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# New exact traveling wave solutions for the Klein–Gordon–Zakharov equations

Yadong Shang [a,*], Yong Huang [b], Wenjun Yuan [a]

[a] School of Mathematics and Information Science, Guangzhou University, Guangzhou, Guangdong 510006, People's Republic of China
[b] Institute of Computer Educational Softwares, Guangzhou University, Guangzhou, Guangdong 510006, People's Republic of China

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ABSTRACT

Based on the extended hyperbolic functions method, we obtain the multiple exact explicit solutions of the Klein–Gordon–Zakharov equations. The solutions obtained in this paper include (a) the solitary wave solutions of bell-type for $u$ and $n$, (b) the solitary wave solutions of kink-type for $u$ and bell-type for $n$, (c) the solitary wave solutions of a compound of the bell-type and the kink-type for $u$ and $n$, (d) the singular traveling wave solutions, (e) periodic traveling wave solutions of triangle function types, and solitary wave solutions of rational function types. We not only rederive all known solutions of the Klein–Gordon–Zakharov equations in a systematic way but also obtain several entirely new and more general solutions. The variety of structures of the exact solutions of the Klein–Gordon–Zakharov equations is illustrated.



## 1. Introduction

In the theoretical investigation of the dynamics of strong Langmuir turbulence in plasma physics, various Zakharov equations take an important role [1,2]. In this paper, we consider the following Klein–Gordon–Zakharov equations:

$$u_{tt} - u_{xx} + u + \alpha n u = 0, \tag{1}$$

$$n_{tt} - n_{xx} = \beta(|u|^2)_{xx}, \tag{2}$$

with $u$ a complex function and $n$ a real function, where $\alpha$, $\beta$ are two nonzero real parameters. This system describes the interaction of the Langmuir wave and the ion acoustic wave in a high frequency plasma [1,2].

In recent years, there have been many works on the qualitative research of the global solutions for the Klein–Gordon–Zakharov equations (1)–(2) [3–6]. Chen Lin considered orbital stability of solitary waves for the Klein–Gordon–Zakharov equations in [7]. More recently, some exact solutions for the Zakharov equations are obtained by using different methods [8–14]. These solutions are not general and by no means exhaust all possibilities. They are only some particular solutions within some specific parameters choices.

The aim of this paper is to find the new and more general explicit and exact special solutions of the Klein–Gordon–Zakharov equations (1)–(2). We obtain various of explicit and exact special solutions of the Klein–Gordon–Zakharov equations (1)–(2) by using the extended hyperbolic functions method presented in [15] by author. These solutions include that of the solitary wave solutions of bell-type for $u$ and $n$, the solitary wave solutions of kink-type for $u$ and bell-type for $n$, the solitary wave solutions of a compound of the bell-type and the kink-type for $u$ and $n$, the singular traveling wave solutions, the periodic traveling wave solutions of triangle functions type, and solitary wave solutions of rational function type.

* Corresponding author.
*E-mail addresses:* gzydshang@126.com, ydshang@263.net (Y. Shang).

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This paper is organized as follows: In Section 2, we present the concrete scheme of the extended hyperbolic functions method for nonlinear wave equations. In Section 3, the method proposed in Section 2 is used to find the explicit and exact special solutions of the Klein–Gordon–Zakharov equations (1)–(2).

## 2. The extended hyperbolic function method

In this section, we briefly review the extended hyperbolic function method, a general method presented in [15] by the first author of this paper based on the methods given by the first author in [16], Conte et al. in [17], and Zhang et al. in [18]. The solitary wave solutions, the singular traveling wave solutions, periodic wave solutions of triangle function types, and traveling wave solutions of rational function type are constructed uniformly by this method.

Given nonlinear partial differential equation, for instance, in two variables, as follows:

$$P(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0, \tag{3}$$

where $P$ is in general a nonlinear function of its arguments, the subscripts denote the partial derivatives. Let $u(x, t) = u(\xi)$, $\xi = kx + \omega t$, then Eq. (3) reduces to a nonlinear ordinary differential equation(ODE)

$$Q(u, u', u'', \ldots) = 0. \tag{4}$$

First of all, based on the fact that solitary wave solutions of nonlinear wave equations are generally polynomials of the $\operatorname{sech}\xi$ function, we suppose that the solution of the ODE (4) is of the form

$$u(x, t) = u(\xi) = \sum_{i=0}^{n} a_i (v(\xi))^i, \tag{5}$$

where the coefficients $a_i$ $(i = 1, 2, \ldots, n)$ are constants to be determined and $v = v(\xi)$ satisfies a nonlinear ordinary differential equation of first order

$$v' = \frac{dv}{d\xi} = v\sqrt{a + bv^2}, \quad a, b \in R. \tag{6}$$

The polynomial degree $n$ can be determined via balancing the highest order derivative terms and the nonlinear terms in ODE. Substituting (5) into (4) and using (6) repeatedly, we obtain a set of nonlinear algebraic equations for $a_i$ $(i = 0, 1, 2 \ldots, n)$, $a, b, k, \omega$. With the aid of the computer program Mathematica or Maple 4 [19], we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i$ $(i = 0, 1, 2 \ldots, n)$, $a, b, k, \omega$. Note that the ODE (6) has the following eight kinds of general solutions

$$v(\xi) = -\sqrt{\frac{a}{b}}\operatorname{csch}\sqrt{a}(\xi + \xi_0), \quad a > 0, b > 0, \tag{7a}$$

$$v(\xi) = \sqrt{\frac{-a}{b}}\sec\sqrt{-a}(\xi + \xi_0), \quad a < 0, b > 0, \tag{7b}$$

$$v(\xi) = \sqrt{\frac{a}{-b}}\operatorname{sech}\sqrt{a}(\xi + \xi_0), \quad a > 0, b < 0, \tag{7c}$$

$$v(\xi) = \sqrt{\frac{-a}{b}}\csc\sqrt{-a}(\xi + \xi_0), \quad a < 0, b > 0, \tag{7d}$$

$$v(\xi) = \exp[\sqrt{a}(\xi + \xi_0)], \quad a > 0, b = 0, \tag{7e}$$

$$v(\xi) = \cos\sqrt{-a}(\xi + \xi_0) + \mathrm{i}\sin\sqrt{-a}(\xi + \xi_0), \quad a < 0, b = 0, \tag{7f}$$

$$v(\xi) = \pm\frac{1}{\sqrt{b}(\xi + \xi_0)}, \quad a = 0, b > 0, \tag{7g}$$

$$v(\xi) = \pm\frac{i}{\sqrt{-b}(\xi + \xi_0)}, \quad a = 0, b < 0. \tag{7h}$$

So the multiple exact special solutions of nonlinear partial differential equation (3) are obtained by making use of (5) and (7).

Secondly, note that many solitary wave solutions of nonlinear wave equations are polynomials of the $\tanh\xi$ function, we suppose that the ODE (4) has solution of the form (5) and $v = v(\xi)$ satisfies a nonlinear ordinary differential equation of first order

$$v' = \frac{dv}{d\xi} = a + bv^2, \quad a, b \in R. \tag{8}$$

The balance constant $n$ can be obtained by means of the leading order term analysis. Substituting (5) into (4) and utilizing (8) repeatedly, we obtain a set of nonlinear algebraic equations with respect to $a_i$ $(i = 0, 1, 2 \ldots, n)$, $a, b, k, \omega$. With the aid of the computer program Mathematica or Maple 4 [19], we solve the set of nonlinear algebraic equations and obtain all the constants $a_i$ $(i = 0, 1, 2 \dots, n)$, $a$, $b$, $k$, $\omega$. Note that the ODE (8) has the following six kinds of general solutions

$$v(\xi) = \operatorname{sgn}(a)\sqrt{\frac{a}{b}}\tan[\sqrt{ab}(\xi + \xi_0)], \quad ab > 0, \tag{9a}$$

$$v(\xi) = -\operatorname{sgn}(a)\sqrt{\frac{a}{b}}\cot[\sqrt{ab}(\xi + \xi_0)], \quad ab > 0, \tag{9b}$$

$$v(\xi) = \operatorname{sgn}(a)\sqrt{\frac{a}{-b}}\tanh[\sqrt{-ab}(\xi + \xi_0)], \quad ab < 0, \tag{9c}$$

$$v(\xi) = \operatorname{sgn}(a)\sqrt{\frac{a}{-b}}\coth[\sqrt{-ab}(\xi + \xi_0)], \quad ab < 0, \tag{9d}$$

$$v(\xi) = -\frac{1}{b(\xi + \xi_0)}, \quad a = 0, b > 0, \tag{9e}$$

$$v(\xi) = a(\xi + \xi_0), \quad a \in R, b = 0, \tag{9f}$$

The multiple exact special solutions of nonlinear partial differential equation (3) are obtained by making use of (5) and (9).

In 1992, Conte et al. presented an indirect method to find more new solitary wave solutions of nonlinear partial differential equations that can be expressed as a polynomial in two elementary functions which satisfy a projective Riccati equation [17]. In [18], Zhang *et al.* proposed the hyperbolic function method based upon the fact that many solitary wave solutions have the format of hyperbolic functions. They expressed the solitary wave solutions of the nonlinear wave equations as the combination of hyperbolic functions and obtained many new exact solitary wave solutions. In [20], Yan presented the generally projective Riccati equation method. More recently, Chen and Ding improved the projective Riccati equation method in [21] and obtained some new solitary wave solutions to the nonlinear evolution equation. In order to obtain some more general exact solutions, we assume that the solutions of the ODE (4) is of the form

$$u(x, t) = u(\xi) = \sum_{i=0}^{n} a_i (f(\xi))^i + \sum_{j=1}^{n} b_j (f(\xi))^{j-1} g, \tag{10}$$

where the coefficients $a_i$ $(i = 0, 1, 2 \dots, n)$ and $b_j$ $(j = 1, 2, \dots, n)$ are constants to be determined. The functions $f$ and $g$ satisfy the coupled Riccati equations

$$f'(\xi) = -f(\xi)g(\xi), \qquad g'(\xi) = 1 - rf(\xi) - g^2(\xi), \tag{11a}$$

$$f'(\xi) = -f(\xi)g(\xi), \qquad g'(\xi) = -1 + rf(\xi) - g^2(\xi), \tag{11b}$$

respectively. Furthermore, we can obtain their first integrals as given

$$g^2(\xi) = 1 - 2rf(\xi) + (b^2 - a^2 + r^2)f^2(\xi), \tag{12a}$$

$$g^2(\xi) = -1 + 2rf(\xi) + (b^2 + a^2 - r^2)f^2(\xi), \tag{12b}$$

respectively. The balance constant $n$ can be determined by the analysis of the leading order term. Substituting (10) into (4) and making use of (11)–(12), eliminating any derivative of $(f, g)$ and any power of $g$ higher than one and setting the coefficients of the different powers of $f$ and $g$ to zero, we obtain a set of nonlinear algebraic equations with all parameters which are to be determined. With the aid of the computer program Mathematica or Maple 4 [19], we can solve the set of nonlinear algebraic equations and obtain all the constants $a_i$ $(i = 0, 1, 2, \dots, n)$, $b_j$ $(j = 1, 2, \dots, n)$, $k$, $\omega$. Note that the ODEs (11) have the following special solutions

$$f(\xi) = \frac{1}{a\cosh\xi + b\sinh\xi + r}, \qquad g(\xi) = \frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}, \tag{13a}$$

$$f(\xi) = \frac{1}{a\cos\xi + b\sin\xi + r}, \qquad g(\xi) = \frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r}, \tag{13b}$$

respectively. So we also obtain the multiple exact special solutions of nonlinear partial differential equation (3) by combining (10) with (13).

## 3. New explicit and exact solutions of the Klein–Gordon–Zakharov equations

In this section, we shall apply the method developed in Section 2 to the Klein–Gordon–Zakharov equations (1)–(2) and give it a series of explicit exact solutions.

To begin with, let us consider the following gauge transformation

$$u(x, t) = e^{i(kx + \omega t + \xi_0)}\varphi(x, t), \tag{14}$$

where $\varphi(x,t)$ is a real–valued function, $k$, $\omega$ are two real constants to be determined, $\xi_0$ is an arbitrary constant. Substituting (14) into (1)–(2), we get

$$\varphi_{tt} - \varphi_{xx} + (k^2 - \omega^2 + 1)\varphi + \alpha n\varphi = 0, \tag{15}$$

$$\omega\varphi_t - k\varphi_x = 0, \tag{16}$$

$$n_{tt} - n_{xx} = \beta(\varphi^2)_{xx}. \tag{17}$$

In view of (16) we suppose

$$\varphi(x,t) = \varphi(\xi) = \varphi(\omega x + kt + \xi_1), \tag{18}$$

where $\xi_1$ is an arbitrary constant. Substituting (18) into (15), we infer that

$$n(x,t) = \frac{(\omega^2 - k^2)\varphi''(\xi)}{\alpha\varphi(\xi)} + \frac{(\omega^2 - k^2 - 1)}{\alpha}. \tag{19}$$

Therefore, we can also assume

$$n(x,t) = \psi(\xi) = \psi(\omega x + kt + \xi_1). \tag{20}$$

Substituting (20) into (17) and integrating the resultant equation twice with respect to $\xi$, one obtains

$$\psi(\xi) = \frac{\beta\omega^2\varphi^2(\xi)}{k^2 - \omega^2} + C, \tag{21}$$

where $C$ is an integration constant. Substituting (21) into (15), we get

$$\varphi''(\xi) + \frac{(k^2 - \omega^2 + 1 + \alpha C)}{(k^2 - \omega^2)}\varphi(\xi) + \frac{\alpha\beta\omega^2}{(k^2 - \omega^2)^2}\varphi^3(\xi) = 0 \tag{22}$$

Let $l = \frac{(k^2-\omega^2+1+\alpha C)}{(k^2-\omega^2)}$, $m = \frac{\alpha\beta\omega^2}{(k^2-\omega^2)^2}$, thus (22) becomes the Liénard equation

$$\varphi''(\xi) + l\varphi(\xi) + m\varphi^3(\xi) = 0. \tag{23}$$

In the following we will discuss how to solve exactly the Liénard equation (23).

Firstly, we suppose that the Liénard equation (23) has a solution of the form (5) with $v$ satisfying Eq. (6). We require $b$ is different from zero in order to obtain nontrivial real solutions. It is easy to know that $n = 1$ by the analysis of the leading order term. So we assume that

$$\varphi(\xi) = a_0 + a_1 v(\xi) \tag{24}$$

substituting (24) into (23) and making use of (6), we obtain a set of nonlinear algebraic equations in $a_0, a_1, a, b, m, l$

$$\begin{aligned} &a_1^3 m + 2a_1 b = 0,\\ &3a_0 a_1^2 m = 0,\\ &a_1 a + l a_1 + 3a_1 a_0^2 m = 0,\\ &a_0^3 m + l a_0 = 0. \end{aligned} \tag{25}$$

In order to obtain an nontrivial solution, we assume that $a_1 \neq 0$. Solving Eq. (25), we have

$$a_0 = 0, \qquad a_1 = \pm\sqrt{-\frac{2b}{m}}, \qquad a = -l, \tag{26}$$

By combining (7), (14), (18), (20), (21), (24) and (26), we have

**Theorem 1.** (1) *Suppose that $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1) < 0$, $\alpha\beta > 0$, then the Klein–Gordon–Zakharov equations (1)–(2) has explicit and exact solitary wave solutions of bell-type for $u(x,t)$ and bell-type for $n(x,t)$*

$$u_1(x,t) = \pm\sqrt{\frac{2(k^2-\omega^2+1+\alpha C)(k^2-\omega^2)}{-\alpha\beta\omega^2}}\,\mathrm{sech}\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{\omega^2-k^2}}(\xi+\xi_1)\right]\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}, \tag{27}$$

$$n_1(x,t) = C - \frac{2(k^2-\omega^2+\alpha C+1)}{\alpha}\,\mathrm{sech}^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{\omega^2-k^2}}(\xi+\xi_1)\right], \tag{28}$$

*where $k, \omega, C$ are arbitrary constants and $\xi = \omega x + kt$, $\xi_0, \xi_1$ are two arbitrary constants. The solution (27) indicates that $u_1(x,t)$ is an envelope solitary wave solution of bell-shape, while (28) shows that $n_1(x,t)$ is a solitary wave solution of bell-shape. The solution $n_1$ is called the Langmuir whistler soliton or the Langmuir pit soliton according to whether $n_1$ is positive or negative.*

(2) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)<0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact singular traveling wave solutions for* $u(x,t)$ *and* $n(x,t)$

$$u_2(x,t)=\pm\sqrt{\frac{2(k^2-\omega^2+1+\alpha C)(k^2-\omega^2)}{\alpha\beta\omega^2}}\operatorname{csch}\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{\omega^2-k^2}}(\xi+\xi_1)\right]e^{i(kx+\omega t+\xi_0)}, \tag{29}$$

$$n_2(x,t)=C+\frac{2(k^2-\omega^2+\alpha C+1)}{\alpha}\operatorname{csch}^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{\omega^2-k^2}}(\xi+\xi_1)\right], \tag{30}$$

*where* $k,\omega,C$ *are arbitrary constants and* $\xi=\omega x+kt$, $\xi_0,\xi_1$ *are two arbitrary constants. The singularity will occur at* $\xi=-\xi_1$ *in solutions of* (29)–(30) *and it represents that the distortions arise from the Langmuir wave of the ion* $n$ *and the ion acoustic wave intensity* $u$ *due to instability.*

(3) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)>0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type given as*

$$u_3(x,t)=\pm\sqrt{\frac{2(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)}{-\alpha\beta\omega^2}}\sec\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{k^2-\omega^2}}(\xi+\xi_1)\right]e^{i(kx+\omega t+\xi_0)}, \tag{31}$$

$$n_3(x,t)=C-\frac{2(k^2-\omega^2+\alpha C+1)}{\alpha}\sec^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{k^2-\omega^2}}(\xi+\xi_1)\right], \tag{32}$$

*and*

$$u_4(x,t)=\pm\sqrt{\frac{2(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)}{-\alpha\beta\omega^2}}\csc\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{k^2-\omega^2}}(\xi+\xi_1)\right]e^{i(kx+\omega t+\xi_0)}, \tag{33}$$

$$n_4(x,t)=C-\frac{2(k^2-\omega^2+\alpha C+1)}{\alpha}\csc^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{k^2-\omega^2}}(\xi+\xi_1)\right], \tag{34}$$

*where* $k,\omega,C$ *are arbitrary parameters and* $\xi_0,\xi_1$ *are two arbitrary constants. These two sets of solutions suggest that the periodic unstable distortions arise from the Langmuir wave intensity* $n$ *and the ion acoustic wave intensity* $u$.

(4) *Suppose that* $k^2-\omega^2+\alpha C+1=0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions of bell-type for* $u(x,t)$ *and bell-type for* $n(x,t)$ *given as*

$$u_5(x,t)=\pm\sqrt{\frac{2(k^2-\omega^2)^2}{-\alpha\beta\omega^2}}\frac{1}{(\omega x+kt+\xi_1)}e^{i(kx+\omega t+\xi_0)}, \tag{35}$$

$$n_5(x,t)=C-\frac{2(k^2-\omega^2)}{\alpha}\frac{1}{(\omega x+kt+\xi_1)^2}, \tag{36}$$

*where* $k,\omega$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants.*

*Secondly, we suppose that the Liénard equation* (23) *has a solution of the form* (5) *with* $v$ *satisfying* (8). *To obtain nontrivial real solutions we also require* $b\neq 0$. *It is easy to know that* $n=1$ *by the analysis of the leading order term. We assume that*

$$\varphi(\xi)=b_0+b_1v(\xi). \tag{37}$$

*Substituting* (37) *into Eq.* (23) *and making use of* (8), *we obtain a set of nonlinear algebraic equations in* $b_0,b_1,a,b,m,l$

$$\begin{aligned}&2b_1b^2+mb_1^3=0,\\&3mb_0b_1^2=0,\\&3mb_0^2b_1+2ab_1b+lb_1=0,\\&mb_0^3+lb_0=0.\end{aligned} \tag{38}$$

*We assume that* $b_1\neq 0$ *to obtain nontrivial solution. Solving* (38), *we obtain*

$$b_0=0,\qquad b_1=\pm\sqrt{\frac{2b^2}{-m}},\qquad a=-\frac{l}{2b}, \tag{39}$$

*By combining* (9), (14), (18), (20), (21), (37) *and* (39), *we have*

**Theorem 2.** (1) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)<0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type as given*

$$u_6(x,t)=\pm\sqrt{\frac{(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)}{\alpha\beta\omega^2}}\tan\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{2(\omega^2-k^2)}}(\xi+\xi_1)\right]\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)} \tag{40}$$

$$n_6(x,t)=C+\frac{(k^2-\omega^2+\alpha C+1)}{\alpha}\tan^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{2(\omega^2-k^2)}}(\xi+\xi_1)\right], \tag{41}$$

*where* $k,\omega$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants.*

(2) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)<0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type as given*

$$u_7(x,t)=\pm\sqrt{\frac{(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)}{\alpha\beta\omega^2}}\cot\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{2(\omega^2-k^2)}}(\xi+\xi_1)\right]\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)} \tag{42}$$

$$n_7(x,t)=C+\frac{(k^2-\omega^2+\alpha C+1)}{\alpha}\cot^2\left[\sqrt{\frac{(k^2-\omega^2+\alpha C+1)}{2(\omega^2-k^2)}}(\xi+\xi_1)\right], \tag{43}$$

*where* $k,\omega$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants.*

(3) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)>0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions of kink-type for* $u(x,t)$ *and bell-type for* $n(x,t)$ *as given*

$$u_8(x,t)=\pm\sqrt{\frac{(k^2-\omega^2+\alpha C+1)(k^2-\omega^2)}{-\alpha\beta\omega^2}}\tanh[\sqrt{\frac{k^2-\omega^2+\alpha C+1}{2(k^2-\omega^2)}}(\xi+\xi_1)]\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)} \tag{44}$$

$$n_8(x,t)=C-\frac{k^2-\omega^2+\alpha C+1}{\alpha}\tanh^2\left[\sqrt{\frac{k^2-\omega^2+\alpha C+1}{2(k^2-\omega^2)}}(\xi+\xi_1)\right], \tag{45}$$

*where* $k,\omega,C$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants. Here the solitary wave solutions* $n_8(x,t)$ *are dark solitons that means the intensity of Langmuir wave is increased as a whole but is decreased in part.*

(4) *Suppose that* $(k^2-\omega^2)(k^2-\omega^2+\alpha C+1)>0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact singular traveling wave solutions for* $u(x,t)$ *and* $n(x,t)$ *as given*

$$u_9(x,t)=\pm\sqrt{\frac{(k^2-\omega^2+\alpha C+1)(k^2-\omega^2)}{-\alpha\beta\omega^2}}\coth\left[\sqrt{\frac{k^2-\omega^2+\alpha C+1}{2(k^2-\omega^2)}}(\xi+\xi_1)\right]\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)} \tag{46}$$

$$n_9(x,t)=C-\frac{k^2-\omega^2+\alpha C+1}{\alpha}\coth^2\left[\sqrt{\frac{k^2-\omega^2+\alpha C+1}{2(k^2-\omega^2)}}(\xi+\xi_1)\right], \tag{47}$$

*where* $k,\omega,C$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants.*

(5) *Suppose that* $k^2-\omega^2+\alpha C+1=0, \alpha\beta<0$, *then the Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions of bell-type for* $u(x,t)$ *and bell-type for* $n(x,t)$ *as given*

$$u_{10}(x,t)=\pm\sqrt{\frac{2(k^2-\omega^2)^2}{-\alpha\beta\omega^2}}\frac{1}{(\omega x+kt+\xi_1)}\mathrm{e}^{\mathrm{i}(kx+\omega t+\xi_0)}, \tag{48}$$

$$n_{10}(x,t)=C-\frac{2(k^2-\omega^2)}{\alpha}\frac{1}{(\omega x+kt+\xi_1)^2}, \tag{49}$$

*where* $k,\omega$ *are arbitrary constants and* $\xi_0,\xi_1$ *are two arbitrary constants.*

*Now we suppose that the Liénard equation* (23) *has a solution of the form* (10) *with* $f$ *and* $g$ *satisfying the coupled Riccati equation* (11a) *and the first integral* (12a). *It is easy to know* $n=1$. *So we assume that*

$$\varphi(\xi)=c_0+c_1f(\xi)+c_2g(\xi), \tag{50}$$

*where the coefficients $c_0, c_1, c_2$ are constants to be determined and satisfy $c_1^2 + c_2^2 \neq 0$. Substituting (50) into the Liénard equation (23) and making use of (11a) and (12a), we can obtain a system of nonlinear algebraic equations with $c_0, c_1, c_2, r, m, l$*

$$\begin{aligned}
&m(c_1^3 + 3c_1c_2^2(r^2 + b^2 - a^2)) + 2c_1(r^2 + b^2 - a^2) = 0,\\
&2c_2(r^2 + b^2 - a^2) + m(3c_1^2c_2 + c_2^3(r^2 + b^2 - a^2)) = 0,\\
&m(3c_0c_1^2 + 3c_0c_2^2(r^2 + b^2 - a^2) - 6c_1c_2^2r) - 3c_1r = 0\\
&-c_2r + 6mc_0c_1c_2 - 2mc_2^3r = 0,\\
&c_1 + lc_1 + m(3c_1c_2^2 + 3c_0^2c_1 - 6c_0c_2^2r) = 0\\
&lc_2 + m(3c_0^2c_2 + c_2^3) = 0,\\
&lc_0 + m(c_0^3 + 3c_0c_2^2) = 0.
\end{aligned} \tag{51}$$

*Solving Eq. (51), we get a set of solutions*

$$c_0 = 0, \qquad c_1 = \pm\sqrt{\frac{a^2 - b^2 - r^2}{2m}}, \qquad c_2 = \pm\sqrt{-\frac{1}{2m}} \tag{52}$$

*provided that $l = \frac{1}{2}$, while $a$, $b$, and $r$ are arbitrary constants satisfy $r^2 + b^2 - a^2 > 0$.*

*We can also obtain another two set of solutions*

$$c_0 = 0, \qquad c_1 = \pm\sqrt{\frac{2(a^2 - b^2)}{m}}, \qquad c_2 = 0, \qquad r = 0 \tag{53}$$

*when $l = -1$, and*

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = \pm\sqrt{-\frac{2}{m}}, \qquad r = 0 \tag{54}$$

*while $l = 2$. By combining (13a), (50) and (52)–(54) we find that, when $l = \frac{1}{2}$, $m < 0$, the Liénard (23) possesses solutions of the form*

$$\varphi(\xi) = \pm\sqrt{\frac{a^2 - b^2 - r^2}{2m}}\frac{1}{a\cosh\xi + b\sinh\xi + r} \pm \sqrt{-\frac{1}{2m}}\frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}, \tag{55}$$

*where $a$, $b$, and $r$ are arbitrary constants such that $r^2 + b^2 - a^2 > 0$,*

$$\varphi(\xi) = \pm\sqrt{\frac{2(a^2 - b^2)}{m}}\frac{1}{a\cosh\xi + b\sinh\xi} \tag{56}$$

*when $l = -1$, $a, b, m$ satisfies $(a^2 - b^2)m > 0$, and*

$$\varphi(\xi) = \pm\sqrt{-\frac{2}{m}}\frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi} \tag{57}$$

*when $l = 2$, $m < 0$.*

*Finally, We suppose that the Liénard equation (23) has a solution (50) with $f$ and $g$ satisfying the coupled Riccati equation (11b) and the first integral (12b). Substituting (50) into (23) and making use of (11b), (12b), one obtains a set of nonlinear algebraic equations with $c_0, c_1, c_2, r, m, l$*

$$\begin{aligned}
&m(c_1^3 + 3c_1c_2^2(b^2 + a^2 - r^2)) + 2c_1(b^2 + a^2 - r^2) = 0,\\
&2c_2(b^2 + a^2 - r^2) + m(3c_1^2c_2 + c_2^3(b^2 + a^2 - r^2)) = 0,\\
&m(3c_0c_1^2 + 3c_0c_2^2(b^2 + a^2 - r^2) + 6c_1c_2^2r) + 3c_1r = 0\\
&c_2r + m[6c_0c_1c_2 + 2c_2^3r] = 0,\\
&-c_1 + lc_1 + m(-3c_1c_2^2 + 3c_0^2c_1 - 6c_0c_2^2r) = 0\\
&lc_2 + m(3c_0^2c_2 - c_2^3) = 0,\\
&lc_0 + m(c_0^3 - 3c_0c_2^2) = 0.
\end{aligned} \tag{58}$$

*Solving Eq. (58), we obtain*

$$c_0 = 0, \qquad c_1 = \pm\sqrt{-\frac{b^2 + a^2 - r^2}{2m}}, \qquad c_2 = \pm\sqrt{\frac{-1}{2m}}, \tag{59}$$

*provided that* $l = -\frac{1}{2}$, $m < 0$. *We also obtain two solutions*

$$c_0 = 0, \qquad c_1 = \pm\sqrt{\frac{-2(b^2+a^2)}{m}}, \qquad c_2 = 0, \qquad r = 0, \tag{60}$$

*when* $l = 1$, $m < 0$, *and*

$$c_0 = 0, \qquad c_1 = 0, \qquad c_2 = \pm\sqrt{\frac{-2}{m}}, \qquad r = 0, \tag{61}$$

*when* $l = -2$, $m < 0$. *By combining* (13b), (50), (59) *and* (61), *we obtain the solutions of the Liénard equation* (23) *given as*

$$\varphi(\xi) = \pm\sqrt{-\frac{b^2+a^2-r^2}{2m}}\frac{1}{a\cos\xi + b\sin\xi + r} \pm \sqrt{\frac{-1}{2m}}\frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r} \tag{62}$$

*when* $l = -\frac{1}{2}$, $m < 0$, *where* $a, b, r$ *are arbitrary constants such that* $b^2 + a^2 - r^2 > 0$, *and*

$$\varphi(\xi) = \pm\sqrt{\frac{-2}{m}}\frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi} \tag{63}$$

*when* $l = -2$, $m < 0$, *and*

$$\varphi(\xi) = \pm\sqrt{\frac{-2(b^2+a^2)}{m}}\frac{1}{a\cos\xi + b\sin\xi} \tag{64}$$

*when* $l = 1$, $m < 0$, *where* $a, b$ *are arbitrary constants.*

*From* (55)–(57) and (62)–(64), we have

**Theorem 3.** (1) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions of a compound of kink-type and bell-type for all* $u(x, t)$, $n(x, t)$

$$u_{11}(x,t) = \pm\left[\sqrt{\frac{(a^2-b^2-r^2)(k^2-\omega^2)^2}{2\alpha\beta\omega^2}}\frac{1}{a\cosh\xi + b\sinh\xi + r} + \sqrt{\frac{(k^2-\omega^2)^2}{-2\alpha\beta\omega^2}}\frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}\right]e^{i(kx+\omega t+\xi_0)} \tag{65}$$

$$n_{11}(x,t) = C - \frac{(k^2-\omega^2)}{2\alpha}\left[\sqrt{r^2+b^2-a^2}\frac{1}{a\cosh\xi + b\sinh\xi + r} + \frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi + r}\right]^2, \tag{66}$$

*where* $k, \omega, r, a, b$ *are arbitrary constants such that* $\alpha\beta < 0$, $r^2 + b^2 - a^2 > 0$, $\frac{1}{2}(k^2 - \omega^2) + \alpha C + 1 = 0$, $\xi = \omega x + kt + \xi_1$.

(2) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions*

$$u_{12}(x,t) = \pm\left[\sqrt{\frac{2(a^2-b^2)(k^2-\omega^2)^2}{\alpha\beta\omega^2}}\frac{1}{a\cosh\xi + b\sinh\xi}e^{i(kx+\omega t+\xi_0)}\right] \tag{67}$$

$$n_{12}(x,t) = C + \frac{2(b^2-a^2)(k^2-\omega^2)}{\alpha}\frac{1}{(a\cosh\xi + b\sinh\xi)^2}, \tag{68}$$

*where* $k, \omega, a, b$ *are arbitrary constants such that* $\alpha\beta(b^2 - a^2) > 0$, $2(k^2 - \omega^2) + \alpha C + 1 = 0$, $\xi = \omega x + kt + \xi_1$.

(3) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact solitary wave solutions*

$$u_{13}(x,t) = \pm\sqrt{\frac{2(k^2-\omega^2)^2}{-\alpha\beta\omega^2}}\frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi}e^{i(kx+\omega t+\xi_0)}, \tag{69}$$

$$n_{13}(x,t) = C - \frac{2(k^2-\omega^2)}{\alpha}\left[\frac{a\sinh\xi + b\cosh\xi}{a\cosh\xi + b\sinh\xi}\right]^2, \tag{70}$$

*where* $k, \omega$ *are arbitrary constants such that* $\alpha\beta < 0$, $\omega^2 - k^2 + \alpha C + 1 = 0$, $\xi = \omega x + kt + \xi_1$.

(4) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type*

$$u_{14}(x,t) = \pm\left[\sqrt{\frac{(r^2-b^2-a^2)(k^2-\omega^2)^2}{2\alpha\beta\omega^2}}\frac{1}{a\cos\xi + b\sin\xi + r} + \sqrt{\frac{(k^2-\omega^2)^2}{-2\alpha\beta\omega^2}}\frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r}\right]e^{i(kx+\omega t+\xi_0)} \tag{71}$$

$$n_{14}(x,t) = C - \frac{(k^2-\omega^2)}{2\alpha}\left[\frac{\sqrt{b^2+a^2-r^2}}{a\cos\xi + b\sin\xi + r} + \frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi + r}\right]^2, \tag{72}$$

*where* $k, \omega, a, b, r$ *are arbitrary parameters such that* $\alpha\beta < 0$, $b^2+a^2-r^2 > 0$, $\frac{3}{2}(k^2-\omega^2)+\alpha C + 1 = 0$, $\xi = \omega x + kt + \xi_1$.

(5) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type*

$$u_{15}(x,t) = \pm\sqrt{\frac{-2(k^2-\omega^2)^2(a^2+b^2)}{\alpha\beta\omega^2}}\frac{1}{a\cos\xi + b\sin\xi}e^{i(kx+\omega t+\xi_0)} \tag{73}$$

$$n_{15}(x,t) = -\frac{1}{\alpha} - \frac{2(k^2-\omega^2)(b^2+a^2)}{\alpha}\frac{1}{(a\cos\xi + b\sin\xi)^2}, \tag{74}$$

*where* $k, \omega, a, b$ *are arbitrary constants such that* $\alpha\beta < 0$, $\xi = \omega x + kt + \xi_1$.

(6) *The Klein–Gordon–Zakharov equations* (1)–(2) *has explicit and exact periodic wave solutions of triangle functions type*

$$u_{16}(x,t) = \pm\sqrt{\frac{-2(k^2-\omega^2)^2}{\alpha\beta\omega^2}}\frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi}e^{i(kx+\omega t+\xi_0)} \tag{75}$$

$$n_{16}(x,t) = C - \frac{2(k^2-\omega^2)}{\alpha}\left[\frac{b\cos\xi - a\sin\xi}{a\cos\xi + b\sin\xi}\right]^2, \tag{76}$$

*where* $k, \omega, a, b$ *are arbitrary constants such that* $\alpha\beta < 0$, $3(k^2-\omega^2)+\alpha C + 1 = 0$, $\xi = \omega x + kt + \xi_1$.

## 4. Summary and conclusions

In summary, we adopt the extended hyperbolic functions method, to obtain multiple exact traveling wave solutions of the Klein–Gordon–Zakharov equations. We obtain some more general solitary wave solutions of the Klein–Gordon–Zakharov equations. It not only produces the same solutions as originally by [6,7] and [10] but also can pick up what we believe to be new solutions missed by other authors. The results indicate the Klein–Gordon–Zakharov equations admit multiple exact traveling wave solutions with two or three arbitrary parameters. The type of exact solitary wave solution is different along with different value of arbitrary parameters. So we can choose appropriate parameter value to obtain solutions we need in practical problems. The method can also be employed to solve a large number of other nonlinear evolution equations, such as nonlinear reaction–diffusion equation, the long-short wave resonance equation, the shallow water wave equation, Whitham–Broer–Kaup equation, variant Boussinesq equation, double Sine–Gardon equation, Dodd–Bullough–Mikhailov equation et al. In particular cases, when $a = 1, b = 0$ in (11a), (12a) and (13a), this method becomes hyperbolic function-method presented in [17]. In case of $a = 0, b = 1$ in (11a), (12a) and (13a), we obtain some singular traveling solutions corresponding to the solitary wave solutions of the Klein–Gordon–Zakharov equations (1)–(2) obtained in case $a = 1, b = 0$. We would like to point out that $g$ in Eq. (7) of [20] is in full agreement with $f$ in (13a) of this paper but $f$ in Eq. (7) of [20] is different from $g$ in (13a) of this paper. It is worthwhile emphasizing that $\sigma_i$ $(i = 1, 2)$ in (11) (respectively $\sigma_i$ $(i = 3, 4)$ in (12) of [21] are special cases of $f$ in (13a)) (respectively (13b)) of the present paper but $g$ in (13a) and (13b) of this paper are more general than $\tau_i$ $(i = 1, 2)$ in (11) and $\tau_i$ $(i = 3, 4)$ in (12) of [21].

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