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Classification of irreducible modules of certain subalgebras of free boson vertex algebra

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Abstract

Let $M(1)$ be the vertex algebra for a single free boson. We classify irreducible modules of certain vertex subalgebras of $M(1)$ generated by two generators. These subalgebras correspond to the $\mathcal{W}(2, 2p-1)$-algebras with central charge $1-6(p-1)^2/p$ where p is a positive integer, $p \geqslant 2$. We also determine associated Zhu's algebras.

1. Introduction

Let $M(1)$ be the vertex algebra generated by a single free boson. For every $z \in \mathbb{C}$, this vertex algebra contains a Virasoro vertex operator subalgebra with central charge $1-12z^2$ (cf. [20,25]). Therefore, $M(1)$ can be treated as a vertex operator algebra of rank $1-12z^2$.

The vertex operator algebra $M(1)$ has a family of irreducible $\mathbb{Z}_{\geqslant 0}$-graded (untwisted) modules $M(1,\lambda)$, $\lambda \in \mathbb{C}$, and a $\mathbb{Z}_2$-twisted irreducible module $M(1)^{\theta}$. Some subalgebras of $M(1)$ have property that any irreducible module for such subalgebra can be constructed from twisted or untwisted modules for $M(1)$. In particular, this is true for the orbifold vertex operator algebra $M(1)^+$ (cf. [6]). Another interesting example of such vertex operator algebra was studied by W. Wang in [29,30]. He showed that $\mathcal{W}(2,3)$-algebra with central charge -2 can be realized as a subalgebra of $M(1)$, and that every irreducible $\mathcal{W}(2,3)$-module is obtained from $M(1)$-modules $M(1,\lambda)$.

Recall that for any vertex operator algebra V, Zhu in [31] constructed an associative algebra $A(V)$ such that there is one-to-one correspondence between the irreducible V-mo-

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dules and the irreducible $A(V)$-modules. In the cases mentioned above the authors explicitly determine corresponding Zhu's algebras as certain quotients of the polynomial algebra $\mathbb{C}[x, y]$.

In the present paper we shall investigate certain vertex subalgebras of $M(1)$ generated by two generators. Let us describe these algebras. Let ω be the Virasoro element in $M(1)$ which generates a subalgebra isomorphic to the simple Virasoro vertex operator algebra $W_0 \cong L(c_{p,1}, 0)$ (cf. [15,28]) where $c_{p,1} = 1 - 6(p-1)^2/p$, $p \geqslant 2$. This vertex operator algebra can be extended by the primary vector $H \in M(1)$ of level $2p-1$ which is defined using Schur polynomials. This new extended vertex operator algebra, which we denote by $\overline{M(1)}$, is known in the physics literature as $\mathcal{W}(2, 2p-1)$-algebra (cf. [7,8,19]).

We also study another description of the vertex operator algebra $\overline{M(1)}$. It is well-known (cf. [9,12]) that some vertex subalgebras of $M(1)$ can be defined as kernels of screening operators. In our particular case we consider two screening operators Q and $\widetilde{Q}$ defined using the formalism of (generalized) vertex operator algebras (see Section 2). Then $\operatorname{Ker}_{M(1)} Q$ is isomorphic to the vertex operator algebra $L(c_{p,1}, 0)$, and $\operatorname{Ker}_{M(1)} \widetilde{Q}$ is isomorphic to $\overline{M(1)}$. These cohomological characterizations provide some deeper information on the structure of $\overline{M(1)}$. It turns out that $\overline{M(1)}$ is a completely reducible module for the Virasoro algebra with central charge $c_{p,1}$. This is important since the larger vertex operator algebra $M(1)$ is not completely reducible.

We prove, as the main result, that every irreducible $\overline{M(1)}$-module can be obtained as an irreducible subquotient of a certain $M(1)$-module $M(1, \lambda)$. In order to prove this result, we determine explicitly Zhu's algebra $A(\overline{M(1)})$. It is isomorphic to the commutative, associative algebra $\mathbb{C}[x, y]/\langle P(x, y)\rangle$ where $\langle P(x, y)\rangle$ is the ideal in $\mathbb{C}[x, y]$ generated by polynomial

$$P(x, y) = y^2 - \frac{(4p)^{2p-1}}{(2p-1)!^2}\left(x + \frac{(p-1)^2}{4p}\right)\prod_{i=0}^{p-2}\left(x + \frac{i}{4p}(2p-2-i)\right)^2.$$

This implies that the irreducible $\mathbb{Z}_{\geqslant 0}$-graded $\overline{M(1)}$-modules are parameterized by the solutions of the equation $P(x, y) = 0$. The determination of the polynomial $P(x, y)$ is the central problem of our construction. When $p = 2$ (cf. [30]), this polynomial can be constructed from a level-six singular vector in a generalized Verma module over $\mathcal{W}(2, 3)$-algebra. In general case, the complicated structure of $\mathcal{W}(2, 2p-1)$-algebras (cf. [7,19]) makes also the calculation of singular vectors extremely difficulty.

Since we are primary concentrated to the problem of classification of irreducible representations, we only want to understand the structure of Zhu's algebra $A(\overline{M(1)})$. Fortunately, this structure can be described without explicit knowledge of relations among generators of $\overline{M(1)}$.

In order to find relations in $A(\overline{M(1)})$, we use a characterization of the subalgebra W_0 as the kernel of the operator $Q|M(1)$. Using this, we show that the elements $H_i H$ for $i \geqslant -2p$ belong to the subalgebra W_0. It turns out that these facts completely determine all relations in Zhu's algebra $A(\overline{M(1)})$ (cf. Section 6).

It is interesting, that the irreducible representations of Zhu's algebra $A(\overline{M(1)})$, and therefore of the vertex operator algebra $\overline{M(1)}$ are parameterized by points of a rational curve. Similar structures were found for some other irrational vertex operator (super)algebras (cf. [2]).

2. Lattice and free boson vertex algebras

We make the assumption that the reader is familiar with the axiomatic theory of vertex (operator) algebras and their representations (cf. [4,13–15,24,31]).

In this section, we shall recall some properties of the lattice and free boson vertex algebras. The details can be found in [3–5,14,20,25,26]. Using language of generalized vertex operator algebras (cf. [4,18,26]), we construct screening operators $\widetilde{Q}$ and Q acting on the vertex algebra $M(1)$. Moreover, we choose the Virasoro element $\omega \in M(1)$ such that $M(1)$ becomes a vertex operator algebra of rank $c_{p,1}$. At the end of this section we shall present a result describing the structure of $M(1)$ as a module for the Virasoro algebra with central charge $c_{p,1}$.

Let $p \in \mathbb{Z}_{\geqslant 0}$, $p \geqslant 2$. Let $\widetilde{L} = \mathbb{Z}\beta$ be a rational lattice of rank one with nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ given by

$$\langle \beta, \beta \rangle = \frac{2}{p}.$$

Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \widetilde{L}$. Extend the form $\langle \cdot, \cdot \rangle$ on $\widetilde{L}$ to $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ be the affinization of $\mathfrak{h}$. Set $\hat{\mathfrak{h}}^+ = t\mathbb{C}[t] \otimes \mathfrak{h}$; $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$. Then $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ are abelian subalgebras of $\hat{\mathfrak{h}}$. Let $U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-)$ be the universal enveloping algebra of $\hat{\mathfrak{h}}^-$. Let $\lambda \in \mathfrak{h}$. Consider the induced $\hat{\mathfrak{h}}$-module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C}_\lambda \simeq S(\hat{\mathfrak{h}}^-) \quad \text{(linearly)},$$

where $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on $\mathbb{C}$, $\mathfrak{h}$ acting as $\langle h, \lambda \rangle$ for $h \in \mathfrak{h}$ and c acts on $\mathbb{C}$ as multiplication by 1. We shall write $M(1)$ for $M(1, 0)$. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ write $h(n) = t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$. Then $M(1)$ is a vertex algebra which is generated by the fields $h(z)$, $h \in \mathfrak{h}$, and $M(1, \lambda)$, for $\lambda \in \mathfrak{h}$, are irreducible modules for $M(1)$.

We shall choose the following Virasoro element $M(1)$:

$$\omega = \frac{p}{4}\beta(-1)^2 + \frac{p-1}{2}\beta(-2).$$

The subalgebra of $M(1)$ generated by ω is isomorphic to the simple Virasoro vertex operator algebra $W_0 \cong L(c_{p,1}, 0)$ where $c_{p,1} = 1 - 6(p-1)^2/p$. Let

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

Thus $M(1)$ is a module for the Virasoro algebra (which we shall denote by *Vir*) with central charge $c_{p,1}$. In other words, $M(1)$ becomes a Feigin–Fuchs module for the Virasoro algebra (cf. [11]).

It is clear that $L(0)$ defines a $\mathbb{Z}_{\geqslant 0}$-graduation on $M(1) = \bigoplus_{m\in\mathbb{Z}_{\geqslant 0}} M(1)_m$. We shall write $\mathrm{wt}(a) = m$ if $v \in M(1)_m$. Thus $M(1)$ becomes a vertex operator algebra of rank $c_{p,1}$ with the Virasoro element ω.

Standard way for constructing vertex subalgebras of $M(1)$ is given by the concept of screening operators acting on $M(1)$ (cf. [9,12]). We shall construct some particular screening operators using language of generalized vertex operator algebras (cf. [4]).

As in [4,26] (see also [14,20]), we have the generalized vertex algebra

$$V_{\widetilde{L}} = M(1) \otimes \mathbb{C}\big[\widetilde{L}\big],$$

where $\mathbb{C}[\widetilde{L}]$ is a group algebra of $\widetilde{L}$ with a generator e^{β}. For $v \in V_{\widetilde{L}}$ let

$$V(v,z) = \sum_{s\in\frac{1}{p}\mathbb{Z}} v_s z^{-s-1}$$

be the corresponding vertex operator (for precise formulae see [4]).

Clearly, $M(1)$ is a vertex subalgebra of $V_{\widetilde{L}}$.

The Virasoro element $\omega \in M(1) \subset V_{\widetilde{L}}$ is also a Virasoro element in $V_{\widetilde{L}}$ implying that $V_{\widetilde{L}}$ has a structure of a generalized vertex operator algebra of rank $c_{p,1}$.

We have the following decomposition:

$$V_{\widetilde{L}} = \bigoplus_{m\in\mathbb{Z}} M(1) \otimes e^{m\beta}.$$

Remark 2.1. If $p = 2$ then $V_{\widetilde{L}}$ is a vertex operator superalgebra which can be constructed using Clifford algebras (cf. [9,20]).

Define $\alpha = p\beta$. Then $\langle\alpha,\alpha\rangle = 2p$, implying that $L = \mathbb{Z}\alpha \subset \widetilde{L}$ is an even lattice. Therefore the subalgebra $V_L \subset V_{\widetilde{L}}$ has a structure of a vertex operator algebra with the Virasoro element ω. In particular, for $v, w \in V_L$, we have $Y(v,z)w = \sum_{n\in\mathbb{Z}} v_n w z^{-n-1}$.

Clearly,

$$M(1) \subset V_L \subset V_{\widetilde{L}}.$$

Define the Schur polynomials $S_r(x_1, x_2, \ldots)$ in variables $x_1, x_2, \ldots$ by the following equation:

$$\exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n\right) = \sum_{r=0}^{\infty} S_r(x_1, x_2, \ldots) y^r. \tag{2.1}$$

For any monomial $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ we have an element

$$h(-1)^{n_1} h(-2)^{n_2} \cdots h(-r)^{n_r} \mathbf{1}$$

in $M(1)$ for $h \in \mathfrak{h}$. Then for any polynomial $f(x_1, x_2, \ldots)$, $f(h(-1), h(-2), \ldots)\mathbf{1}$ is a well-defined element in $M(1)$. In particular, $S_r(h(-1), h(-2), \ldots)\mathbf{1} \in M(1)$ for $r \in \mathbb{Z}_{\geqslant 0}$. Set $S_r(h)$ for $S_r(h(-1), h(-2), \ldots)\mathbf{1}$.

We shall now list some relations in the generalized vertex operator algebra $V_{\widetilde{L}}$.

Let $\gamma, \delta \in \widetilde{L}$. Instead of recalling the exact Jacobi identity in $V_{\widetilde{L}}$, we shall only say that in the case

$$\langle \gamma, \delta \rangle \in 2\mathbb{Z},$$

the Jacobi identity gives the following formulas (cf. [4,18,26]):

$$Y(e^{\gamma}, z)e^{\delta} = \sum_{n \in \mathbb{Z}} e^{\gamma}_n e^{\delta} z^{-n-1}, \tag{2.2}$$

$$\left[e^{\gamma}_n, e^{\delta}_m\right] = \sum_{i=0}^{\infty} \binom{n}{i} \left(e^{\gamma}_i e^{\delta}\right)_{n+m-i} \quad (m, n \in \mathbb{Z}). \tag{2.3}$$

The following relations in the generalized vertex operator algebra $V_{\widetilde{L}}$ are of great importance:

$$e^{\gamma}_i e^{\delta} = 0 \quad \text{for } i \geqslant -\langle \gamma, \delta \rangle. \tag{2.4}$$

Especially, if $\langle \gamma, \delta \rangle \geqslant 0$, we have $e^{\gamma}_i e^{\delta} = 0$ for $i \in \mathbb{Z}_{\geqslant 0}$, and if $\langle \gamma, \delta \rangle = -n < 0$, we get

$$e^{\gamma}_{i-1} e^{\delta} = S_{n-i}(\gamma) e^{\gamma+\delta} \quad \text{for } i \in \{0, \ldots, n\}. \tag{2.5}$$

From the Jacobi identity in the (generalized) vertex operator algebras V_L and $V_{\widetilde{L}}$ follows:

$$\left[L(n), e^{\alpha}_m\right] = -m e^{\alpha}_{m+n}, \tag{2.6}$$

$$\left[L(n), e^{-\alpha}_m\right] = \left(2p(n+1) - m\right) e^{-\alpha}_{n+m}, \tag{2.7}$$

$$L(n)e^{\alpha} = \delta_{n,0} e^{\alpha} \quad (n \geqslant 0), \tag{2.8}$$

$$L(n)e^{-\alpha} = \delta_{n,0}(2p-1)e^{-\alpha} \quad (n \geqslant 0), \tag{2.9}$$

$$L(n)e^{-\beta} = \delta_{n,0} e^{-\beta} \quad (n \geqslant 0), \tag{2.10}$$

$$\left[L(n), e^{-\beta}_r\right] = -r e^{-\beta}_{r+n} \quad \left(r \in \frac{1}{p}\mathbb{Z}\right). \tag{2.11}$$

Define

$$Q = e^{\alpha}_0 = \mathrm{Res}_z\, Y(e^{\alpha}, z),$$
$$\widetilde{Q} = e^{-\beta}_0 = \mathrm{Res}_z\, Y(e^{-\beta}, z).$$

From (2.6) and (2.11) we see that the operators Q and $\widetilde{Q}$ commute with the Virasoro operators $L(n)$. We are interested in the action of these operators on $M(1)$. In fact, Q and $\widetilde{Q}$ are the screening operators, and therefore $\operatorname{Ker}_{M(1)} Q$ and $\operatorname{Ker}_{M(1)} \widetilde{Q}$ are vertex subalgebras of $M(1)$ (for details see Section 14 in [9] and reference therein).

Some properties of the screening operators Q and $\widetilde{Q}$ are given by the following lemma.

Lemma 2.1. *For $p > 1$ we have*:

(i) $[Q, \widetilde{Q}] = 0$.
(ii) $\widetilde{Q} e^{n\alpha} \neq 0$, $n \in \mathbb{Z}_{>0}$.
(iii) $\widetilde{Q} e^{-n\alpha} = 0$, $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. First we note that $\langle \alpha, \beta \rangle = 2 \in \mathbb{Z}$. Then the commutator formulae (2.3) gives that

$$[Q, \widetilde{Q}] = [e_0^{\alpha}, e_0^{-\beta}] = (e_0^{\alpha} e^{-\beta})_0 = (\alpha(-1) e^{\alpha-\beta})_0 = \frac{p}{p-1} (L(-1) e^{\alpha-\beta})_0.$$

Since $(L(-1)u)_0 = 0$ in every generalized vertex operator algebra, we conclude that $[Q, \widetilde{Q}] = 0$. This proves (i).

Relation (ii) follows from (2.5), and relation (iii) from (2.4). □

We shall now investigate the action of the operator Q. Since operators Q^j, $j \in \mathbb{Z}_{>0}$, commute with the action of the Virasoro algebra, they are actually intertwiners between Feigin–Fuchs modules inside the vertex operator algebra V_L.

Recall that a vector in V_L is called primary if it is a singular vector for the action of the Virasoro algebra.

Since $e^{-n\alpha}$ is a primary vector in V_L for every $n \in \mathbb{Z}_{\geqslant 0}$, we have that $Q^j e^{-n\alpha}$ is either zero or a primary vector. Fortunately, the question of non-triviality of intertwiners between Feigin–Fuchs modules is well studied in the literature. So using arguments from [8,21] together with the methods developed in [27] and [10] one can see that the following lemma holds.

Lemma 2.2. $Q^j e^{-n\alpha} \neq 0$ *if and only if* $j \in \{0, \ldots, 2n\}$.

Next, we shall present the theorem describing a structure of the vertex operator algebra $M(1)$ as a module for the Virasoro vertex operator algebra $L(c_{p,1}, 0)$. Again, the theorem can be proved using a standard analysis in the theory of Feigin–Fuchs modules (see [8,10, 11,21]).

Theorem 2.1.

(i) *The vertex operator algebra $M(1)$, as a module for the vertex operator algebra $L(c_{p,1}, 0)$, is generated by the family of singular and cosingular vectors $\widetilde{Sing} \cup \widetilde{CSing}$, where*

$$\widetilde{Sing} = \{u_n \mid n \in \mathbb{Z}_{\geqslant 0}\}; \qquad \widetilde{CSing} = \{w_n \mid n \in \mathbb{Z}_{>0}\}.$$

These vectors satisfy the following relations:

$$u_n = Q^n e^{-n\alpha}, \qquad Q^n w_n = e^{n\alpha}, \qquad U(Vir)u_n \cong L\big(c_{p,1}, n^2 p + np - n\big).$$

(ii) *The submodule generated by vectors* u_n, $n \in \mathbb{Z}_{\geqslant 0}$, *is isomorphic to*

$$[Sing] \cong \bigoplus_{n=0}^{\infty} L\big(c_{p,1}, n^2 p + np - n\big).$$

(iii) *The quotient module is isomorphic to*

$$M(1)/[Sing] \cong \bigoplus_{n=1}^{\infty} L\big(c_{p,1}, n^2 p - np + n\big).$$

(iv) $Qu_0 = Q\mathbf{1} = 0$, *and* $Qu_n \neq 0$, $Qw_n \neq 0$ *for every* $n \geqslant 1$.

Theorem 2.1 immediately gives the following result.

Proposition 2.1. *We have*

$$L(c_{p,1}, 0) \cong W_0 = \mathrm{Ker}_{M(1)}\, Q.$$

3. The vertex operator algebra $\overline{M(1)}$

Recall that the Virasoro vertex operator algebra $L(c_{p,1}, 0)$ is the kernel of the screening operator Q. But we have already seen that there is another screening operator $\widetilde{Q}$ acting on $M(1)$. Define the following vertex algebra

$$\overline{M(1)} = \mathrm{Ker}_{M(1)}\, \widetilde{Q}.$$

Since $\widetilde{Q}$ commutes with the action of the Virasoro algebra, we have that

$$L(c_{p,1}, 0) \cong W_0 \subset \overline{M(1)}.$$

This implies that $\overline{M(1)}$ is a vertex operator subalgebra of $M(1)$ in the sense of [13] (i.e., $\overline{M(1)}$ has the same Virasoro element as $M(1)$).

The following theorem will describe the structure of the vertex operator algebra $\overline{M(1)}$ as a $L(c_{p,1}, 0)$-module.

Theorem 3.1. *The vertex operator algebra* $\overline{M(1)}$ *is isomorphic to* $[Sing]$ *as a* $L(c_{p,1}, 0)$-*module, i.e.,*

$$\overline{M(1)} \cong \bigoplus_{n=0}^{\infty} L\big(c_{p,1}, n^2 + np - n\big).$$

Proof. By Theorem 2.1 we know that the $L(c_{p,1},0)$-submodule generated by the set $\widetilde{Sing}$ is completely reducible. So to prove the assertion, it suffices to show that the operator $\widetilde{Q}$ annihilates vector $v \in \widetilde{Sing} \cup \widetilde{CSing}$ if and only if $v \in \widetilde{Sing}$. Let $v \in \widetilde{Sing}$, then $v = Q^n e^{-n\alpha}$ for certain $n \in \mathbb{Z}_{\geqslant 0}$. Since by Lemma 2.1, $\widetilde{Q} e^{-n\alpha} = 0$, we have that

$$\widetilde{Q} v = \widetilde{Q} Q^n e^{-n\alpha} = Q^n \widetilde{Q} e^{-n\alpha} = 0.$$

Let now $v \in \widetilde{CSing}$. Then there is $n \in \mathbb{Z}_{>0}$ such that $Q^n v = e^{n\alpha}$. Assume that $\widetilde{Q} v = 0$. Then we have that

$$0 = Q^n \widetilde{Q} v = \widetilde{Q} Q^n v = \widetilde{Q} e^{n\alpha},$$

contradicting Lemma 2.1(ii). This proves the theorem. □

Remark 3.1. It is very interesting that although $M(1)$ is not completely reducible $L(c_{p,1},0)$-module, its subalgebra $\overline{M(1)}$ is completely reducible.

Next we shall prove that the vertex operator algebra $\overline{M(1)}$ is generated by only two generators.

Motivated by formulae (18) in [21], we define the following three (non-zero) elements in V_L:

$$F = e^{-\alpha}, \qquad H = QF, \qquad E = Q^2 F.$$

From relations (2.6)–(2.9) we see that

$$L(n)E = \delta_{n,0}(2p-1)E, \quad L(n)F = \delta_{n,0}(2p-1)F,$$
$$L(n)H = \delta_{n,0}(2p-1)H \quad (n \geqslant 0),$$

i.e., E, F and H are primary vectors in V_L. In fact, H is a primary vector in $M(1)$.

Lemma 3.1. *In the vertex operator algebra V_L the following relations hold*:

(i) $Q^3 F = 0$.
(ii) $E_i E = F_i F = 0$, *for every* $i \geqslant -2p$.
(iii) $Q(H_i H) = 0$, *for every* $i \geqslant -2p$.
(iv) $H = S_{2p-1}(\alpha)$.

Proof. Relation (i) is a special case of Lemma 2.2. Let now $i \in \mathbb{Z}$, $i \geqslant -2p$. From (2.4) we have that $F_i F = e_i^{-\alpha} e^{-\alpha} = 0$.

Next we observe that Q acts as a derivation on V_L, that is

$$Q(a_n b) = (Qa)_n b + a_n(Qb) \quad \text{for every } a, b \in V_L,\ n \in \mathbb{Z}. \tag{3.1}$$

Then using (i) and (3.1) we see that $E_i E$ is proportional to $Q^4(F_i F)$, which implies that $E_i E = 0$. This proves (ii).

Relation (iii) follows form (ii) and the fact that $Q(H_i H)$ is proportional to $Q^3(F_i F)$.

Relation (iv) is a direct consequence of (2.5). □

Theorem 3.2. *The vertex operator algebra $\overline{M(1)}$ is generated by ω and $H = S_{2p-1}(\alpha)$.*

Proof. Let U be the vertex subalgebra of $\overline{M(1)}$ generated by ω and H. We need to prove that $U = \overline{M(1)}$. Let W_n by the (irreducible) *Vir*-submodule of $\overline{M(1)}$ generated by vector u_n. Then $W_n \cong L(c_{p,1}, n^2 p + np - n)$. Using Lemma 2.2 we see that

$$\mathrm{Ker}_{\overline{M(1)}}\, Q^n \cong \bigoplus_{i=0}^{n-1} W_i.$$

It suffices to prove that $u_n \in U$ for every $n \in \mathbb{Z}_{\geqslant 0}$. We shall prove this claim by induction. By definition we have that $u_0, u_1 (= H) \in U$. Assume that we have $k \in \mathbb{Z}_{\geqslant 0}$ such that $u_n \in U$ for $n \leqslant k$. In other words, the inductive assumption is $\bigoplus_{i=0}^{k} W_i \in U$.

We shall now prove that $u_{k+i} \in U$. Set $j = -2kp - 1$. By Lemma 2.2 we have

$$Q^{2k+2} e^{-(k+1)\alpha} = Q^{2k+2}\bigl(e_j^{-\alpha} e^{-k\alpha}\bigr) \neq 0.$$

Next we notice that

$$Q^{k+1}(H_j u_k) = Q^{k+1}\bigl(Qe^{-\alpha}\bigr)_j \bigl(Q^k e^{-k\alpha}\bigr) = \frac{1}{2k+1} Q^{2k+2}\bigl(e_j^{-\alpha} e^{-k\alpha}\bigr),$$

which implies that

$$Q^{k+1}(H_j u_k) \neq 0.$$

So we have found vector $H_j u_k \in U$ such that

$$\begin{aligned} \mathrm{wt}(H_j u_k) &= (2p-1) + \bigl(k^2 p + kp - k\bigr) - j - 1 = (k+1)^2 p + (k+1)p - (k+1) \\ &= \mathrm{wt}(u_{k+1}). \end{aligned}$$

This implies that

$$H_j u_k \in \bigoplus_{i=0}^{k+1} W_i \quad \text{and} \quad H_j u_k \notin \bigoplus_{i=0}^{k} W_i.$$

Since $Q^{k+1}(\bigoplus_{i=0}^{k} W_i) = 0$ and $\mathrm{wt}(H_j u_k) = \mathrm{wt}(u_{k+1})$ we conclude that there is a constant C, $C \neq 0$, such that

$$H_j u_k = C u_{k+1} + u', \quad u' \in \bigoplus_{i=0}^{k} W_i \subset U.$$

Since $H_j u_k \in U$, we conclude that $u_{k+1} \in U$.

Therefore, the claim is verified, and the proof of the theorem is complete. □

Remark 3.2. If $p = 2$, then the elements E, F, H span the triplet algebra studied by M. Gaberdiel and H. Kausch (cf. [16,17,22]). In this case $\overline{M(1)}$ is isomorphic to $\mathcal{W}$-algebra $\mathcal{W}(2,3)$ with $c = -2$. Its irreducible modules were classified by W. Wang in [29,30].

In general, E, F and H span the $\mathcal{W}(2, 2p-1, 2p-1, 2p-1)$ algebra with central charge $c_{p,1}$ (cf. [8,21]). Our vertex operator algebra $\overline{M(1)}$ is isomorphic to the $\mathcal{W}$ algebra $\mathcal{W}(2, 2p-1)$ investigated in a number of physics papers (cf. [7,8,19]).

The following lemma will imply that for $i \geqslant -2p$ vector $H_i H$ can be constructed using only the action of the Virasoro operators $L(n)$ on the vacuum vector $\mathbf{1}$.

Lemma 3.2. *We have*:

$$H_i H \in W_0 \cong L(c_{p,1}, 0) \quad \text{for every } i \geqslant -2p.$$

In particular, $H_{-1} H \in W_0$.

Proof. The proof follows from Proposition 2.1 and Lemma 3.1(iii). □

Remark 3.3. In the case $p = 2$, the fact that $H_{-1} H \in W_0$ can be proved directly using a singular vector of level 6 in a generalized Verma module associated to $\mathcal{W}(2,3)$-algebra with central charge $c = -2$ (cf. [16,30]). This singular vector implies that in $\overline{M(1)}$ the following relation holds:

$$H_{-1} H = \frac{1}{4}\left(\frac{19}{36} L(-3)^2 + \frac{8}{9} L(-2)^3 + \frac{14}{9} L(-2) L(-4) - \frac{44}{9} L(-6)\right)\mathbf{1}.$$

Lemma 3.2 indicates the existence of similar relation of level $2(2p-1)$ in the general case, but we don't now the explicit formulae. Instead of looking for such formulae, we use the realization of the vertex operator algebra $\overline{M(1)}$ inside the lattice vertex operator algebra V_L, and the description of the Virasoro vertex operator algebra $L(c_{p,1}, 0)$ from Proposition 2.1.

4. Spanning sets for $\overline{M(1)}$ and $A(\overline{M(1)})$

In this section shall find a spanning set for $\overline{M(1)}$ and for Zhu's algebra $A(\overline{M(1)})$. First we recall the definition of Zhu's algebra for vertex operator algebras. Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. We shall always assume that

$$V = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} V_n, \quad \text{where } V_n = \{a \in V \mid L(0)a = nv\}.$$

For $a \in V_n$, we shall write $\mathrm{wt}(a) = n$.

Definition 4.1. We define the bilinear maps $*: V \otimes V \to V$, $\circ: V \otimes V \to V$ as follows:

$$a * b := \mathrm{Res}_z\, Y(a,z)\frac{(1+z)^{\mathrm{wt}(a)}}{z} b = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(a)}{i} a_{i-1}b,$$
$$a \circ b := \mathrm{Res}_z\, Y(a,z)\frac{(1+z)^{\mathrm{wt}(a)}}{z^2} b = \sum_{i=0}^{\infty} \binom{\mathrm{wt}(a)}{i} a_{i-2}b.$$

Extend to $V \otimes V$ linearly, denote $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$.

Denote by $[a]$ the image of a in V under the projection of V into $A(V)$. We have:

Theorem 4.1 [31].

(i) *The quotient space $(A(V), *)$ is an associative algebra with unit element $[\mathbf{1}]$.*
(ii) *Let $M = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} M(n)$ be a $\mathbb{Z}_{\geqslant 0}$-graded V-module. Then the top level $M(0)$ of M is a $A(V)$-module under the action $[a] \mapsto o(a) = a_{\mathrm{wt}(a)-1}$ for homogeneous $a \in V$.*
(iii) *Let (U, π) be an irreducible $A(V)$-module. Then there exists an irreducible $\mathbb{Z}_{\geqslant 0}$-graded V-module $L(U) = \bigoplus_{n\in\mathbb{Z}_{\geqslant 0}} L(U)(n)$ such that the top level $L(U)(0)$ of $L(U)$ is isomorphic to U as $A(V)$-module.*
(iv) *There is one-to-one correspondence between the irreducible $A(V)$-modules and the irreducible $\mathbb{Z}_{\geqslant 0}$-graded V-modules.*

We shall need some information about the commutators $[H_n, H_m]$ for $m, n \in \mathbb{Z}$.

Lemma 4.1. *For any $m, n \in \mathbb{Z}$, commutators $[H_n, H_m]$ are expressed as (infinite) linear combination of*

$$L(p_1)\cdots L(p_s), \quad p_1, \dots, p_s \in \mathbb{Z},\ s \leqslant 2p-1.$$

In particular, for every vector $v \in \overline{M(1)}$ we have

$$[H_n, H_m]v = fv, \quad \textit{for certain } f \in U(\mathit{Vir}).$$

Proof. By the commutator formulae in vertex (operator) algebras follows:

$$[H_m, H_n] = \sum_{i=0}^{\infty} \binom{m}{i} (H_i H)_{m+n-i}.$$

From Lemma 3.2 follows that $H_i H \in W_0$ for every nonnegative integer i. Thus, $H_i H$ is an element of the Virasoro vertex operator algebra $W_0 \cong L(c_{p,1}, 0)$. In fact, $\mathrm{wt}(H_i H) \leqslant 2(2p-1)$, which implies that $H_i H$ can be expressed as (finite) linear combination

$$L(-n_1)\cdots L(-n_s)\mathbf{1}, \quad n_i \geqslant 2,\ n_1 + \cdots + n_s \leqslant 2(2p-1),\ s \leqslant 2p-1.$$

Therefore, $[H_m, H_n]$ can be expressed as (infinite) linear combination of

$$L(p_1)\cdots L(p_s), \quad p_1, \ldots, p_s \in \mathbb{Z},\ s \leqslant 2p-1.$$

This proves the first assertion. The second assertion follows from the first assertion and from the simple observation that if $v \in \overline{M(1)}$, and $m, n \in \mathbb{Z}$, then $[H_m, H_n]v$ is well-defined. □

Remark 4.1. Using different arguments, a result similar to our Lemma 4.1 was noticed in the physics literature (cf. [7,19,23]).

Lemma 4.1 shows that the structure of the vertex operator algebra $\overline{M(1)}$ is similar to the structure of the vertex operator algebra $M(1)^+$ studied in [6].

So, using this lemma and a completely analogous proofs to the proofs of Proposition 3.4 and Theorem 3.5 in [6], one obtains the following theorem.

Theorem 4.2.

(i) *The vertex operator algebra $\overline{M(1)}$ is spanned by the following vectors*

$$L(-m_1)\cdots L(-m_s)H_{-n_1}\cdots H_{-n_t}\mathbf{1},$$

where $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s \geqslant 2$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant \mathbf{1}$.

(ii) *Zhu's algebra $A(\overline{M(1)})$ is spanned by the set*

$$\{[\omega]^{*s} * [H]^{*t} \mid s, t \geqslant 0\}.$$

In particular, Zhu's algebra $A(\overline{M(1)})$ is isomorphic to a certain quotient of the polynomial algebra $\mathbb{C}[x, y]$, where x and y correspond $[\omega]$ and $[H]$.

The fact that Zhu's algebra $A(\overline{M(1)})$ is commutative, enable us to study irreducible highest weight representations of the vertex operator algebra $\overline{M(1)}$. For given $(r, s) \in \mathbb{C}^2$, let $\mathbb{C}_{r,s}$ be the one dimensional module of $A(\overline{M(1)})$, with $[\omega]$ acting as the scalar r and $[H]$ as the scalar s. Therefore every irreducible $A(\overline{M(1)})$-module is one-dimensional, and it is isomorphic to a module $\mathbb{C}_{r,s}$ for certain $(r, s) \in \mathbb{C}^2$. Then Theorem 4.1 implies that every irreducible $\mathbb{Z}_{\geqslant 0}$-graded $\overline{M(1)}$-module is isomorphic to a certain module $L(\mathbb{C}_{r,s})$. So irreducible representations of $\overline{M(1)}$ are parameterized by certain subset of $\mathbb{C}^2$. In the following sections we will prove that this subset is a rational curve.

5. Representations of $\overline{M(1)}$

In this section we identify a family of irreducible $\overline{M(1)}$-modules. These modules are parameterized by points $(r, s) \in \mathbb{C}^2$ satisfying one algebraic equation.

By construction, the vertex operator algebra $\overline{M(1)}$ is a subalgebra $M(1)$. We now that for every $\lambda \in \mathfrak{h}(\cong \mathbb{C})$, $M(1, \lambda)$ is an irreducible $M(1)$-module with the highest weight vector v_λ. Thus $M(1, \lambda)$ is a $\overline{M(1)}$-module. Denote by $\widetilde{V}_\lambda$ the $\overline{M(1)}$-submodule generated by vector v_λ.

Set $H(n) = H_{n+2p-2}$, and $H(z) = \sum_{n\in\mathbb{Z}} H(n)z^{-n-2p+1}$.

First we recall the following result proved by the author in [1] for the purpose of studying $\mathcal{W}_{1+\infty}$-algebra.

Proposition 5.1 [1, Proposition 3.1]. *Let $h \in \mathfrak{h}$, and $r \in \mathbb{Z}_{\geqslant 0}$. Let $u = S_r(h(-1), h(-2), \ldots)\mathbf{1}$. Set $Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$. Then we have*

(1) $u_n v_\lambda = 0$ *for* $n > r - 1$,
(2) $u_{r-1} v_\lambda = \binom{\langle\lambda,h\rangle}{r} v_\lambda$.

Now Proposition 5.1 directly implies the following result.

Proposition 5.2. *For every $\lambda \in \mathfrak{h}$, $\widetilde{V}_\lambda$ is a $\mathbb{Z}_{\geqslant 0}$-graded $\overline{M(1)}$-module. The top level $\widetilde{V}_\lambda(0)$ is one-dimensional and generated by v_λ. Let $t = \langle\lambda, \alpha\rangle$. For every $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$L(n)v_\lambda = \delta_{n,0}\frac{1}{4p}t\big(t - 2(p-1)\big)v_\lambda, \tag{5.1}$$

$$H(n)v_\lambda = \delta_{n,0}\binom{t}{2p-1}v_\lambda. \tag{5.2}$$

By slightly abusing language, we can say that $\widetilde{V}_\lambda$ is a highest weight $\overline{M(1)}$-module with respect to the Cartan subalgebra $(L(0), H(0))$, and the highest weight is $(u(t), v(t))$ where $t = \langle\lambda, \alpha\rangle \in \mathbb{C}$ and

$$u(t) = \frac{1}{4p}t\big(t - 2(p-1)\big), \qquad v(t) = \binom{t}{2p-1}. \tag{5.3}$$

It is important to notice that for every $t \in \mathbb{C}$

$$u(t) = u\big(2(p-1) - t\big), \qquad v(t) = -v\big(2(p-1) - t\big).$$

Moreover, the mapping $t \mapsto (u(t), v(t))$ is a injection.

Now we notice that the top level of $\overline{M(1)}$-module $\widetilde{V}_\lambda(0) = \mathbb{C}v_\lambda$ has to be an irreducible module for Zhu's algebra $A(\overline{M(1)})$. We have the following isomorphism of $A(\overline{M(1)})$-modules:

$$\widetilde{V}_\lambda(0) \cong \mathbb{C}_{r,s}, \quad \text{where } r = u(t),\ s = v(t),\ t = \langle \lambda, \alpha \rangle.$$

Then one can show that the induced irreducible $\overline{M(1)}$-module $L(\widetilde{V}_\lambda(0)) \cong L(\mathbb{C}_{r,s})$ is isomorphic to the irreducible quotient of $\widetilde{V}_\lambda$.

Define $P(x, y) \in \mathbb{C}[x, y]$ by

$$P(x, y) = y^2 - C_p\left(x + \frac{(p-1)^2}{4p}\right)\prod_{i=0}^{p-2}\left(x + \frac{i}{4p}(2p-2-i)\right)^2, \tag{5.4}$$

where $C_p = (4p)^{2p-1}/(2p-1)!^2$.

In the following lemma we shall see that the highest weights of $\overline{M(1)}$-modules $\widetilde{V}_\lambda$ ($\lambda \in \mathfrak{h}$) coincide with the solutions of the equation $P(x, y) = 0$. The proof is similar to the proof of Lemma 4.4 in [30].

Lemma 5.1. *Solutions of the equation*

$$P(x, y) = 0 \tag{5.5}$$

are parameterized by

$$(x, y) = \big(u(t), v(t)\big), \quad t \in \mathbb{C}. \tag{5.6}$$

Proof. It is easy to verify that for every $t \in \mathbb{C}\, P(u(t), v(t)) = 0$.

Let now (x, y) be any solution of (5.5). Then there is $t_0 \in \mathbb{C}$ such that

$$x = u(t_0) = u\big(2(p-1) - t_0\big).$$

By substituting $x = u(t_0)$ in Eq. (5.5) we get that $y^2 = \binom{t_0}{2p-1}^2$, which implies that

$$y = \binom{t_0}{2p-1} = v(t_0) \quad \text{or} \quad y = -\binom{t_0}{2p-1} = v\big(2(p-1) - t_0\big).$$

So there is a unique $t \in \mathbb{C}$ such that (5.6) holds. □

Theorem 5.1. *Assume that $(r, s) \in \mathbb{C}^2$ so that $P(r, s) = 0$. Then*

(i) *$\mathbb{C}_{r,s}$ is an irreducible $A(\overline{M(1)})$-module.*
(ii) *$L(\mathbb{C}_{r,s})$ is an irreducible $\overline{M(1)}$-module.*

Proof. Using Lemma 5.1 we see that there is a unique $\lambda \in \mathfrak{h}$ so that $\mathbb{C}_{r,s} \cong \widetilde{V}_\lambda(0)$, where $\widetilde{V}_\lambda(0)$ is an $A(\overline{M(1)})$-module constructed in Proposition 5.2. So $\mathbb{C}_{r,s}$ is an irreducible $A(\overline{M(1)})$-module. Assertion (ii) follows from Theorem 4.1. □

6. Zhu's algebra $A(\overline{M(1)})$ and the classification of irreducible modules

In this section we shall prove our main result saying that the modules constructed in Section 5 provide all irreducible $\mathbb{Z}_{\geqslant 0}$-graded $\overline{M(1)}$-modules. Our proof will use the theory of Zhu's algebras.

We are now going to determine Zhu's algebra for the vertex operator algebra $\overline{M(1)}$. We have already proved that $A(\overline{M(1)})$ is isomorphic to a certain quotient of the polynomial algebra $\mathbb{C}[x, y]$, where x corresponds to $[\omega]$ and y to $[H]$. Now we shall find all relations in $A(\overline{M(1)})$.

Lemma 6.1. *In Zhu's algebra $A(\overline{M(1)})$, we have*:

$$P\big([\omega],[H]\big) = 0.$$

Proof. Lemma 3.2 implies that for every $i > -2p$

$$H_{i-1}H = f_i\big(L(-2), L(-3), \ldots\big)\mathbf{1}$$

for certain polynomial $f_i \in \mathbb{C}[x_1, x_2, \ldots]$. This implies that in $A(\overline{M(1)})$, we have

$$[H_{i-1}H] = g_i\big(\big[L(-2)\mathbf{1}\big]\big) = g_i\big([\omega]\big)$$

for certain polynomial $g_i \in \mathbb{C}[x]$ such that $\deg(g_i) \leqslant 2p - 1$. The definition of the multiplication in $A(\overline{M(1)})$ gives that

$$[H] * [H] = \sum_{i=0}^{2p-1} \binom{2p-1}{i} [H_{i-1}H] = \sum_{i=0}^{2p-1} \binom{2p-1}{i} g_i\big([\omega]\big).$$

Let $g(x) = \sum_{i=0}^{2p-1} \binom{2p-1}{i} g_i(x)$. So we have proved that there is polynomial $g \in \mathbb{C}[x]$ so that

$$[H] * [H] = [H]^2 = g([\omega]), \quad \deg(g) \leqslant 2p - 1. \tag{6.1}$$

Now we shall determine the polynomial g explicitly. Recall that if $(r, s) \in \mathbb{C}^2$ such that $P(r, s) = 0$, then $\mathbb{C}_{r,s}$ is an irreducible $A(\overline{M(1)})$-module (Theorem 5.1). Let us now evaluate both sides of (6.1) on $A(\overline{M(1)})$-modules $\mathbb{C}_{r,s}$. We get that $s^2 = g(r)$ for every $(r, s) \in \mathbb{C}^2$ such that $P(r, s) = 0$. This implies that for every $r \in \mathbb{C}$

$$g(r) = s^2 = s^2 - P(r, s) = C_p\left(r + \frac{(p-1)^2}{4p}\right)\prod_{i=0}^{p-2}\left(r + \frac{i}{4p}(2p - 2 - i)\right)^2,$$

where $C_p = (4p)^{2p-1}/(2p-1)!^2$. In this way we have proved that

$$g(x) = y^2 - P(x, y) = C_p\left(x + \frac{(p-1)^2}{4p}\right)\prod_{i=0}^{p-2}\left(x + \frac{i}{4p}(2p-2-i)\right)^2.$$

Using (6.1) we get

$$P\big([\omega], [H]\big) = [H]^2 - g\big([\omega]\big) = 0,$$

as desired. □

Now we are in the position to find all relations in Zhu's algebra $A(\overline{M(1)})$.

Theorem 6.1. *Zhu's algebra* $A(\overline{M(1)})$ *is isomorphic to the commutative, associative algebra* $\mathbb{C}[x, y]/\langle P(x, y)\rangle$.

Proof. By Theorem 4.2 we have a surjective algebra homomorphism

$$\Phi : \mathbb{C}[x, y] \to A\big(\overline{M(1)}\big),$$
$$x \mapsto [\omega],$$
$$y \mapsto [H].$$

It suffices to prove that $\operatorname{Ker}\Phi = \langle P(x, y)\rangle$.

Lemma 6.1 gives that $\langle P(x, y)\rangle \subseteq \operatorname{Ker}\Phi$.

Assume now that $K(x, y) \in \operatorname{Ker}\Phi$. Note that $P(x, y)$ has degree 2 in y. Using the division algorithm we get

$$K(x, y) = A(x, y)P(x, y) + R(x, y),$$

where $A(x, y), R(x, y) \in \mathbb{C}[x, y]$ so that $R(x, y)$ has degree at most **1** in y. So we can write $R(x, y) = B(x)y + C(x)$, where $B(x), C(x) \in \mathbb{C}[x]$. Since $P(x, y), K(x, y) \in \operatorname{Ker}\Phi$ we have that $R(x, y) \in \operatorname{Ker}\Phi$. We now evaluate polynomial $R(x, y)$ on $A(\overline{M(1)})$-modules and obtain

$$R\big(u(t), v(t)\big) = 0 \quad \text{for every } t \in \mathbb{C},$$

where polynomials $u(t), v(t)$ are defined by (5.3). Therefore

$$B\big(u(t)\big)v(t) = -C\big(u(t)\big) \quad \text{for every } t \in \mathbb{C}. \tag{6.2}$$

Assume that $B(x) \neq 0$. Then polynomial

$$B\big(u(t)\big)v(t) = B\left(\frac{1}{4p}t(t - 2p + 2)\right)\binom{t}{2p-1}$$

has odd degree in t. On the other hand, if polynomial $-C(u(t))$ is nontrivial, it must have even degree in t. This is a contradiction because of (6.2). So $B(x) = 0$. Using (6.2) we also get that $C(u(t)) = 0$ for every $t \in \mathbb{C}$, which implies that $C(x) = 0$. In this way we have proved that

$$R(x, y) = B(x)y + C(x) = 0,$$

and therefore

$$K(x, y) = P(x, y)A(x, y) \in \langle P(x, y) \rangle.$$

So $\operatorname{Ker} \Phi = \langle P(x, y) \rangle$, and the theorem holds. □

Theorem 6.2. *The set*

$$\left\{ L(\mathbb{C}_{r,s}) \mid (r, s) \in \mathbb{C}^2,\ P(r, s) = 0 \right\} \tag{6.3}$$

provides all non-isomorphic irreducible $\mathbb{Z}_{\geqslant 0}$-graded modules for the vertex operator algebra $\overline{M(1)}$.

Proof. Since Zhu's algebra $A(\overline{M(1)})$ is commutative, Theorem 6.1 implies that the set

$$\left\{ \mathbb{C}_{r,s} \mid (r, s) \in \mathbb{C}^2,\ P(r, s) = 0 \right\}$$

provides all irreducible modules for Zhu's algebra $A(\overline{M(1)})$. Then Theorem 4.1 implies that the set (6.3) provides all irreducible $\mathbb{Z}_{\geqslant 0}$-graded $\overline{M(1)}$-modules. □

Remark 6.1. Theorem 6.2 shows that the irreducible $\overline{M(1)}$-modules are parameterized by the solutions of the equation $P(x, y) = 0$. We have observed that all solutions of this equation can be written in the form $(u(t), v(t))$ $(t \in \mathbb{C})$ which are exactly the highest weights of $\overline{M(1)}$-submodules of $M(1, \lambda)$ constructed in Proposition 5.2. This leads to the conclusion (as in [30]) that every irreducible $\overline{M(1)}$-module can be identified starting from modules for the vertex operator algebra $M(1)$.

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