Classical integrability of two-dimensional non-linear sigma models

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Abstract

The conditions under which a general two-dimensional non-linear sigma model is classically integrable are given. These requirements are found by demanding that the equations of motion of the theory are expressible as a zero curvature relation. Some new integrable two-dimensional sigma models are then presented.

1. Introduction

As is well known, a two-dimensional theory is classically integrable if its equations of motion can be represented as a zero curvature relation. That is, a Lax pair \((A_0, A_1)\) can be found such that the commutator

\[
\left[ \partial_0 + A_0(\lambda), \partial_1 + A_1(\lambda) \right] = 0
\]

yields the equations of motion of the model under consideration for all values of the spectral parameter \(\lambda\). Here, the two-dimensional coordinates are \((\tau, \sigma)\) with \(\partial_0 = \frac{\partial}{\partial \tau}\) and \(\partial_1 = \frac{\partial}{\partial \sigma}\). In the rest of the Letter, however, we will use the complex coordinates \((z = \tau + i \sigma, \bar{z} = \tau - i \sigma)\) together with \(\partial = \frac{\partial}{\partial z}\) and \(\bar{\partial} = \frac{\partial}{\partial \bar{z}}\).

The classical example of this setting is the principal chiral non-linear sigma model in two dimensions \([1,2]\). The action of this theory is given by

\[
S = \int d\tau d\bar{\tau} \eta_{kl} \eta^{kl}(X) \bar{\eta}^{kl}(X) \partial X^k \bar{\partial} X^l,
\]
where \( \eta_{kl} \) is an invertible invariant bilinear form of the Lie algebra \( \mathcal{G} \) defined by \( [T_i, T_j] = f^k_{ij} T_k \). That is, \( \eta_{ij} f^i_{kj} + \eta_{kj} f^i_{ij} = 0 \). The vielbeins \( e^k_i \) satisfy the Cartan–Maurer relation
\[
\partial_i e^k_i - \partial_j e^k_j + f^k_{mn} e^m_i e^n_j = 0. \tag{1.3}
\]
The equations of motion of the principal chiral sigma model can be cast in the form
\[
\partial(e^k_i \partial X^i) + \tilde{\partial}(e^k_i \partial X^i) = 0. \tag{1.4}
\]
This expression is reached after using the properties of \( \eta_{ij} \) and \( e^k_i \).

The Lax pair for this model is constructed in the following manner: consider the commutator
\[
\left[ \frac{1}{2} + \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right] \frac{1}{2} - \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right] (e^j_i \partial X^j) T_k \right]
= \left\{ \begin{array}{l}
-\lambda \left[ \partial(e^k_i \partial X^i) + \tilde{\partial}(e^k_i \partial X^i) \right] \\
+ \left( \frac{1}{2} \pm \sqrt{\lambda^2} \right) \partial \left( e^j_i \partial X^j \right) - \tilde{\partial} \left( e^j_i \partial X^j \right) + f^k_{mn} (e^m_i \partial X^i) (e^n_j \partial X^j) \right\} T_k. \tag{1.5}
\right.
\]

If this commutator vanishes for all values of \( \lambda \) then the term proportional to \( -\lambda \) yields the equations of motion of the principal chiral sigma model. The second term is identically zero due to the Cartan–Maurer relations between the vielbeins \( e^k_i \).

In summary, the equations of motion are completely specified in terms of the currents
\[
I^i_j = e^j_i \partial X^j, \quad \tilde{I}^i_j = e^j_i \tilde{\partial} X^j. \tag{1.6}
\]
These currents satisfy the Bianchi identities \( \partial \tilde{I}^i_j - \tilde{\partial} I^i_j + f^i_{jk} I^j_k = 0 \). A convenient way of interpreting the Bianchi identities stems from the expressions
\[
\partial X^i = (e^{-1})^j_i I^j_i, \quad \tilde{\partial} X^i = (e^{-1})^j_i \tilde{I}^j_i. \tag{1.7}
\]
The Bianchi identities are nothing else than the integrability conditions of these last relations. Namely,
\[
\partial \tilde{\partial} X^i - \tilde{\partial} \partial X^i = 0. \tag{1.8}
\]
This remark will be of use in what follows. Finally, a curvature which can be written as a linear combination of the equations of motion and the Bianchi identities is found.

The problem of solving the equations of motion is therefore brought to the consideration of the first order linear differential system
\[
\left[ \partial + \left( \frac{1}{2} + \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right) \right] \left( e^k_i \partial X^i \right) T_k \right]\Psi = 0, \quad \left[ \tilde{\partial} + \left( \frac{1}{2} - \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right) \right] \left( e^k_i \tilde{\partial} X^i \right) T_k \right]\Psi = 0. \tag{1.9}
\]
The function \( \Psi \) obviously depends on the spectral parameter \( \lambda \). By choosing the plus sign of the term \( \sqrt{\frac{1}{4} + \lambda^2} \), we notice that for \( \lambda = 0 \) a particular solution to this system is given by \( \Psi(\lambda = 0) = g^{-1} \). Here \( g(X) \) is a Lie group element corresponding to the Lie algebra \( \mathcal{G} \) and we have \( (e^k_i \partial X^i) T_k = g^{-1} \partial g \) and \( (e^k_i \tilde{\partial} X^i) T_k = g^{-1} \partial g \). Hence, the information regarding the field \( g \) of the non-linear sigma model is extracted from the function \( \Psi \) for a given value of the spectral parameter.

The existence of the Lax pair (and the spectral parameter) can be interpreted as a sign for hidden symmetries in the classical theory [3]. One can show that the transformation \( \delta g = g \Psi e^{\epsilon} T_i \Psi^{-1} \), where \( e^{\epsilon} \) are infinitesimally
constants, preserves the equation of motion \( \partial (g^{-1} \mathcal{A} g) + \bar{\partial} (g^{-1} \partial g) = 0 \). Indeed, the Lax pair implies that

\[
\delta (g^{-1} \mathcal{A} g) = \frac{-1 + 2 \lambda \pm 2 \sqrt{1 + \lambda^2}}{1 + 2 \lambda \pm 2 \sqrt{1 + \lambda^2}} \partial (\Psi_{\epsilon i} T_i \Psi^{-1}) \quad \text{and} \quad \delta (g^{-1} \partial g) = \frac{-1 - 2 \lambda \pm 2 \sqrt{1 + \lambda^2}}{1 - 2 \lambda \pm 2 \sqrt{1 + \lambda^2}} \bar{\partial} (\Psi_{\epsilon i} T_i \Psi^{-1}).
\]

Consequently, one has \( \partial [\delta (g^{-1} \mathcal{A} g)] + \bar{\partial} [\delta (g^{-1} \partial g)] = 0 \) and the equation of motion remains invariant.

### 2. Generalisation

We would like now to generalise the previous discussion to any two-dimensional non-linear sigma model. Our aim is to determine the conditions under which a theory is integrable. We start from the action

\[
S = \int dz \bar{z} Q_{ij} (X) \partial X^i \bar{\partial} X^j. \tag{2.1}
\]

The tensor \( Q_{ij} \) has symmetric and anti-symmetric parts. The equations of motion resulting from the variation of this action are given by

\[
\mathcal{E}_l \equiv \partial_l Q_{ij} \partial X^i \bar{\partial} X^j - \partial (Q_{lj} \bar{\partial} X^j) - \bar{\partial} (Q_{il} \partial X^i) = 0. \tag{2.2}
\]

By contracting the equations of motion with the matrix \( K_{n}^{ml} (X) \), one gets

\[
K_{n}^{ml} \mathcal{E}_l = (K_{n}^{ml} \partial_l Q_{ij} + Q_{lj} \bar{\partial} K_{n}^{ml} + Q_{il} \partial_j K_{n}^{ml}) \partial X^i \bar{\partial} X^j - \bar{\partial} (K_{n}^{ml} Q_{lj} \bar{\partial} X^j) - \partial (K_{n}^{ml} Q_{il} \partial X^i) = 0. \tag{2.3}
\]

In general, \( K_{n}^{ml} \) could also depend on some parameters which will be interpreted as the spectral parameters. Our next step is the introduction of the two currents defined by

\[
\partial X^i = \alpha^i_j (X) A^j, \quad \bar{\partial} X^i = \beta^i_j (X) \bar{A}^j, \tag{2.4}
\]

where \( \alpha^i_j \) and \( \beta^i_j \) are two invertible matrices. The introduction of these currents is purely for a book-keeping purpose. It also allows for a rapid comparison with the coordinate-free formulation of some known integrable non-linear sigma models (such as the principal chiral model and the \( G/H \) coset sigma models). In terms of these currents, the equations of motion take the form

\[
K_{n}^{ml} \mathcal{E}_l = (K_{n}^{ml} \partial_l Q_{ij} + Q_{lj} \bar{\partial} K_{n}^{ml} + Q_{il} \partial_j K_{n}^{ml}) \alpha^i_p \beta^j_q A^p \bar{A}^q - \bar{\partial} (K_{n}^{ml} Q_{lj} \beta^j_q \bar{A}^p) - \partial (K_{n}^{ml} Q_{il} \alpha^i_p A^q) = 0. \tag{2.5}
\]

There are various ways of expressing the equations of motion. However, this last form is the most convenient for our purpose.

The above definition of the two currents \( A^i \) and \( \bar{A}^i \) leads to some Bianchi identities. These are found from the integrability condition

\[
B^i \equiv \partial \bar{\partial} X^i - \bar{\partial} \partial X^i = \partial (\beta^i_j \bar{A}^j) - \bar{\partial} (\alpha^i_p A^p) = 0. \tag{2.6}
\]

By contracting this last equation with the matrix \( L_{mi}^{n} (X) \), one gets

\[
L_{mi}^{n} B^i = \partial (L_{mi}^{n} \beta^i_{pq} \bar{A}^q) - \bar{\partial} (L_{mi}^{n} \alpha^i_{pq} A^p) - (\partial_j L_{mi}^{n} - \partial_i L_{mj}^{n}) \alpha^j_{pq} A^p \bar{A}^q = 0. \tag{2.7}
\]
Like $K_{n}^{m}$, the tensor $L_{n}^{m}$ could in principle depend on some parameters too. We have therefore expressed the
equations of motion in terms of two currents subject to some Bianchi identities. We now turn our attention to the
construction of the Lax pair.

Let us consider the two differential operator as defined by

$$D_{j} = \delta_{j}^{i} \partial + V_{j}^{i}, \quad \overline{D}_{j} = \delta_{j}^{i} \overline{\partial} + \overline{V}_{j}^{i},$$

(2.8)

where $V_{j}^{i}$ and $\overline{V}_{j}^{i}$ are two gauge connections. The curvature of these two operators is given by

$$F_{k}^{i} = D_{j}^{i} \overline{D}_{k}^{j} - \overline{D}_{j}^{i} D_{k}^{j} = \delta \overline{V}_{k}^{i} - \overline{\partial} V_{k}^{i} + V_{j}^{i} \overline{V}_{k}^{j} - \overline{V}_{j}^{i} V_{k}^{j}.$$  

(2.9)

Following the case of the principal chiral sigma model, we require that this curvature satisfies

$$F_{k}^{i} = K_{k}^{ij} \mathcal{E}_{i} + L_{k}^{i} \mathcal{B}^{i},$$

(2.10)

By matching the terms involving the two derivatives $\partial$ and $\overline{\partial}$ on both sides of Eq. (2.10), one is forced to choose
the two gauge fields as

$$V_{k}^{i} = (K_{k}^{ij} Q_{j} + L_{k}^{i}) \alpha_{m}^{2} X^{m} = (K_{k}^{ij} Q_{j} + L_{k}^{i}) \partial X^{i},$$

$$\overline{V}_{k}^{i} = (-K_{k}^{ij} Q_{j} + L_{k}^{i}) \beta_{i}^{2} X^{m} = (-K_{k}^{ij} Q_{j} + L_{k}^{i}) \overline{\partial} X^{i}.$$  

(2.11)

Substituting back $V_{k}^{i}$ and $\overline{V}_{k}^{i}$ in (2.10), leads to the following differential equation

$$\begin{align*}
(K_{k}^{ij} \delta_{i} Q_{mn} + Q_{in} \delta_{m} K_{k}^{ij} + Q_{ml} \delta_{n} K_{k}^{ij}) - (\delta_{m} L_{kn}^{i} - \delta_{n} L_{km}^{i} + L_{jm}^{i} L_{kn}^{j} - L_{jn}^{i} L_{km}^{j}) & \\
= (K_{k}^{ij} K_{k}^{ji} - K_{k}^{ij} K_{k}^{ji}) Q_{in} Q_{ml} + (K_{k}^{ij} L_{kn}^{j} - L_{jn}^{i} K_{k}^{ij}) Q_{ml} + (K_{k}^{ij} L_{kn}^{i} - L_{jn}^{i} K_{k}^{ji}) Q_{ln}.
\end{align*}$$

(2.12)

This last equation is, as expected, independent of $\alpha_{i}^{j}$ and $\beta_{i}^{j}$. Therefore the choice of $A_{i}^{j}$ and $\overline{A}_{i}^{j}$ has no influence on
the construction. We conclude that the curvature $F_{k}^{i}$, corresponding to the gauge fields $V_{k}^{i}$ and $\overline{V}_{k}^{i}$ in (2.11), can be
written as a linear combination of the equations of motion and the Bianchi identities if Eq. (2.12) is fulfilled. This
relation depends on the ansatz one takes for the differential operators $D_{j}$ and $\overline{D}_{j}$ and is, therefore, only a sufficient
condition for the integrability of non-linear sigma models. Furthermore, the quantities $K_{k}^{ij}$ and $L_{jk}^{i}$ must depend
on at least one spectral parameter $\lambda$. If the equation $F_{k}^{i} = 0$ holds for all values of $\lambda$ then one obtains $\mathcal{E}_{i} = 0$ and
$\mathcal{B}^{i} = 0$. Notice that none of the tensors $Q_{ij}$, $K_{k}^{ij}$ and $L_{jk}^{i}$ is, a priori, known. We should mention that the range of
values where the two indices $i$ and $k$, in $K_{k}^{ij}$ and $L_{jk}^{i}$, take values is also not known. The only requirement imposed
on these tensors is a sort of non-degeneracy condition of the form $K_{k}^{ij} \mathcal{E}_{i} = L_{k}^{i} \mathcal{B}^{i} = 0 \Rightarrow \mathcal{E}_{i} = \mathcal{B}^{i} = 0$.

The linear system that replaces the equations of motion is therefore given by

$$\begin{align*}
[\partial + (K^{ij} Q_{ji} + L_{j}) \partial X^{i}] \Psi &= 0, & [\overline{\partial} + (-K^{ij} Q_{ji} + L_{j}) \overline{\partial} X^{i}] \Psi &= 0,
\end{align*}$$

(2.13)

where $K^{ij}$ and $L_{j}$ denote two matrices whose entries are the elements $K_{k}^{ij}$ and $L_{jk}^{i}$, respectively. Let us assume that
there exists a set $\{\lambda_{0}\}$ of the spectral parameters such that $K^{ij}(\lambda_{0}) = 0$ and $L_{j}(\lambda_{0}) = \overline{L}_{j}$. For this set of spectral
parameters, a particular solution to the linear system is given by

$$\Psi(\lambda_{0}) = \exp(M),$$

(2.14)

where the matrix $M(X)$ is related to $\overline{L}_{j}(X)$ through

$$\overline{L}_{j} = -\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \text{ad}(M)^{n} \partial_{j} M.$$  

(2.15)
Here $\text{ad}(A)B = [A, B]$ and $\text{ad}(A)^n B = [A, [A, [A, \ldots [A, B]] \ldots]$ with $A$ appearing $n$ times. Therefore, the information regarding the fields $X^i$ of the non-linear sigma model can be retrieved from $\Psi(\lambda_0)$ provided that the set $\{\lambda_0\}$ exists.

So far there is no Lie algebra structure in our construction. This is due to the fact that our gauge connections $V^i_j$ and $\tilde{V}^i_j$ are not necessarily evaluated in a Lie algebra. Let us now choose a gauge connection that takes value in the Lie algebra $\mathcal{H}$ generated by $[H_i, H_j] = h_{ij}^k H_k$. Namely, $W = W^i H_i$ and $\tilde{W} = \tilde{W}^i H_i$. In a similar manner, we require that

$$[\tilde{a} + W, \tilde{a} + \tilde{W}] = (\partial \tilde{W}^i - \tilde{a} W^i + h^i_{jk} W^j \tilde{W}^k) H_i = (\mu^{il} \mathcal{E}_l + \rho^l_\mathcal{B}^l) H_i,$$

where $\mu^{il}$ and $\rho_l^i$ are the equivalent of $K^{ij}$ and $L^{ij}_{kl}$, respectively. The contracted equations of motion, $\mu^{ml} \mathcal{E}_l$, and the contracted Bianchi identities, $\rho^m_l \mathcal{B}^l$, are given by

$$\mu^{ml} \mathcal{E}_l = (\mu^{ml} \partial_0 Q_{ij} + Q_{ij} \partial_0 \mu^{ml} + Q_{ij} \partial_0 \mu^{ml}) A^n \tilde{A}^q - \partial(\mu^{ml} Q_{ij} \beta^m_0 A^p) - \partial(\mu^{ml} Q_{ij} \alpha^m_0 A^p) = 0,$$

$$\rho^m_l \mathcal{B}^l = (\rho^m_l \tilde{B}^l \tilde{A}^q) - \partial(\rho^m_l \tilde{A}^q A^p) - (\partial \rho^m_l \tilde{B}^l A^p) = 0.$$

A direct inspection of Eq. (2.16) leads to the gauge connections

$$W^i = (\mu^{il} Q_{ij} + \rho^l_0) A^n \partial X^j,$$

$$\tilde{W}^i = (-\mu^{il} Q_{ij} + \rho^l_0) \tilde{A}^q \partial X^j.$$

The differential equation we obtain in this case is given by

$$(\mu^{il} \partial_0 Q_{mn} + Q_{mn} \partial_0 \mu^{il} + Q_{ml} \partial_0 \mu^{il}) - (\partial_m \mu^i_l - \partial_n \mu^i_l + h^i_{jk} \rho^j_0 \rho^k_0)$$

$$= h^i_{jk} \mu^{kl} Q_{lm} + h^i_{jk} \mu^{kl} (\rho^j_0 Q_{lm} + \rho^k_0 Q_{ml}).$$

Here also, the dependence on the spectral parameter $\lambda$ could only be in the tensors $\mu^{ij}$ and $\rho^l_0$. Moreover, for the non-degeneracy condition $\mu^{il} \mathcal{E}_l = \rho^l_0 \mathcal{B}^l = 0 \Rightarrow \mathcal{E}_l = \mathcal{B}^l = 0$ to hold, it is reasonable to require the tensors $\mu^{il}$ and $\rho^l_0$ to be invertible. In this case, the dimension of the Lie algebra $\mathcal{H}$ is the same as the dimension of the target space manifold. It is also worth mentioning that a comparison between the Lie algebra construction (2.16) and the general setting of Eq. (2.10) reveals that $K^{ij}_{kl} = (H_n)^i_j \mu^{ml}$ and $L^{ij}_{kl} = (H_n)^i_j \rho^l_0$, where $(H_n)^i_j$ are the elements of the matrix $H_n$. This shows that one can, a priori, take any matrix representation for the Lie algebra $\mathcal{H}$. One provides, in this case, an interpretation for the range of values of the indices $i$ and $k$ in $K^{ij}_{kl}$ and $L^{ij}_{kl}$.

There are some interesting geometrical structures appearing in Eq. (2.12) (and similarly in Eq. (2.19)). These structures are the Lie derivative of $Q_{ij}$ with respect to $K^{ij}_{kl}$ and the curvature of the differential operator $(\nabla_i)^l_0 = \delta^l_0 \partial_i + L^{ij}_{kl}$. Using the properties of these two geometrical objects, one can derive some consistency conditions for the validity of our differential equations. According to our investigation, the form of these conditions is too involved and does not lead to any new insight. Finally, it is worth mentioning that Eq. (2.19) is a generalisation of an equation that appears in the context of Poisson–Lie duality in two-dimensional sigma models [4,5]. This equation is given by $\mu^{il} \partial_0 Q_{mn} + Q_{mn} \partial_0 \mu^{il} + Q_{ml} \partial_0 \mu^{il} = h^i_{jk} \mu^{kl} Q_{lm} Q_{mt}$ and has been solved by Klimčík and Ševera. This suggests a possible link between classical integrability an T-duality in sigma models. This relationship has been noticed and exploited in [6–8] for a very particular class of classically integrable sigma models. The precise understanding of the full connection between integrability and Poisson–Lie duality is certainly worth exploring. The first step towards this goal would be to investigate whether a Poisson–Lie dualisable sigma model is integrable. We hope to return to this issue in a future work.
3. Examples

It is easier to consider the case described by the gauge connections $W^i$ and $\bar{W}^i$ in (2.18) and the differential Eq. (2.19). Let us first check that one obtains some known integrable two-dimensional non-linear sigma models. Indeed, the Lax pair construction (1.5) for the principal chiral sigma model is found by taking

$$\mu^{ij} = \lambda \eta^{ik}(e^{-1})_{kl}^j, \quad \rho^i = \left(\frac{1}{2} + \frac{1}{4} + \lambda^2\right) e^i_n, \quad h^i_{jk} = f^i_{jk},$$

(3.1)

where $\eta^{ij}$ is the inverse of the bilinear form $\eta_{ij}$. Similarly, all the Lax pairs corresponding to the other known integrable sigma models can be constructed in a similar manner. In particular those defined on a symmetric space (coset models). These are known to be integrable and their equations of motion correspond to those of gauged principal chiral sigma models. Their Lax pair formulation is carried out in a coordinate-free fashion (as is the case for the principal chiral sigma model). This formulation still holds if one chooses a system of coordinates (which is the case in our construction). A gauge fixing procedure is then needed in order to have the right number of fields (coordinates) in the sigma model as well as an invertible target space metric.

By examining Eq. (2.19) one can realise that there are new integrable sigma models. A straightforward solution to this equation is found by taking

$$\mu^{ij} = \alpha^{ik}v^j_k, \quad v^i_k = Q_{mn} + Q_{nk}g^{nm} + Q_{mk}g^{nm} = 0, \quad h^i_{jk} = 0, \quad \rho^i_j = \partial_j A^i,$$

(3.2)

where $A^i(X)$ is an arbitrary function and $\alpha^{ik}$ is a constant tensor. This means that the matrix $v^i_k$ is a Killing vector of the tensor $Q_{ij}$ and the Lie algebra $\mathcal{H}$ is Abelian. In this case Eq. (2.19) is satisfied for any arbitrary tensors $\alpha^{ij}$ and $A^i$. Notice that neither the equations of motion nor the Bianchi identities have, for this class of sigma models, terms quadratic in the currents $A^i$ and $\bar{A}^i$. The Lax pair construction for these types of sigma models is given by

$$\left[\partial + \left(\omega^{jk}v^i_k Q_{ij} \partial X^j + \partial A^i\right) H_i, \partial + \left(-\omega^{mn}v^i_n Q_{ij} \partial X^j + \partial A^m\right) H_m\right]$$

$$= \left\{\omega^{mn}\left[-\partial (v^i_n Q_{ij} \partial X^j) - \partial (v^j_q Q_{ij} \partial X^j)\right] - \partial (\partial \mathcal{H} - \partial A^m)\right\} H_m.$$

(3.3)

When this curvature vanishes, the term proportional to the arbitrary tensor $\omega^{mn}$ yields the equations of motion while the second term is identically zero. We have chosen here to use directly $\partial X^i$ and $\partial X^i$ instead of the currents $A^i$ and $\bar{A}^i$.

We conclude that any two-dimensional non-linear sigma model possessing as many isometries as the dimension of the manifold is classically integrable. This class of non-linear sigma models include all those corresponding to group manifolds. We notice also that, for a given sigma model, one can find an infinite number of Lax pairs: one for each choice of $(\mathcal{A}', \alpha^{ij})$. Moreover, one has a Lax pair construction involving an arbitrary number of spectral parameters.

An example of this class is given by a generalisation of the principal chiral sigma model. We consider the action

$$S = \int d\sigma d\bar{\sigma} \Omega_k e^i_k(X) e_j^j(X) \partial X^i \partial \bar{X}^j,$$

(3.4)

where $\Omega_k$ can be any constant matrix (not necessarily the bilinear form $\eta_{kl}$). The underlying Lie algebra is still $\mathcal{G}$, defined in the introductory section. The Killing vector $v^i_j$ corresponding to the tensor $Q_{ij} = \Omega_i e^i_j e^j_k$ is given by $v^i_j = R^i_j (e^{-1})^j_k$, where $R^i_j$ is defined by $g^{-1} T_i g = R^i_j T_j$ and verifies the relations $\partial_k R^i_j = f^i_{mn} R^m_n e^j_k$ and $f^i_{jk} R^j_k R^k_n = f^i_{mn} R^m_n$. Here $g(X)$ is an element in the Lie group corresponding to the Lie algebra $\mathcal{G}$. The vielbeins are given by the usual expression $e^i_j T_i = (g^{-1} \partial_j g) e^i_j T_i$.

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2 Since the zero curvature condition leads to $v^i_j \xi = 0$ (and not $\xi = 0$), we require the Killing vectors $v^i_j$ to be invertible. This in turn implies that the range of values of the indices $i$ and $j$ in $v^i_j$ is the same.
At this point some remarks are due. Firstly, since the principal chiral sigma model is a special case of the action (3.4), we have therefore found a new Lax pair formulation for this model. It is surprising that this formulation relies on an Abelian Lie algebra instead of the usual non-Abelian one. Secondly, there is another model based on the Lie algebra $G = SU(2)$ and which is known to be classically integrable [9]. This model corresponds to taking a diagonal $\Omega_{ij}$ with $\Omega_{11} = \Omega_{22} \neq \Omega_{33}$. In this case also our Lax pair construction is different from that of [9]. Finally, we should mention that an earlier attempt to study the classical integrability of the model (3.4) did not produce any new integrable systems [10].

The second example we consider here is described by the action

$$S = \int dz \, d\bar{z} \, Q^{ij}(\chi) \partial \chi_i \bar{\partial} \chi_j,$$

(3.5)

where $Q^{ij}$ is given through its inverse by

$$Q_{ij} = \eta_{ij} + f_{ik}^j \chi_k.$$

(3.6)

This model is the non-Abelian T-dual of the chiral principal sigma model [11–14]. It is not known whether this model is classically integrable. The study of this model sheds, therefore, some light on whether duality preserves integrability.

The equations of motion stemming from our action can be cast in the form

$$\partial \bar{J}^j - \bar{\partial} J^i + f_{ij}^k J^i \bar{J}^j = 0,$$

(3.7)

where the two currents are given by [15]

$$J^i = Q^{ij} \partial \chi_i, \quad \bar{J}^j = -Q_{ij} \bar{\partial} \chi_j.$$

(3.8)

This definition leads to the Bianchi identities ($\partial \bar{\partial} \chi_i - \bar{\partial} \partial \chi_i = 0$)

$$\partial \bar{J}^j + \bar{\partial} J^i + \eta^{ik} f_{ij}^k \chi_i \left( \partial \bar{J}^j - \bar{\partial} J^i + f_{mn}^j J^m J^n \right) = 0.$$

(3.9)

The Lax pair construction is found through the commutator

\[
\left[ \partial + \left( \frac{1}{2} + \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right) J^k T_k, \bar{\partial} + \left( \frac{1}{2} - \lambda \pm \sqrt{\frac{1}{4} + \lambda^2} \right) \bar{J}^j T_j \right]
\]

\[
= -\lambda \left[ \partial \bar{J}^j + \bar{\partial} J^i + \eta^{ik} f_{ki}^j \chi_i \left( \partial \bar{J}^j - \bar{\partial} J^i + f_{mn}^j J^m J^n \right) \right]
\]

\[
+ \left( \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda^2} \right) \delta^j_i + \lambda \eta^{jk} f_{ik}^j \chi_l \right] \left( \partial \bar{J}^l - \bar{\partial} J^i + f_{mn}^j J^m J^n \right) T_j.
\]

(3.10)

If this commutator is to vanish then the term proportional to $-\lambda$ yields the Bianchi identities while the term proportional to $\left( \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda^2} \right)$ leads to the equations of motion. Therefore, the non-Abelian dual of the principal chiral sigma model is also classically integrable.

In summary, we have given in this Letter the necessary constraints for a non-linear sigma model to be classically integrable. The equations found are not empty and lead to some unexpected integrable theories. The geometrical nature of these equations might be employed to find more solutions.

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