denoted \( \text{PGl}(n, k) \). See also [Qu97, p. 308, Lemma 2]. In our case we will use the shorthand \( \text{PGO}(n, k) \).

The subgroup of \( \text{Aut}(M_n(k), \star) \subset \text{PGl}(n, k) \) that preserves the rank \( n_1 \) diagonal idempotents \( u_i \in J \) consists of those elements that are represented by matrices in block diagonal form with each block of size \( n_i \). Now we just need to show that the following composition of morphisms induces an isomorphism,

\[
\text{SO}^{n_1} \times \text{O}^{n_2} \times \cdots \times \text{O}^{n_d} \to (\text{GL}^{n_1} \times \cdots \times \text{GL}^{n_d}) \cap \text{GO}_n \to C_{n_*}^0 \to (A_1, \ldots, A_d) \mapsto (A_1, \ldots, A_d).
\]

The second morphism is induced from the quotient \( \text{GO}_n \to \text{PGO}_n \). If we replace \( \text{SO}^{n_1} \) with \( \text{O}^{n_1} \), then this composition would be surjective with kernel of order two: The identity, together with \( (-I_1, \ldots, -I_{n_d}) \). Since \( n_1 \) is odd, \( \text{O}^{n_1}/(-I_{n_1}) \) is isomorphic to \( \text{SO}^{n_1} \), so we get the required isomorphism.

Case \( r = 1 \): Here \( C^1 \cong k \times k \) and conjugation is the exchange map. Similar to the \( r = 0 \) case, we represent elements of the automorphism group \( \text{Aut}(M_n(k \times k), \star) \cong \mathbb{Z}/2\mathbb{Z} \times \text{PGL}_n \) by \( n \times n \) matrices (up to a scalar), together with an element of \( \mathbb{Z}/2\mathbb{Z} \). So the elements of this group that fix the idempotents \( (u_i) \) are the ones represented by matrices in block diagonal form with blocks of size \( n_i \). This gives us the group we want, where the scalars \( \text{GL}_1 \) are diagonally embedded into \( \text{GL}^{n_1} \times \cdots \times \text{GL}^{n_d} \). Note that we do not need to assume \( n_1 \) is odd for this description.

Case \( r = 2 \): Here \( C^2 \cong M_2(k) \). This case is similar to the other two, where we represent elements of the automorphism group \( \text{Aut}(M_n(C^2), \star; n) \) by \( 2n \times 2n \) symplectic matrices, up to a scalar. The
only symplectic scalar matrices are plus and minus the identity, so by using the same block diagonal description, we get the desired group. Note that we do not need to assume $n_1$ is odd for this description. \qed

**Proposition 2.3.** We have isomorphisms

$$C_3^2 := \text{Aut}(J^3; 3) \cong F_4,$$

$$C_{1,2}^3 := \text{Aut}(J^3; 1, 2) \cong \text{Spin}(9),$$

$$C_{1,1}^3 := \text{Aut}(J^3; 1, 1, 1) \cong \text{Spin}(8).$$

**Proof.** The first two isomorphisms may be found in [SV00, 7.2, 7.1], or also in [Ja68, Chapter IX]. In fact, the final isomorphism $\text{Aut}(J^3; 1, 1, 1) \cong \text{Spin}(8)$ is stated as an exercise [Ja68, p. 378]. Details are given in [Mac08b, Proposition 4.4]. \qed

3. Jordan algebras with frames

In this section we begin generalizing the results of [Mac08a].

In the above definition of $\mathcal{H}(\phi, q)$, if $n = \sum n_i$, and we are given a diagonalization of $q$, then there is a canonical choice of $(n_1, \ldots, n_d)$-frame given by the diagonal matrices, where, for example, $u_1$ is $\text{diag}(1, 1, \ldots, 1, 0, \ldots, 0)$. This is called the diagonal $n_*$-frame.

This choice of $(n_1, \ldots, n_d)$-frame, in general, depends on the diagonalization of $q$. We will describe precisely how this depends on the diagonalization in Corollary 3.5.

If we multiplied $q = q_1 \perp \cdots \perp q_d$ by a non-zero scalar, then this has no effect on the involution $\sigma_q$, and in particular no effect on the map $\mathcal{H}$. So we lose no generality in assuming $q_1$ has determinant 1, since $n_1$ is odd.

We will continue to use the following notation. Let $\phi$ be a Pfister $r$-form over $k$ with associated composition algebra $C$. We have positive integers $n_1, \ldots, n_d$ with $n_1$ odd, and $n = \sum n_i$. Let $q_1$ and $q_i'$ be non-degenerate quadratic forms over $k$ of dimension $n_i$ for $1 \leq i \leq d$, each of which have a fixed diagonalization, and assume $q_1$ and $q_1'$ have determinant 1. Let $q = q_1 \perp \cdots \perp q_d$ and $q' = q_1' \perp \cdots \perp q_d'$ be the quadratic forms on $V_0$ and $V_0'$ respectively. Then we can define a hermitian form $h$ (resp. $h'$) on $V = C \otimes V_0$ (resp. $V' = C \otimes V_0'$) over $C$ by $h(\alpha \otimes x, \beta \otimes y) = \bar{\alpha} q(x, y) \beta$ (similarly for $h'$). Also, let $h_i$ and $h'_i$ be the hermitian forms corresponding to the subforms $q_i$ and $q_i'$.

First we will need the following lemma. For its proof, recall that the Witt ring $W(k)$ of a field $k$ is the set of isometry classes of non-degenerate quadratic forms over $k$ modulo hyperbolic forms, with addition $\perp$ and multiplication $\otimes$. Then the fundamental ideal is the ideal $I(k) \subset W(k)$ of classes of even-dimensional forms [La05, Chapter II].

**Lemma 3.1.** Let $\phi$ be an $r$-Pfister form, and let $q_1$ and $q_1'$ be quadratic forms over $k$ of the same odd dimension $n_1$ such that $\det(q_1) = \det(q_1') = 1$, as above. If $\phi \otimes q_1 \equiv \phi \otimes \lambda q_1'$ for some $\lambda \in k^*$, then $\lambda$ is represented by $\phi$.

**Proof.** By [La05, Corollary II.2.2] we know that $q_1 \equiv q_1' \equiv (d) \mod I^2(k)$ in the Witt ring, where $d = (-1)^{n_1+1}$. Then by multiplying by $\phi$ we get that

$$\phi \otimes q_1 \equiv \phi \otimes q_1' \equiv d\phi \mod I^{r+2}.$$ 

Now by using the given isometry we get that $d \phi \equiv \lambda d \phi \mod I^{r+2}$, and so $\phi \otimes (\lambda) \in I^{r+2}$. By the “Hauptsatz” (see [La05, X.5]) this implies that $\phi \otimes (\lambda)$ is hyperbolic, which implies the result. \qed

**Proposition 3.2.** Assume $r \neq 3$ and $n \geq 3$. The following are equivalent:

1. $\phi \otimes q_i \equiv \phi \otimes q_i'$ as quadratic forms, for all $1 \leq i \leq d$. 
(2) \( h_i \cong h'_i \) as hermitian forms over \( C \) for all \( 1 \leq i \leq d \).

(3) \((M_n(C), \sigma_n)\) is isomorphic as an algebra with involution to \((M_n(C), \sigma_{n'})\) through an isomorphism that preserves the canonical \((n_1, \ldots, n_d)\)-frame.

(4) \( \mathcal{H}(\phi, q_1 \perp \cdots \perp q_d) \cong \mathcal{H}(\phi', q'_1 \perp \cdots \perp q'_d) \) as \( G'_{n_1,\ldots,n_d} \)-torsors over \( k \).

**Proof.** The equivalence of (1) and (2) is done in [Sch85, 10.1.1, 10.1.7]. In fact, it is stated there for \( \mathbb{C} \) division, but the statements extend to \( \mathbb{C} \) split. For example, the trace form \( q_{h}(x) := h(x, x) \) is a quadratic form on \( V \) over \( k \) equal to \( \phi \otimes q \), regardless of whether \( C \) is split or not.

The equivalence of (2) and (3) is an extension of [KMRT98, Proposition 4.2, p. 43] (or [Mac08a, Proposition 3.2]) to include \((n_1, \ldots, n_d)\)-frames. We know that the equivalence of involutions on \( \text{End}_C(V) \cong M_n(C) \) and hermitian forms over \( C \) is given by \( \sigma_h \leftrightarrow h \). So (2) implies that the \((n_1, \ldots, n_d)\)-frame is preserved by the involution and hence (3). Assuming (3), then [KMRT98, Proposition 4.2] gives us that \( h \cong \lambda h' \) for some non-zero \( \lambda \), and since the \((n_1, \ldots, n_d)\)-frame is preserved, \( h_1 \cong h'_1 \). But we have assumed that \( q_1 \) and \( q'_1 \) have determinant 1, so by the equivalence for (1) and (2) plus the above Lemma 3.1, \( \lambda \) is represented by \( \phi \), and this implies \( h_i \cong h'_i \) for each \( 1 \leq i \leq d \), which is (2).

The equivalence of (3) and (4) follows from the correspondence between algebras with involution and special Jordan algebras of degree \( \geq 3 \) (see [Ja68, p. 210]), since the canonical \((n_1, \ldots, n_d)\)-frames are preserved by this correspondence. \( \square \)

Notice that the equivalence of (2) and (3) crucially used that \( q_1 \) and \( q'_1 \) have determinant 1, which we can only assume because \( n_1 \) is odd.

**Remark 3.3.** This gives another proof of the fact from [Ja68, p. 204], which uses the coordinatization theorem, that Jordan algebras \( \mathcal{H}(\phi, q) \) and \( \mathcal{H}(\phi, q') \) are isomorphic for \( \phi \) hyperbolic, and in fact the isomorphism preserves the diagonal idempotents.

**Remark 3.4.** The equivalence of (1), (2) and (3) is also proved in [HKRT96, Theorem 15] in the case of \( r = 1 \) and \( n_\bullet = (3) \).

**Corollary 3.5.** For \( n_1 + \cdots + n_d \geq 3 \) and \( n_1 \) odd, then \( \mathcal{H}(\phi, q_1 \perp \cdots \perp q_d) \cong \mathcal{H}(\phi', q'_1 \perp \cdots \perp q'_d) \) as \( G'_{n_\bullet} \)-torsors if and only if \( \phi \cong \phi' \) and \( \phi \otimes q_i \cong \phi' \otimes q'_i \) for all \( 1 \leq i \leq d \).

**Proof.** The \( r \neq 3 \) cases are covered in Proposition 3.2, noting from [Mc04, 16.11] that isomorphic reduced simple Jordan algebras of degree \( \geq 3 \) have isomorphic composition algebras, and hence isometric norm forms \( \phi \) and \( \phi' \).

In the \( r = 3 \) case, as usual we insist that \( n = 3 \). There are 3 cases to consider: \( n_\bullet = (1, 1, 1), (1, 2) \) or \( (3) \). The first case follows from [SV00, 5.8.1] or [AJ57, Theorem 5], the second case follows from [SV00, 5.8.1], while the \( (1, 1, 1) \) case is exactly the statement of [AJ57, Theorem 9]. \( \square \)

Following the notation of [GMS03], we will let \( Q_n(k) \) denote the set of isometry classes of \( n \)-dimensional non-degenerate quadratic forms over \( k \), \( Q_{n,1}(k) \) the set of isometry classes of \( n \)-dimensional non-degenerate quadratic forms over \( k \) with determinant 1, and \( Pf_r(k) \) will be the set of isometry classes of \( r \)-fold Pfister forms over \( k \). Using Corollary 3.5 we will abuse notation, and consider \( \mathcal{H} \) as a morphism of functors, for \( n \geq 3 \):

\[
\mathcal{H}(k) : Pf_r(k) \times Q_{n_1,1}(k) \times Q_{n_2}(k) \times \cdots \times Q_{n_d}(k) \to H^1(k, G'_{n_1,\ldots,n_d}),
\]

\[
(\phi, (a_1, \ldots, a_{n_1}), \ldots, (a_{n-n_d+1}, \ldots, a_n)) \mapsto \text{Sym}(M_n(C_\phi), \sigma(a_1, \ldots, a_n)).
\]

The Jordan algebra on the right-hand side is equipped with the diagonal \( n_\bullet \)-frame. This is well defined, which is to say, the isomorphism class of the \( n_\bullet \)-frame doesn’t depend on the choices of diagonalizations, by the previous corollary.
3.6. Reduced Jordan algebras with frames

Recall that a Jordan algebra over $k$ that has a $(1, 1, \ldots, 1)$-frame is called reduced.

**Lemma 3.7.** Let $J$ be a central simple Jordan algebra with an $(n_1, \ldots, n_d)$-frame where $n_1$ is odd. If either $r = 0$, 2 or $\gcd(n_i) = 1$ then $J$ is reduced.

**Proof.** If $r = 0$ or 2 and $n$ is odd, then [Mac08a, Lemma 3.3] shows that $J$ is reduced.

If $r = 0$ or 2 and $n$ is even, then both idempotents $u_1$ and $\sum_{i>1} u_i$, from the $(n_1, \ldots, n_d)$-frame, are of odd rank. Then consider the Peirce subspaces $J_1(u_1)$ and $J_1(\sum_{i>1} u_i)$, of elements for which multiplication by the idempotent is the identity (in other words, $J_1(u) := \{x \in J \mid xu = x\}$). These subspaces are closed under multiplication and the idempotent acts as the identity, so they are Jordan algebras of degrees $n_1$ and $\sum_{i>1} n_i$ respectively. The Jordan algebras are simple because their primitive idempotents are connected, since $J$ is simple [Mc04, 20.2.4]. So by [Mac08a, Lemma 3.3] they are both reduced, and hence $J$ is reduced.

Finally, from [Mc04, 20.2.4] we know that for any central simple Jordan algebra with $(n_1, \ldots, n_d)$-frame we can decompose the frame further into idempotents of equal rank. So if the greatest common divisor of the $n_i$’s is 1, then this rank must be 1, and hence the algebra is reduced. □

We will use the following terminology from [Ga09, 5].

**Definition 3.8.** Let $A, B$ be functors from the category Fields/$k_0$ of field extensions of $k_0$ to the category of Sets. Then a morphism of functors $F : A \to B$ is called surjective at 2, if for every extension $k/k_0$ and $x \in B(k)$, there is an extension $L/k$ of odd degree such that $x_L = F(L)(y)$ for some $y \in A(L)$.

The next proposition shows that the functor $\mathcal{H}$ as defined at the end of Section 3 is surjective at 2.

**Proposition 3.9.** Let $(J; u_1, \ldots, u_d)$ be a central simple Jordan algebra over $k$ of degree $n$ with an $(n_1, \ldots, n_d)$-frame with $n_1$ odd. Then there is an odd degree extension $L/k$ such that $(J; u_1, \ldots, u_d)_L$ is in the image of $\mathcal{H}(L)$.

**Proof.** First we assume that $n = \sum_i n_i$ is odd. In this case we can apply [Mac08a, Lemma 3.4] to get an odd extension $L/k$ that reduces $J$. Then by [Al47, p. 561] we can decompose each $u_i$ into pairwise orthogonal (absolutely) primitive idempotents, giving a $(1, \ldots, 1)$-frame. These idempotents are connected since $J$ is simple [Mc04, 20.2.4], so we can apply the Jacobson coordinatization theorem [Mc04, 17.2.1]. This gives us an isomorphism from $J_L$ to a Jordan algebra in the image of $\mathcal{H}(L)$ sending these primitive idempotents to the canonical primitive idempotents $e_j$ for $1 \leq j \leq n$. Now it is clear that the given $(n_1, \ldots, n_d)$-frame is sent to the canonical $(n_1, \ldots, n_d)$-frame in $\mathcal{H}(L)$, as required.

For $n$ even, $n_1$ and $\sum_{i>1} n_i$ are odd, so consider the Peirce spaces $J_1(u_1)$ and $J_1(\sum_{i>1} u_i)$ as in the proof of Lemma 3.7. They are central simple Jordan algebras of odd degree, so there is an odd extension $L/k$ that reduces both of them. By decomposing each $u_i$ into pairwise orthogonal rank 1 idempotents, as in the $n$ odd case, the same argument applies. □

4. Cohomological invariants

We will follow the notation of [GMS03] (see also [Mac08b, 5], [Se95], [Ga09]) and write $H^*(k)$ or even $H^*$ for the Galois cohomology ring $H^*(\text{Gal}(k_s/k), \mathbb{Z}/2\mathbb{Z})$. For $a \in k^*/(k^*)^2$, we will denote the corresponding element $(a) \in H^1(k)$ so that we have $(a \cdot b) = (a) + (b)$. We let $Q_{r,1}(k)$ be the pointed set of $k$-isometry classes of $n$-dimensional quadratic forms of determinant 1. And we let $P_l(k)$ be the pointed set of $k$-isometry classes of $r$-Pfister forms. We will write $\text{Inv}(G) = \text{Inv}_{k_0}(H^1(\cdot, G))$ for the group of cohomological invariants in $H^*$ of $G$-torsors.
To define the Stiefel–Whitney classes of a quadratic form \( q \) over \( k \), take a diagonalization \( q \cong \langle a_1, \ldots, a_n \rangle \). Then define \( w_i(q) \) to be the \( i \)th degree part of the product

\[
w(q) = \prod_{j=1}^{n} (1 + (a_j)) \in H^*(k).
\]

This product is called the **total Stiefel–Whitney class**. It is independent of the diagonalization, which can be shown by a chain equivalence argument (see [Mi70]), so \( w \) is a cohomological invariant of quadratic forms.

It is shown in [GMS03, 17.3] that \( \text{Inv}(\mathcal{O}_n) = \text{Inv}(\mathcal{O}_n) \) is a free \( H^*(k_0) \)-module which has a basis consisting of the set \( \{1, w_1, \ldots, w_n\} \).

For \( n \)-Pfister forms, \( \text{Inv}(\text{Pf}_r) \), has an \( H^*(k_0) \)-basis consisting of \( \{1, e_r\} \).

**Remark 4.1.** For each \( r \geq 1 \), the invariant \( e_r \) classifies the \( n \)-Pfister form up to isometry, and hence the isomorphism class of its associated composition algebra [EL72, Main Theorem], [Vo96]. Elman and Lam show that the invariant in Milnor \( K \)-theory mod 2 classifies Pfister forms, but we need the Milnor conjecture to see that the invariant in Galois cohomology \( H^*(k) \) also classifies Pfister forms.

The Stiefel–Whitney classes, on the other hand, do not in general classify quadratic forms. Similarly, the invariants we will define for \( G_{n_1}^{r_1} \) will not, in general, classify the torsors up to isomorphism. See Section 4.10 for further discussion.

### 4.2. Cohomological invariants of Jordan algebras with frames

Now we will use the results above on the structure of Jordan algebras of degree \( \geq 3 \) with \( (n_1, \ldots, n_d) \)-frame to understand cohomological invariants of \( G_{n_1}^{r_1} \)-torsors with mod 2 coefficients.

**Corollary 4.3.** We have an injective map of \((\text{mod } 2)\) invariants

\[
\text{Inv}(G_{n_1}^{r_1} \cdots G_{n_d}^{r_d}) \hookrightarrow \text{Inv}(\text{Pf}_r) \otimes \text{Inv}(\mathcal{O}_{n_1,1}) \otimes \cdots \otimes \text{Inv}(\mathcal{O}_{n_d}).
\]

**Proof.** By [Ga09, Lemma 5.3] we can use the surjectivities from Proposition 3.9 to induce an injective map on invariants. Then from [GMS03, Ex. 16.5] or [Ga09, 6.7], we can express the invariants of the direct product \( \text{Pf}_r \times \mathcal{O}_{n_1,1} \times \cdots \times \mathcal{O}_{n_d} \) as the tensor product of the invariants of each factor. \( \square \)

Each of the factors of the right-hand side in the above corollary are well understood by Section 4. Now it is a matter of deciding which of these invariants occur as \( G_{n_i}^{r_i} \)-invariants. In other words, we wish to determine the image of the injective map in Corollary 4.3. It turns out to be the constant invariants together with all multiples of \( e_r \), the degree \( r \) invariant of \( \text{Pf}_r \).

**Theorem 4.4.** Let \( r = 0, 1, 2 \) or 3, and \( n_1, \ldots, n_d \) positive integers with \( n_1 \) odd, such that \( n = \sum n_i \geq 3 \) and if \( r = 3 \) then \( n = 3 \). For \( 1 \leq i \leq d \) let \( 0 \leq r_i \leq n_i \) be integers. Then the invariant

\[
e_r \otimes w_{r_1} \otimes \cdots \otimes w_{r_d} \in \text{Inv}(\text{Pf}_r) \otimes \text{Inv}(\mathcal{O}_{n_1,1}) \otimes \cdots \otimes \text{Inv}(\mathcal{O}_{n_d})
\]

extends uniquely to a \( G_{n_1}^{r_1} \)-invariant of degree \( r + \sum r_i \), which we will call \( \nu_{r_1} \)....
If the invariants extend, then by Corollary 4.3 they are unique. First we will show how to construct the invariants on reduced $G^n_r$-torsors, and then use [Ga09, Proposition 7.2] to extend them to all $G^n_r$-torsors.

Lemma 4.5. Let $\phi$ be an $r$-fold Pfister form. Let $q_i$ and $q'_i$ be quadratic forms over $k$ for $1 \leq i \leq d$, and $0 \leq r_i \leq \dim(q_i)$ integers. If $\phi \otimes q_i \cong \phi \otimes q'_i$ for all $i$, then

$$e_r(\phi)w_{r_1}(q_1) \cdots w_{r_d}(q_d) = e_r(\phi)w_{r_1}(q'_1) \cdots w_{r_d}(q'_d) \in H^r + \sum r_i(k).$$

Proof. Firstly, notice that for $d = 1$ this follows from [Mac08a, Lemma 4.4], simply by equating the homogeneous parts in the graded ring $H^*(k)$. For $d > 1$, we can repeatedly apply the $d = 1$ case to get the result, since $H^*(k)$ is commutative. □

Proof of Theorem 4.4. First we will show that the invariants $e_r \otimes w_{r_1} \otimes \cdots \otimes w_{r_d}$ extend to invariants on $k$-isomorphism classes of reduced Jordan algebras with $(n_1, \ldots, n_d)$-frame.

As in the proof of Proposition 3.9, by the Jacobson coordinatization theorem we can write any such isomorphism class as $J = H(\phi, q_1, \ldots, q_d)$ with canonical $(n_1, \ldots, n_d)$-frame. We will use the notation $q = q_1 \perp \cdots \perp q_d$ as usual, and in particular that $\det(q_1) = 1$. By Corollary 3.5 we have that $\phi$ and $\phi \otimes q_i$ are determined up to isometry by the $(n_i)$-frame isomorphism class of $(J; q_1, \ldots, q_d)$. Then by Lemma 4.5 we can define

$$v_{r_1, \ldots, r_d}(J; u_1, \ldots, u_d) := e_r(\phi)w_{r_1}(q_1) \cdots w_{r_d}(q_d) \in H^r + \sum r_i(k)$$
on the image of $H(k)$ in $G^n_r$-torsors. So we have extended $e_r \otimes w_{r_1} \otimes \cdots \otimes w_{r_d}$ to reduced torsors.

To finish the proof, recall that by Proposition 3.9 we know $H$ is surjective at 2. We are in the situation of [Ga09, Proposition 7.2], which states that if $F : A \to B$ is an injective morphism of functors that is surjective at 2, then every mod 2 invariant of $A$ extends to $B$. So these invariants may be extended to all $G^n_r$-torsors. By Corollary 4.3, we know that $v_{r_1, \ldots, r_d}$ is the unique invariant extending $e_r \otimes w_{r_1} \otimes \cdots \otimes w_{r_d}$. □

Remark 4.6. Notice that if $r_1$ is odd, then $v_{r_1, \ldots, r_d}$ is the zero invariant, since the odd Stiefel–Whitney classes of a determinant 1 quadratic form are zero (see [GMS03, 17.4]).

Now we can state and prove the main theorem.

Theorem 4.7. Let $n_1 + \cdots + n_d \geq 3$ with $n_1$ odd. Then $\text{Inv}_{k_0}^\text{e}(G^n_{n_1, \ldots, n_d})$ is a free $H^*(k_0)$-module, where the basis is the set consisting of 1 together with the invariants $\{v_{r_1, \ldots, r_d}\}$, for all choices of $0 \leq r_i \leq n_i$ with $r_1$ even.

Proof. For $r = 0$ this is shown in [GMS03, ch. VI], since we have an isomorphism $G^n_{n_1, \ldots, n_d} \cong \text{SO}_{n_1} \times \text{O}_{n_2} \times \cdots \times \text{O}_{n_d}$ (see above, Theorem 2.2). Notice that in this case $v_{0, \ldots, 0} = 1$, causing a redundancy in the set of basis elements. Also, the injection in Corollary 4.3 is an isomorphism for $r = 0$.

So take $r > 0$. From Corollary 4.3 and [GMS03, 18.1] we know that every $G^n_{n_1, \ldots, n_d}$-invariant restricts to

$$1 \otimes a + e_r \otimes b \in \text{Inv}^\text{e}(\text{Pf}_r) \otimes \text{Inv}(\text{O}_{n_1,1} \times \cdots \times \text{O}_{n_d}),$$

for some uniquely defined $a, b \in \text{Inv}(\text{O}_{n_1,1} \times \cdots \times \text{O}_{n_d})$. We know from the $r = 0$ case that any such $b$ is in the $H^*(k)$-span of the products of Stiefel–Whitney classes $w_{r_1} \otimes \cdots \otimes w_{r_d}$ where $r_1$ is even. So by Theorem 4.4, $e_r \otimes b$ is the restriction of some $G^n_{n_1, \ldots, n_d}$-invariant in the $H^*(k)$-span of $\{v_{r_1, \ldots, r_d}\}$. So all that remains to show is that if $1 \otimes a$ is the restriction of a $G^n_{n_1, \ldots, n_d}$-invariant, then $a$ is constant.

Let $a'$ be a $G^n_{n_1, \ldots, n_d}$-invariant that restricts to $1 \otimes a$. If we let $\phi_s$ be the split Pfister $r$-form, then for any quadratic forms $q$ and $q'$, we know $H(\phi_s, q)$ and $H(\phi_s, q')$ are split, and are isomorphic as Jordan $k$-algebras with canonical $(n_s)$-frames, by Corollary 3.5. So on reduced algebras with $(n_s)$-frames,
Proof of Theorem 0.1. This follows directly from Theorem 4.7. The expression \( p(t) \) counts the degrees of the invariants \( v_{r_1,...,r_n} \) whenever they are non-trivial (see Definition 5.1 below), which is always the case unless \( r_1 \) is odd, or \( r = 0 \) and \( r_i = 0 \) for all \( i \). So to count the \( r_1 \) part of the degree, notice that 
\[
(1 + t^2 + t^4 + \cdots + t^{n_1-1}) = (1 + t + \cdots + t^{n_1})/(1 + t).
\]
\[
\Box
\]

Remark 4.8. This theorem may be extended to include the \( n = 2 \) and \( n = 1 \) cases. See [Mac08b] for details.

Remark 4.9. Let us note that \( G_{1,1,1}^1 \cong \mathbb{Z}/2\mathbb{Z} \ltimes \text{GL}_2^2 \), where \( \mathbb{Z}/2\mathbb{Z} \) acts by inverse transpose. According to Theorem 4.7 the invariants of this group have an \( H^\ast(k_0) \)-basis with degrees 0, 1, 2, and 3. If instead we consider the semi-direct product with a trivial \( \mathbb{Z}/2\mathbb{Z} \) action (which gives the direct product) then \( \text{Inv}(\mathbb{Z}/2\mathbb{Z} \ltimes \text{GL}_2^2) = \text{Inv}(\mathbb{Z}/2\mathbb{Z}) \). In fact, if we defined the \( \mathbb{Z}/2\mathbb{Z} \) action by permuting the two copies of \( \text{GL}_1 \), then it is shown in [Gu07, Example 4.2.1] that \( \text{Inv}(\mathbb{Z}/2\mathbb{Z} \ltimes \text{GL}_2^2) = \text{Inv}(\mathbb{Z}/2\mathbb{Z}) \), as well.

4.10. Classification of \( G_{n_*}^r \)-torsors by invariants

One may roughly interpret the invariant \( v_{0,...,0} \) as a measure of how far away the Jordan algebra is from being split, and the other \( v_{r_1,...,r_n} \)'s as measures of how twisted the Jordan algebra is.

One may ask to what extent do these \( v_{r_*} \) invariants determine the \( G_{n_*}^r \)-torsors? The \( d = 1 \) case is discussed in [Mac08a, Remark 4.8]. The answer is closely depends on when the Steifel–Whitney classes of quadratic forms determine their isometry classes. In particular we will use the following lemma.

Lemma 4.11.

1. Fix an \( r \geq 0 \) and \( n \leq 3 \). Let \( \phi \) be an \( r \)-Pfister form over any field \( k \), and let \( q, q' \) be quadratic forms of dimension \( n \) over \( k \). Then \( \phi \otimes q \cong \phi \otimes q' \) if and only if
\[
e_r(\phi)w(q) = e_r(\phi)w(q') \in H^\ast(k).
\]

2. Fix an \( r \geq 0 \) and \( n \geq 4 \). There exists a field \( k \), an \( r \)-Pfister form \( \phi \) over \( k \), and quadratic forms \( q, q' \) over \( k \) of dimension \( n \) such that \( \phi \otimes q \not\cong \phi \otimes q' \) but
\[
e_r(\phi)w(q) = e_r(\phi)w(q') \in H^\ast(k).
\]

Proof. The “only if” part of (1) is true for any \( n \), as proved in Lemma 4.5(1).

(1) For \( n = 3 \), we can write any quadratic form as \( q = \lambda(-a,-b,ab) \). Notice that \( e_r(\phi)w(q) \) determines the isometry classes of \( \phi \) and \( \phi \otimes (\lambda) \), by considering the degree \( r \) and \( r+1 \) parts, respectively. Also, a quick computation shows that these together with \( e_r(\phi)w_2(q) \) determines the isometry class of \( \phi \otimes (a,b) \). Now by choosing any non-zero \( \lambda' \in k^\ast \) that is represented by \( \phi \otimes (\lambda) \), we see that \( e_r(\phi)w(q) \) determines the isometry class of \( \phi \otimes \lambda'(a,b) \cong \phi \otimes \lambda(a,b) \). Then by Witt cancellation, the isometry class of \( \phi \otimes q \) is determined, as required.

The cases \( n = 1, 2 \) are similar and easier.

(2) For \( n \geq 4 \), we will construct a counter-example as in [Sch67, Beispiel 3.4.1]. Let \( k_0 \) be an algebraically closed field, let \( t_1, \ldots, t_{r+3} \) be indeterminates, and define \( k = k_0(t_1, \ldots, t_{r+3}) \). Then \( \phi := \langle \langle t_1, \ldots, t_r \rangle \rangle \) is a non-hyperbolic \( r \)-Pfister form. Define the following \( n \)-dimensional quadratic forms over \( k \),
\[
q := t_{r+3} \langle \langle t_{r+1}, t_{r+2} \rangle \rangle \perp \langle 1, \ldots, 1 \rangle,
\]
\[
q' := \langle \langle t_{r+1}, t_{r+2} \rangle \rangle \perp \langle 1, \ldots, 1 \rangle.
\]
It is an easy check that their total Stiefel–Whitney classes \( w(q) = w(q') \) are equal. But \( \phi \otimes q \) represents \( t_{r+3} \), and \( \phi \otimes q' \) doesn’t, so they are not isometric, as required.  

We have implicitly used the Milnor conjecture here for part (1), as discussed in Remark 4.1.

**Theorem 4.12.** Let \( r = 0, 1, 2 \) or 3, and let \( n_* = (n_1, \ldots, n_d) \) with \( n_1 \) odd, such that \( \sum n_i = n \geq 3 \) and if \( r = 3 \) then \( n = 3 \).

1. Assume that \( n_i \leq 3 \) for each \( 1 \leq i \leq d \), and that if \( r = 1 \) or 3 then there is an \( i \) such that \( n_i \neq 3 \). Then \( G_{n_*}^r \) -torsors over any field \( k \) are classified by their mod 2 invariants.
2. Assume that there is an \( n_i \geq 4 \). Then there exists a field \( k \) and two non-isomorphic \( G_{n_*}^r \) -torsors over \( k \) that have equal mod 2 invariants.
3. Assume \( r = 1 \) or 3, and that \( n_i \geq 3 \) for each \( 1 \leq i \leq d \). Then there exists a field \( k \) and two non-isomorphic \( G_{n_*}^r \) -torsors over \( k \) that have equal mod 2 invariants.

**Proof.** (1) Since \( n_i \leq 3 \) for all \( i \), Lemma 4.11 together with Corollary 3.5 show that reduced central simple Jordan algebras with \( n_* \) -frame are classified by their mod 2 invariants.

- **Case** \( r = 0 \) or 2: Since \( n_1 \) is odd, all \( G_{n_*}^r \) -torsors are reduced, by Lemma 3.7. So the mod 2 invariants classify the torsors in this case.
- **Case** \( r = 3, n_* \neq (3, \ldots, 3) \): Again by Lemma 3.7, the torsors are always reduced and hence the mod 2 invariants classify them.
- **Case** \( r = 1, n_* \neq (3, \ldots, 3) \): Since \( n_1 \) is odd, and there is an \( n_i \) that is either 1 or 2, we know by Lemma 3.7 that all \( G_{n_*}^r \) -torsors are reduced. So in these cases the mod 2 invariants classify the torsors.

(2) If there is an \( n_i \geq 4 \), then we can use the counter-example from the above Lemma 4.11(2) together with Corollary 3.5 to construct two non-isomorphic reduced central simple Jordan algebras that have equal mod 2 invariants.

(3) By part (2) we can assume that \( n_i = 3 \) for all \( i \).

- **Case** \( r = 3, n_* = (3, \ldots, 3) \): Then for any non-reduced simple exceptional Jordan algebra, there is a “reduced model” with equal invariants [PR96]. So the invariants do not classify the torsors in this case.
- **Case** \( r = 1, n_* = (3, \ldots, 3) \): Let \( k_0 \) be an algebraically closed field. We can find a field extension \( k/k_0 \) and a non-reduced simple Jordan algebra \( J \) over \( k \) with \( n_* \) -frame, \( u_* \). For each \( i \), we have that the Peirce space \( J_1(u_i) \) is a degree 3 simple Jordan algebra. Their reduced models [PR96] give us quadratic forms \( \langle \alpha \rangle \otimes q_i \) whose invariants \( e_i \otimes w \) agree with the invariants \( v_i \) for the Jordan algebra. Here \( \alpha \in k^* \) and \( q_i \) is 3-dimensional. Then all of the mod 2 invariants of the non-reduced algebra \( (J, u_*) \) are equal to the mod 2 invariants of the reduced algebra \( \mathcal{H}(\langle \alpha \rangle, q_1, \ldots, q_d) \). So the mod 2 invariants do not classify the torsors in these cases.  

**5. Essential dimension**

Given an algebraic group \( G \) over \( k_0 \), and a \( G \) -torsor \( E \) over \( k \), the essential dimension of \( E \) is defined to be the minimum transcendence degree over \( k_0 \) of all fields of definition of \( E \). A subfield \( k \supset L \supset k_0 \) is said to be a field of definition if there is a \( G \) -torsor \( E' \) over \( L \) such that \( E'_k \cong E \). One thinks of \( \text{ed}(E) \) as the number of parameters needed to define the torsor \( E \). The essential dimension of an algebraic group is defined to be the supremum of the essential dimensions of all of its torsors [RY00,BF03,CS06].

We can also talk about the essential dimension at a prime \( p \). If \( E \) is a \( G \) -torsor over \( k \), then

\[ \text{ed}(E; p) := \inf \{ \text{ed}(E_L) \mid L/k \text{ a finite extension of degree prime to } p \} . \]

And then \( \text{ed}(G; p) \) is the supremum of \( \text{ed}(E; p) \) over all \( G \) -torsors (see [CS06]). In particular, \( \text{ed}(G; p) \leq \text{ed}(G) \).

We will determine the value of \( \text{ed}(G_{n_1, \ldots, n_d}^r; 2) \).
Definition 5.1. A mod 2 degree $m$ cohomological invariant $a : H^1(\cdot, G) \to H^m(\cdot, \mathbb{Z}/2\mathbb{Z})$ is called non-trivial if for any field extension $k/k_0$ there is another extension $L/k$ and a $G_L$-torsor $E$ such that $a(E) \neq 0$.

Lemma 5.2. If there is a non-trivial mod $p$ degree $m$ cohomological invariant of an algebraic group $G$ over $k_0$, then $\text{ed}(G; p) \geq m$.

Proof. This observation is due to Serre and is stated in [RY00, Lemma 6.9]. Note the definitions of a cohomological invariant and essential dimension used are somewhat different from the ones we used. One could also prove this from a slight modification of [BF03, Corollary 3.6], which states that if there is a degree $m$ cohomological invariant, then the essential dimension is at least $m$. We just need the fact that $H^m(k, \mathbb{Z}/p\mathbb{Z}) \to H^m(L, \mathbb{Z}/p\mathbb{Z})$ is injective when the degree of $L/k$ isn’t divisible by $p$ [Se02, I.2.4 Corollary].

Theorem 5.3. Let $n = \sum n_i$ such that $n_1$ is odd.

(1) Then

$$\text{ed}(G_{n_1, \ldots, n_d}; 2) = r + n - 1.$$  

(2) If $r = 0$ or 2, or $\gcd(n_1) = 1$ then

$$\text{ed}(G_{n_1, \ldots, n_d}) = \text{ed}(G_{n_1, \ldots, n_d}; 2).$$

Proof. (1) For $n \geq 3$, by the surjectivity at 2 in Lemma 3.9 we have that for any $\text{Aut}(J; n_1)$-torsor $(J; u_*)$ over $k$, there is an odd degree extension $L/k$ such that $(J; u_*)_L$ is in the image of $\mathcal{G}(L)$. So by using [BF03, Lemma 1.11] we have that $\text{ed}((J; u_*)_L) \leq \text{ed}((J; u_*)_L) \leq \text{ed}(\text{Pfr}) + \text{ed}(\mathbb{Q}_{n_1, 1}) + \cdots + \text{ed}(\mathbb{Q}_{n_d}) = r + n - 1$. This gives us the upper bound $\text{ed}(G_{n_1, \ldots, n_d}; 2) \leq r + n - 1$.

The lower bound uses the above Lemma 5.2 and the non-triviality of the degree $r + n - 1$ cohomological invariant $v_{n_1, \ldots, n_d}$ from Theorem 4.4. Non-triviality of these invariants is easy, simply by looking at their construction.

(2) If all $G_{n_i}$-torsors were reduced, then the surjectivity at 2 from Lemma 3.9 would be simply surjectivity. In this case the proof of (1) goes through for $\text{ed}(G)$, instead of $\text{ed}(G; 2)$. Indeed for $r = 0$ or 2, or $\gcd(n_1) = 1$, all torsors are reduced by Lemma 3.7.

References


