Indag. Mathem., N.S., 16 (1), 1–19 March 21, 2005

Stability of homogeneous principal bundles with a classical group as the structure group

by Indranil Biswas

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Communicated by Prof. J.J. Duistermaat at the meeting of October 25, 2004

ABSTRACT

Let G be a connected semisimple linear algebraic group defined over an algebraically closed field k and $P \subset G$ a parabolic subgroup without any simple factor. Let H be a connected reductive linear algebraic group defined over the field k such that all the simple quotients of H are of classical type. Take any homomorphism $\rho : P \to H$ such that the image of ρ is not contained in any proper parabolic subgroup of H. Consider the corresponding principal H-bundle $E_P(H) = (G \times H)/P$ over G/P. We prove that $E_P(H)$ is strongly stable with respect to any polarization on G/P.

1. INTRODUCTION

Let G be a connected semisimple linear algebraic group defined over an algebraically closed field k. Let $P \subset G$ be a parabolic subgroup without any simple factor. The quotient G/P is a complete variety. The principal P-bundle over G/P defined by the quotient map $G \to G/P$ will be denoted by E_P.

Let V be a finite dimensional irreducible left P-module. Let $E_P(V) = (G \times V)/P$ be the vector bundle over G/P associated to the principal P-bundle E_P for the P-module V.

The main theorem in [11] states that the vector bundle $E_P(V)$ is stable with respect to any polarization on G/P provided the characteristic of the field k is zero (see [11, p. 136, Theorem 2.4]); the definition of a stable bundle is recalled

MSC: 14L30, 14F05, 14M17

Key words and phrases: Principal bundle, Stable bundle, Homogeneous space

E-mail: indranil@math.tifr.res.in (I. Biswas).

in Section 2. A weaker result was proved earlier in [7]. In [11], Umemura asks the question whether the above mentioned theorem of [11] remains valid when the characteristic of k is positive (see [11, p. 131]).

In [3] it was proved that for any algebraically closed field k the vector bundle $E_P(V)$ over G/P is stable with respect to any polarization on G/P, answering the question of Umemura affirmatively. In other words, Theorem 2.4 of [11] remains valid for all algebraically closed fields.

Fix a homomorphism

$$\rho : P \to H,$$

where H is a connected reductive linear algebraic group defined over the field k. We will assume that the homomorphism ρ is irreducible. This means that the image $\rho(P)$ is not contained in any proper parabolic subgroup of H. Let $E_P(H) = (G \times H)/P$ be the principal H-bundle over G/P obtained by extending the structure group of the principal P-bundle E_P using the homomorphism ρ.

If $k = \mathbb{C}$, then the principal H-bundle $E_P(H)$ is stable with respect to any polarization on G/P [2]. It is natural to ask if this remains valid for other algebraically closed fields.

Assume that all the simple quotients of H are of classical type. We prove that for any algebraically closed field k the principal H-bundle $E_P(H)$ over G/P is stable with respect to any polarization on G/P (Theorem 4.1). In fact, we prove that $E_P(H)$ is strongly stable with respect to any polarization on G/P.

2. SEMISTABILITY OF HOMOGENEOUS BUNDLES

Let k be an algebraically closed field. The characteristic of k will be denoted by p. Let G be a connected semisimple linear algebraic group defined over the field k. Fix a reduced proper parabolic subgroup

$$P \subsetneq G$$

without any simple factor. That P is without any simple factor means that the image of P in any simple quotient of G is a proper parabolic subgroup.

Let

(2.1) $$R_u(P) \subset P$$

be the *unipotent radical*. So $R_u(P)$ is a connected normal reduced unipotent subgroup of P, and the quotient

(2.2) $$L(P) := P/R_u(P),$$

which is called the *Levi quotient*, is a reductive linear algebraic group (see [5, p. 125]).

Let H be a connected reductive linear algebraic group defined over the field k. Let

(2.3) $$\rho : P \to H$$

be a homomorphism with the property that the image of P is not contained in any proper parabolic subgroup of H.

The natural projection $G \to G/P$ defines a principal P-bundle over the projective variety G/P. This principal P-bundle over G/P will be denoted by E_P. Let $E_P(H)$ denote the principal H-bundle over G/P obtained by extending the structure group of the P-bundle E_P using the homomorphism ρ (defined in (2.3)). Therefore, the total space of $E_P(H)$ is the quotient space

(2.4) $\quad E_P(H) = (G \times H)/P,$

where the action of any $\alpha \in P$ sends any $(g, h) \in G \times H$ to $(g\alpha, \rho(\alpha^{-1})h) \in G \times H$. The right translation action of H on $G \times H$ descends to an action of H on the quotient space $E_P(H)$ in (2.4) making $E_P(H)$ a principal H-bundle over G/P. The natural projection $G \to G/P$ induces a projection of $E_P(H)$ to G/P.

Let

(2.5) $\quad E_{L(P)} := \big(G \times L(P)\big)/P$

be the principal $L(P)$-bundle over G/P obtained by extending the structure group of the principal P-bundle E_P using the quotient map $P \to L(P)$ in (2.2).

Take a very ample line bundle ξ over G/P, which is also called a polarization. The *degree* of any line bundle η over G/P is defined to be the degree of the restriction of η to a smooth complete intersection curve obtained by intersecting hypersurfaces of G/P in the complete linear system for ξ (this does not depend on the choice of the complete intersection curve). The degree of a vector bundle W over G/P is defined to be the degree of the line bundle $\bigwedge^r W$, where $r = \text{rank}(W)$. More generally, the degree of a arbitrary coherent sheaf $\widetilde{W}$ over G/P is defined to be

$$\text{degree}(\widetilde{W}) = \sum_{i=0}^{\ell_0} (-1)^i \text{degree}(W_i),$$

where

$$0 \to W_{\ell_0} \to \cdots \to W_1 \to W_0 \to \widetilde{W} \to 0$$

is an exact sequence of coherent sheaves on G/P with each W_i, $i \in [0, \ell_0]$, a vector bundle. If can shown that degree$(\widetilde{W})$ is independent of the choice of the exact sequence.

Note that from this definition it follows immediately that if $\widetilde{W}$ is a torsion sheaf supported on a closed subscheme $S \subset G/P$ such that the codimension of S is at least two, then degree$(\widetilde{W}) = 0$. We also note that the degree of a coherent W on G/P coincides with the intersection number of $c_1(W)$ (the Chern class is considered as an element of the Chow group) and $\dim G/P - 1$ hypersurfaces in G/P in the complete linear system of the very ample line bundle ξ.

Let W be a vector bundle defined over a nonempty Zariski open subset $U \subset G/P$ such that the complement $(G/P) \setminus U$ is of codimension at least two. Then the direct image $\iota_* W$ is a coherent sheaf on G/P, where $\iota : U \to G/P$ is the inclusion map. By degree(W) we will mean degree($\iota_* W$).

A torsionfree coherent sheaf W over G/P is called *stable* if

$$\frac{\text{degree}(W')}{\text{rank}(W')} < \frac{\text{degree}(W)}{\text{rank}(W)}$$

for every coherent subsheaf $W' \subset W$ with $0 < \text{rank}(W') < \text{rank}(W)$ (the degree is defined by fixing a very ample line bundle on G/P). The torsionfree coherent sheaf W is called *semistable* if

$$\frac{\text{degree}(W')}{\text{rank}(W')} \leqslant \frac{\text{degree}(W)}{\text{rank}(W)}$$

for every coherent subsheaf W' of W of positive rank. It is easy to see that a stable sheaf is semistable.

When the characteristic p of the field k is positive, a vector bundle W is called *strongly semistable* if the iterated pull back $(F^n)^* W$ is semistable for all $n \geqslant 0$, where

(2.6) $\qquad F : G/P \to G/P$

is the Frobenius morphism of G/P; here F^0 denotes the identity map of G/P. For notational convenience, if the characteristic p of k is zero, then F will denote the identity map of G/P. If $p = 0$, then a strongly semistable vector bundle will simply mean a semistable vector bundle. Note that this is compatible with the above convention on F.

We will now recall the definition of a (semi)stable principal bundle; see [8–10]. Let G' be a connected reductive linear algebraic group defined over the field k and $E'_{G'}$ a principal G'-bundle over G/P. The G'-bundle $E'_{G'}$ is called *stable* (respectively, *semistable*) if for every triple of the form (P', U, σ), where

(i) $P' \subset G'$ is a reduced maximal parabolic subgroup,
(ii) $U \subset G/P$ is a Zariski open dense subset such that the codimension of complement $(G/P) \setminus U$ is at least two,
(iii) $\sigma : U \to (E'_{G'}/P')|_U$ is a reduction of structure group over U of the G'-bundle $E'_{G'}$, to the subgroup P',

the following inequality holds:

$$\text{degree}(\sigma^* T_{\text{rel}}) > 0$$

(respectively, $\text{degree}(\sigma^* T_{\text{rel}}) \geqslant 0$), where T_{rel} is the relative tangent bundle for the natural projection $E'_{G'}/P' \to G/P$ and the degree is defined after fixing a polarization on G/P (see [9, p. 129, Definition 1.1] and [9, p. 131, Lemma 2.1]).

Remark 2.1. For any vector bundle W over G/P of rank n, there is a corresponding principal $\text{GL}(n, k)$-bundle over G/P defined by the space of all linear isomorphisms of $k^{\oplus n}$ with the fibers of W. It is straight-forward to check that the vector bundle W is stable (respectively, semistable) if and only if the corresponding principal $\mathrm{GL}(n, k)$-bundle over G/P is stable (respectively, semistable).

Remark 2.2. Let $\rho : G' \to G''$ be a quotient reductive group of G' such that the kernel of ρ is contained in the center of G'. Given any principal G'-bundle $E'_{G'}$ over G/P, we have the principal G''-bundle $E'_{G'}(G'')$ over G/P obtained by extending the structure group of $E'_{G'}$ using the projection ρ. It is straight-forward to check that the principal G'-bundle $E'_{G'}$ is stable (respectively, semistable) if and only if the corresponding principal G''-bundle $E'_{G'}(G'')$ is stable (respectively, semistable). Combining this with Remark 2.1 we conclude that a vector bundle W over G/P of rank n is stable (respectively, semistable) if and only if the principal $\mathrm{PGL}(n, k)$-bundle over G/P defined by W is stable (respectively, semistable).

A principal G'-bundle $E'_{G'}$ over G/P is called *strongly stable* (respectively, *strongly semistable*) if the iterated pull back $(F^n)^* E'_{G'}$ is a stable (respectively, semistable) principal G'-bundle for all $n \geqslant 0$, where the map F, as before, is the Frobenius map in (2.6) when the characteristic of k is positive and it is the identity map of G/P when the characteristic of k is zero.

Lemma 2.3. *The principal $L(P)$-bundle $E_{L(P)}$, defined in* (2.5)*, is strongly semistable with respect to any polarization on G/P.*

The principal H-bundle $E_P(H)$, defined in (2.4)*, is strongly semistable with respect to any polarization on G/P.*

Proof. Take any finite-dimensional left $L(P)$-module V. Let $E_{L(P)}(V)$ be the corresponding vector bundle over G/P associated to the principal $L(P)$-bundle $E_{L(P)}$. We recall that $E_{L(P)}(V) = (E_{L(P)} \times V)/L(P)$; the action of any $g \in L(P)$ sends any $(z, v) \in E_{L(P)} \times V$ to $(zg, g^{-1} \cdot v)$. We know that if the $L(P)$-module V is irreducible, then the vector bundle $E_{L(P)}(V)$ is stable with respect to any polarization on G/P [3, p. 135, Theorem 2.1].

Fix a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V \tag{2.7}$$

of the left $L(P)$-module V such that each successive quotient V_i/V_{i-1}, $1 \leqslant i \leqslant \ell$, is an irreducible left $L(P)$-module. Let $E_{L(P)}(V_i)$, $1 \leqslant i \leqslant \ell$, denote the vector bundle over G/P associated to the principal $L(P)$-bundle $E_{L(P)}$ for the left $L(P)$-module V_i. Note that the filtration of $L(P)$-modules in (2.7) gives a filtration of subbundles

$$0 = E_{L(P)}(V_0) \subset E_{L(P)}(V_1) \subset \cdots \subset E_{L(P)}(V_{\ell-1}) \subset E_{L(P)}(V_\ell) = E_{L(P)}(V) \tag{2.8}$$

of the vector bundle $E_{L(P)}(V)$. The quotient vector bundle $E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})$ is identified in an obvious fashion with the vector bundle associated to the principal $L(P)$-bundle $E_{L(P)}$ for the left $L(P)$-module V_i/V_{i-1}.

Let

$$(2.9)\qquad Z\big(L(P)\big)\subset L(P)$$

denote the subgroup–scheme defined by the center of $L(P)$. Note that $Z(L(P))$ in general is nonreduced; for example, if $L(P)=\mathrm{SL}(2,k)$ and the characteristic of k is two, then $Z(L(P))$ is nonreduced.

Henceforth in the proof of the lemma we will assume that V satisfies the condition that the center $Z(L(P))$ (defined in (2.9)) acts trivially on V. Since $Z(L(P))$ acts trivially on the $L(P)$-module V_i/V_{i-1}, the vector bundle

$$E_{L(P)}(V_i/V_{i-1})=E_{L(P)}(V_i)/E_{L(P)}(V_{i-1})$$

associated to $E_{L(P)}$ for the $L(P)$-module V_i/V_{i-1} has the property that its determinant bundle, namely the line bundle $\bigwedge^{\mathrm{top}}E_{L(P)}(V_i/V_{i-1})$ over G/P, is isomorphic to the trivial line bundle. Indeed, this follows immediately from the fact that $L(P)/Z(L(P))$ does not have any nontrivial character. In particular, we have

$$\mathrm{degree}\big(E_{L(P)}(V_i/V_{i-1})\big)=0$$

with respect to any polarization on G/P.

Since V_i/V_{i-1}, $1\leqslant i\leqslant \ell$, is an irreducible $L(P)$-module, the associated vector bundle $E_{L(P)}(V_i/V_{i-1})$ is stable [3, p. 135, Theorem 2.1]. Therefore, (2.8) is a filtration of subbundles of $E_{L(P)}(V)$ such that each successive quotient is a stable vector bundle of degree zero with respect to any polarization on G/P. From this it follows immediately that the vector bundle $E_{L(P)}(V)$ is semistable with respect to any polarization on G/P (the extension of a semistable vector bundle of degree zero by another semistable vector bundle of degree zero is semistable).

Let $F_{L(P)}:L(P)\to L(P)$ denote the Frobenius morphism of $L(P)$. As before, if the characteristic of the field k is zero, then $F_{L(P)}$ will denote the identity map $L(P)$. For any integer $n\geqslant 0$, let $V(n)$ denote the left $L(P)$-module constructed using the composition homomorphism

$$(2.10)\qquad L(P)\xrightarrow{F^n_{L(P)}}L(P)\to \mathrm{GL}(V),$$

where $F^n_{L(P)}$ is the n-fold iteration of the self-map $F_{L(P)}$ (the Frobenius morphism of $L(P)$). Let $E_{L(P)}(V(n))$ denote the vector bundle over G/P associated to the principal $L(P)$-bundle $E_{L(P)}$ for the left $L(P)$-module $V(n)$ defined in (2.10). From this definition of $E_{L(P)}(V(n))$ it follows that the vector bundle $E_{L(P)}(V(1))$ is identified with the pull back $F^*E_{L(P)}(V)$, where F, as in (2.6), is the Frobenius morphism of G/P (it is the identity map of G/P if the field k is of characteristic zero); see [8, p. 287, Remark 3.22]. Consequently, for any in integer $n\geqslant 0$, the vector bundle $E_{L(P)}(V(n))$ is identified with the pull back $(F^n)^*E_{L(P)}(V)$.

Note that since $Z(L(P))$ (defined in (2.9)) acts trivially on V, it also acts trivially on each $V(n)$. Therefore, replacing the $L(P)$-module V by the $L(P)$-module $V(n)$ in the above argument for semistability of $E_{L(P)}(V)$ we conclude that the vector bundle $E_{L(P)}(V(n))$ is semistable with respect to any polarization on G/P. In view of the above remark that $(F^n)^* E_{L(P)}(V) = E_{L(P)}(V(n))$, this implies that the vector bundle $E_{L(P)}(V)$ is strongly semistable with respect to any polarization on G/P.

Setting V to be the Lie algebra of $L(P)$ equipped with the adjoint action of $L(P)$ we conclude that the adjoint vector bundle $\mathrm{ad}(E_{L(P)})$ (of the principal $L(P)$-bundle $E_{L(P)}$) is strongly semistable with respect to any polarization on G/P. This immediately implies that the principal $L(P)$-bundle $E_{L(P)}$ is strongly semistable with respect to any polarization on G/P.

To prove the second part of the lemma, consider the homomorphism ρ in (2.3). Using the condition that the image of ρ is not contained in any proper parabolic subgroup of H it can be deduced that $\rho(R_u(P)) = e$, where $R_u(P)$, as in (2.1), is the unipotent radical, and $e \in H$ is the identity element. Indeed, if $\rho(R_u(P)) \neq e$, then $\rho(R_u(P))$ is a connected nontrivial unipotent subgroup of H. From this it follows that the normalizer of $\rho(R_u(P))$ in H is contained in a proper parabolic subgroup Q of H [5, p. 186, Corollary A]. To deduce this from [5, p. 186, Corollary A] note the following two facts:

(i) for the sequence of subgroups $\{N_i\}_{i\in\mathbb{N}}$ of G in [5, p. 185, §30.3] the inclusion $N_i \subseteq N_{i+1}$ holds for all $i \in \mathbb{N}$, and
(ii) any connected unipotent subgroup of H is contained in a Borel subgroup of H, which in turn follows from the definition of a Borel subgroup (see [5, p. 134, §21.3]).

Since $R_u(P)$ is a normal subgroup of P and the normalizer of $\rho(R_u(P))$ (in H) is contained in Q, we conclude that $\rho(P) \subset Q$. But, by assumption, the image $\rho(P)$ is not contained in any proper parabolic subgroup of H. Therefore, we conclude that $\rho(R_u(P)) = e$.

Since $\rho(R_u(P)) = e$, the homomorphism ρ in (2.3) induces a homomorphism

$$\bar{\rho} : P/R_u(P) = L(P) \to H. \tag{2.11}$$

Let $Z_0(L(P)) \subset L(P)$ be the reduced subgroup defined by the connected component of the center of $L(P)$. Similarly, set $Z_0(H) \subset H$ to be the reduced subgroup defined by the connected component of the center of H. It is easy to see that

$$\bar{\rho}\big(Z_0\big(L(P)\big)\big) \subset Z_0(H), \tag{2.12}$$

where $\bar{\rho}$ is constructed in (2.11). Indeed, if we have

$$\bar{\rho}\big(Z_0\big(L(P)\big)\big) \not\subset Z_0(H),$$

then the centralizer, in H, of the torus $\bar{\rho}(Z_0(L(P)))$ is a proper subgroup of H (note that $\bar{\rho}(Z_0(L(P)))$ is connected as $Z_0(L(P))$ is so). On the other hand, the centralizer of any torus in H is a Levi subgroup of a parabolic subgroup of H. Therefore, the centralizer, in H, of the torus $\bar{\rho}(Z_0(L(P)))$ is contained in a proper parabolic subgroup of H. If $Q \subset H$ is a proper parabolic subgroup containing the centralizer of $\bar{\rho}(Z_0(L(P)))$, then Q contains $\bar{\rho}(L(P))$ as $L(P)$ commutes with $Z_0(L(P))$. But this contradicts the given condition that $\rho(P)$ is not contained in any proper parabolic subgroup of H. Therefore, the inclusion (2.12) is valid.

Since the $L(P)$-bundle $E_{L(P)}$ is strongly semistable and the inclusion (2.12) holds, it follows immediately from [8, p. 288, Theorem 3.23] that the principal H-bundle $E_P(H)$ is strongly semistable (if the characteristic of k is zero, then it directly follows from [8, p. 285, Theorem 3.18]). This completes the proof of the lemma. □

In Section 4 we will show that $E_P(H)$ is strongly stable under the assumption that all the simple quotients of H are of classical type. For that we will need an analog of the socle for a semistable projective bundle (the socle of a semistable vector bundle is the unique maximal polystable subsheaf). The socle for a semistable projective bundle will be constructed in the next section.

3. SOCLE FOR A SEMISTABLE PROJECTIVE BUNDLE

Although throughout the paper we consider principal bundles only over G/P, the socle constructed in this section might have other applications. Therefore, in this section we do the construction of the socle for semistable principal bundles over an arbitrary smooth projective variety.

Let X be an irreducible smooth projective variety defined over the field k. Fix a very ample line bundle (= polarization) on X in order to be able to define degree of coherent sheaves on X. Stability and semistability of principal bundles over X are defined exactly as done in Section 2 for principal bundles over G/P.

Let $\mathbb{P} \to X$ be a projective bundle over X of relative dimension $d-1$. Let $F_{\mathbb{P}}$ denote the principal $\mathrm{PGL}(d,k)$-bundle over X defined by $\mathbb{P}$. So for each closed point $x \in X$, the fiber $(F_{\mathbb{P}})_x$ is the space of all linear isomorphisms from $\mathbb{P}^{d-1}_k$ to $\mathbb{P}_x$; here $\mathbb{P}^{d-1}_k$ denotes the space of all one-dimensional subspaces of $k^{\oplus d}$.

Take a Zariski open dense subset $U \subset X$ such that the complement $X \setminus U$ is of codimension at least two. Let $\mathbb{P}' \subset \mathbb{P}$ be a subprojective bundle defined over U. So for each closed point $x \in U$, the fiber $\mathbb{P}'_x$ is a linear subspace of the projective space $\mathbb{P}_x$. Let $d'-1$ be the relative dimension of projective bundle $\mathbb{P}'$ over U.

Consider the obvious decomposition $k^{\oplus d} = k^{\oplus d'} \oplus k^{\oplus(d-d')}$. Using this decomposition we have a natural inclusion of $\mathbb{P}^{d'-1}_k$ in $\mathbb{P}^{d-1}_k$. The inclusion map sends any line ζ in $k^{\oplus d'}$ to the line in $k^{\oplus d}$ given by $(\zeta, 0) \subset k^{\oplus d'} \oplus k^{\oplus(d-d')}$. Let

$$Q \subset \mathrm{PGL}(d,k)$$

be the maximal parabolic subgroup defined by the space of all automorphisms of $\mathbb{P}^{d-1}_k$ preserving the above defined subspace $\mathbb{P}^{d'-1}_k$.

The above subbundle $\mathbb{P}' \subset \mathbb{P}|_U$ gives a reduction of structure group over U of the principal $\mathrm{PGL}(d,k)$-bundle $F_{\mathbb{P}}$ to the subgroup $Q \subset \mathrm{PGL}(d,k)$. This reduction of structure group

$$F_Q \subset F_{\mathbb{P}}|_U \tag{3.1}$$

is defined by the condition that for any closed point $x \in U$ and any closed point $z \in (F_{\mathbb{P}})_x$ in the fiber, the isomorphism of $\mathbb{P}_k^{d-1}$ with $(F_{\mathbb{P}})_x$ given by z takes the subspace $\mathbb{P}_k^{d'-1} \subset \mathbb{P}_k^{d-1}$ to the subspace $\mathbb{P}'_x \subset \mathbb{P}_x$. The reduction of structure group in (3.1) gives a section

$$\sigma : U \to (F_{\mathbb{P}}/Q)|_U \tag{3.2}$$

of the fiber bundle $(F_{\mathbb{P}}/Q)|_U \to U$.

Given any maximal parabolic subgroup Q' of $\mathrm{PGL}(d,k)$, there is a proper subspace $V(Q') \subset k^{\oplus d}$ such that Q' coincides with the subgroup of $\mathrm{PGL}(d,k)$ that preserves $V(Q')$.

Therefore, from the definition, given in Section 2, of a semistable principal $\mathrm{PGL}(d,k)$-bundle it follows that the principal $\mathrm{PGL}(d,k)$-bundle $F_{\mathbb{P}}$ over X is semistable if and only if for each pair $(U, \mathbb{P}')$ of the above type the inequality

$$\mathrm{degree}(\sigma^* T_{\mathrm{rel}}) \geqslant 0 \tag{3.3}$$

holds, where T_{rel} is the relative tangent bundle for the projection $F_{\mathbb{P}}/Q \to X$ and σ is the reduction of structure group to Q constructed in (3.2).

If there is a vector bundle W over X such that $\mathbb{P}$ is the projective bundle $\mathbb{P}(W)$ (here $\mathbb{P}(W)$ denotes the space of all one-dimensional subspaces in the fibers of W), then there is a unique subbundle

$$W' \subset W|_U$$

of rank d' over the open subset U such that the above subprojective bundle $\mathbb{P}' \subset \mathbb{P}|_U$ is identified with the projective bundle $\mathbb{P}(W')$. Furthermore, we have

$$\mathrm{degree}(\sigma^* T_{\mathrm{rel}}) = \mathrm{degree}(W/W')d' - \mathrm{degree}(W')(d-d'), \tag{3.4}$$

where σ is constructed in (3.2) and T_{rel} is as in (3.3). It may be pointed out that from the identity (3.4) it follows immediately that the principal $\mathrm{PGL}(d,k)$-bundle $F_{\mathbb{P}}$ is semistable (respectively, stable) if and only if the vector bundle W is semistable (respectively, stable); see Remark 2.2.

Now-onwards, in this section assume that the principal $\mathrm{PGL}(d,k)$-bundle $F_{\mathbb{P}}$ over X corresponding to the projective bundle $\mathbb{P}$ is semistable.

We recall that a torsionfree coherent sheaf W'_0 is called *polystable* if it is isomorphic to a direct sum of coherent sheaves $\bigoplus_i W'_{i,0}$, where each $W'_{i,0}$ is stable with

$$\frac{\mathrm{degree}(W'_{i,0})}{\mathrm{rank}(W'_{i,0})} = \frac{\mathrm{degree}(W'_0)}{\mathrm{rank}(W'_0)}$$

for all i. A semistable vector bundle has a unique maximal polystable subsheaf which is called the *socle*. The socle of a semistable vector bundle W_0 is generated by all polystable subsheaves $W'_0 \subseteq W_0$ with

$$\frac{\mathrm{degree}(W'_0)}{\mathrm{rank}(W'_0)} = \frac{\mathrm{degree}(W_0)}{\mathrm{rank}(W_0)}; \tag{3.5}$$

see [6] and [1] for the details.

Let $W' \subset W$ be the socle of a semistable vector bundle W. From the definition of a socle it follows that the quotient W/W' is torsionfree. Indeed, since W' is a polystable subsheaf of W satisfying (3.5), any torsion subsheaf of W/W' can be absorbed in some polystable subsheaf of W satisfying (3.5). Since W/W' is torsionfree it follows immediately that there is a Zariski open dense subset $U' \subset X$ such that

(1) the complement $X \setminus U'$ is of codimension at least two, and
(2) the subsheaf $W' \subset W$ is a subbundle over U'.

The Zariski open subset of X over which the quotient W/W' is a vector bundle can be taken to U'.

Therefore, if $\mathbb{P} = \mathbb{P}(W)$, where W is a semistable vector bundle over X, then $\mathbb{P}$ has a unique maximal subprojective bundle $\mathbb{P}_1 \subset \mathbb{P}$ defined over some open dense subset $U \subset X$ such that

(1) the complement $X \setminus U$ is of codimension at least two,
(2) the projective bundle $\mathbb{P}_1$ is polystable, and
(3) $\text{degree}(\sigma^* T_{\text{rel}}) = 0$, where σ and T_{rel} are as in (3.3) for the subprojective bundle $\mathbb{P}_1$.

To construct this projective bundle $\mathbb{P}_1$, let U denote the open dense subset of X over which the socle W' of W is a subbundle of W. Now set $\mathbb{P}_1$ to be the projective bundle over U defined by the space of all lines in the fibers of W'. We recall that a projective bundle $\mathbb{P}_0$ defined over an open dense subset U_0 of X, with $\text{codim}(X \setminus U_0) \geqslant 2$, is polystable if $\mathbb{P}_0$ contains finitely many subprojective bundles $\mathbb{P}_{i,0}$, $1 \leqslant i \leqslant n$, such that

(1) the relative dimension of the projective bundle $\mathbb{P}_0$ coincides with $\sum_{i=1}^{n} d_{i,0}$, where $d_{i,0}$ is the relative dimension of the projective bundle $\mathbb{P}_{i,0}$,
(2) for each closed point $x \in U_0$, the linear span of the fibers $\{(\mathbb{P}_{i,0})_x\}_{i=1}^{n}$ is $(\mathbb{P}_0)_x$, and
(3) if σ_i, $1 \leqslant i \leqslant n$, is the section constructed as in (3.2) for the subprojective bundle $\mathbb{P}_{i,0} \subset \mathbb{P}_0$, then $\text{degree}(\sigma^* T_{\text{rel}}) = 0$, where T_{rel} is the relative tangent bundle as in (3.3) for $\mathbb{P}_{i,0}$.

However, a general projective bundle is not the projectivization of some vector bundle. Our aim in this section is to prove that given any semistable projective bundle $\mathbb{P}$ there is a unique maximal pair $(U_0, \mathbb{P}_1)$ satisfying the above three conditions.

As before, let $\mathbb{P}$ be a semistable projective bundle over X. Let $\mathbb{P}' \subset \mathbb{P}|_U$ be a polystable subprojective bundle defined over a Zariski open dense subset $U \subset X$

such that the complement $X \setminus U$ is of codimension at least two. Further assume that $\mathbb{P}'$ satisfies the following condition:

$$\text{degree}\big((\sigma')^* T_{\text{rel}}\big) = 0, \tag{3.6}$$

where σ' is the section over U constructed as in (3.2) for the subprojective bundle $\mathbb{P}'$, and T_{rel} is the relative tangent bundle as in (3.3) corresponding to $\mathbb{P}'$.

Let $\mathbb{P}'' \subset \mathbb{P}|_U$ be a stable subprojective bundle defined over the same open subset U such that

$$\text{degree}\big((\sigma'')^* T_{\text{rel}}\big) = 0, \tag{3.7}$$

where σ'' is the section over U constructed as in (3.2) for the subprojective bundle $\mathbb{P}''$, and T_{rel} is defined as in (3.3) for $\mathbb{P}''$.

Proposition 3.1. *Take the above two subprojective bundles $\mathbb{P}'$ and $\mathbb{P}''$ of $\mathbb{P}$ defined over the open subset U. There is a Zariski open dense subset $U' \subset U$, whose complement $U \setminus U'$ is of codimension at least two, such that one of the following two holds:*

(1) *For any closed point $y \in U'$, the fiber $\mathbb{P}''_y$ is contained in the fiber $\mathbb{P}'_y$.*
(2) *For any closed point $y \in U'$, the two subspaces $\mathbb{P}'_y$ and $\mathbb{P}''_y$ of $\mathbb{P}_y$ are disjoint.*

Proof. Let $U'' \subset U \subset X$ be a Zariski open dense subset such that

(i) the complement $X \setminus U''$ is of codimension at least two, and
(ii) $\mathbb{P}'$ and $\mathbb{P}''$ together generate a subprojective bundle of $\mathbb{P}$ over U''.

Such an open subset U'' can be constructed as follows. Let

$$f : \mathbb{P} \to X \tag{3.8}$$

be the natural projection from the total space of the projective bundle. The pulled back projective bundle $f^*\mathbb{P}$ has the property that there is a certain canonically defined vector bundle $\mathcal{V}$ over $\mathbb{P}$ such that $f^*\mathbb{P}$ is identified with $\mathbb{P}(\mathcal{V})$. The vector bundle $\mathcal{V}$ in question is the dual bundle $J^1_{\text{rel}}(\mathcal{O}_{\mathbb{P}})^*$, where $J^1_{\text{rel}}(\mathcal{O}_{\mathbb{P}})$ is the relative first jet bundle of the trivial line bundle; the jet bundle fits in an exact sequence

$$0 \to T_f^* \to J^1_{\text{rel}}(\mathcal{O}_{\mathbb{P}}) \to \mathcal{O}_{\mathbb{P}} \to 0$$

over $\mathbb{P}$, where $T_f^* \subset T^*\mathbb{P}$ is the relative cotangent bundle for the projection f. The two subprojective bundles $\mathbb{P}'$ and $\mathbb{P}''$ define subbundles of $\mathcal{V}$ over $f^{-1}(U)$. More precisely, let $\mathcal{V}'$ (respectively, $\mathcal{V}''$) be the unique subbundle of $\mathcal{V}|_{f^{-1}(U)}$ such that the subprojective bundle of $\mathbb{P}(\mathcal{V})|_{f^{-1}(U)}$ defined by $f^*\mathbb{P}'$ (respectively, $f^*\mathbb{P}''$) coincides with the subprojective bundle $\mathbb{P}(\mathcal{V}')|_{f^{-1}(U)}$ (respectively, $\mathbb{P}(\mathcal{V}'')|_{f^{-1}(U)}$); here f is the projection in (3.8) and U is the open set over which $\mathbb{P}'$ and $\mathbb{P}''$ are defined.

Let $\mathcal{V}_1$ be the unique smallest coherent subsheaf of $\mathcal{V}|_{f^{-1}(U)}$ satisfying the following two conditions:

(1) the subsheaf $\mathcal{V}_1$ contains both $\mathcal{V}'$ and $\mathcal{V}''$, and
(2) the quotient $\mathcal{V}/\mathcal{V}_1$ is torsionfree.

Let $U_1'' \subset f^{-1}(U)$ be the open subset over which the subsheaf $\mathcal{V}_1$ is a subbundle of $\mathcal{V}_{f^{-1}(U)}$. Since $\mathcal{V}/\mathcal{V}_1$ is torsionfree, the codimension of the complement $f^{-1}(U) \setminus U_1''$ is at least two. Since $U_1'' = f^{-1}(f(U_1''))$, the codimension of the complement $U \setminus f(U_1'')$ coincides with the codimension of the complement $f^{-1}(U) \setminus U_1''$ and hence it is at least two. Consequently, the codimension of the complement $X \setminus f(U_1'')$ is at least two (recall that $\operatorname{codim}(X \setminus U) \geqslant 2$). The open subset U'' in the beginning of the proof can be taken to be the image $f(U_1'')$, where f is the projection in (3.8).

Let $\mathbb{P}_1' \subset \mathbb{P}|_{U''}$ be the subprojective bundle over U'' generated by $\mathbb{P}'$ and $\mathbb{P}''$. Therefore, $\mathbb{P}_1'$ is determined by the condition that the subprojective bundle $f^*\mathbb{P}_1' \subset (f^*\mathbb{P})|_{U_1''}$ coincides with the subprojective bundle $\mathbb{P}(\mathcal{V}_1) \subset (f^*\mathbb{P})|_{U_1''}$, where f is the projection in (3.8) and $\mathcal{V}_1$ is constructed above.

Consequently, both $\mathbb{P}'$ and $\mathbb{P}''$ are subprojective bundles of the projective bundle $\mathbb{P}_1'$ defined over the open subset U''.

Let $F_{\mathbb{P}_1'}$ be the principal $\mathrm{PGL}(n_1'+1, \mathbb{C})$-bundle over U'' defined by $\mathbb{P}_1'$, where n_1' is the dimension of a fiber of the natural projection of $\mathbb{P}_1'$ to U''. Let

(3.9) $\quad \sigma_1' : U'' \to F_{\mathbb{P}_1'}/Q_1'$

be the reduction of structure group defined by the subprojective bundle $\mathbb{P}'$ of $\mathbb{P}_1'$ (the construction of this reduction is identical to the construction done in (3.2)); here Q_1' is a maximal parabolic subgroup of $\mathrm{PGL}(n_1'+1, \mathbb{C})$ of type determined by the relative dimension of the projective bundle $\mathbb{P}'$ over U''.

Since the projective bundle $\mathbb{P}$ is semistable, from (3.6) and (3.7) it follows that the projective bundle $\mathbb{P}_1'$ is semistable, and furthermore,

(3.10) $\quad \operatorname{degree}\big((\sigma_1')^* T_{\mathrm{rel}}\big) = 0,$

where T_{rel} is the relative tangent bundle for the natural projection $F_{\mathbb{P}_1'}/Q_1' \to U''$ and σ_1' is the section constructed in (3.9).

Let $\mathbb{P}_1''$ be the projective bundle over U'' whose fiber over any closed point $x \in U''$ is the space of all hyperplanes in the fiber $(\mathbb{P}_1')_x$ that contain the linear subspace $\mathbb{P}_x' \subset (\mathbb{P}_1')_x$. Also define $\mathbb{P}'''$ to be the projective bundle over U'' whose fiber over any closed point $x \in U''$ is the space of all hyperplanes in the fiber $(\mathbb{P}'')_x$. If the subprojective bundle $\mathbb{P}'' \subset \mathbb{P}$ is not contained in the subprojective bundle $\mathbb{P}'$, then over a nonempty Zariski open subset of U'' the above constructed projective bundle $\mathbb{P}_1''$ is a subprojective bundle of $\mathbb{P}'''$. To see this, take any closed point $x \in U''$ and any hyperplane $H \in (\mathbb{P}_1'')_x$ in the projective space $(\mathbb{P}_1')_x$. Consider the linear subspace

$$H \cap (\mathbb{P}'')_x \subset (\mathbb{P}_1')_x$$

(recall that $(\mathbb{P}'')_x$ is a subspace of $(\mathbb{P}'_1)_x$). Since H contains the subspace $\mathbb{P}'_x \subset (\mathbb{P}'_1)_x$, the above intersection $H \cap (\mathbb{P}'')_x$ is a hyperplane in $(\mathbb{P}'')_x$ for the general point H in the total space of $\mathbb{P}''_1$, provided the subprojective bundle $\mathbb{P}'' \subset \mathbb{P}$ is not contained in the subprojective bundle $\mathbb{P}'$.

Since the projective bundle $\mathbb{P}''$ is stable, the above observation combined together with (3.10) and (3.7) yield that the two subspaces $(\mathbb{P}')_x$ and $(\mathbb{P}'')_x$ of $(\mathbb{P}'_1)_x$ are disjoint for the general closed point $x \in U''$, provided the subprojective bundle $\mathbb{P}'' \subset \mathbb{P}$ is not contained in the subprojective bundle $\mathbb{P}' \subset \mathbb{P}$. This completes the proof of the proposition. □

Given any semistable projective bundle $\mathbb{P}$ over X, from Proposition 3.1 it follows that there are finitely many stable subprojective bundles $\mathbb{P}^i \subset \mathbb{P}$, $1 \leqslant i \leqslant n$, defined over open dense subsets whose complements are of codimension at least two such that

(1) for each $1 \leqslant i \leqslant n$, we have

$$\text{degree}(\sigma_i^* T_{\text{rel}}) = 0,$$

where σ_i is the section constructed as in (3.2) for the subprojective bundle $\mathbb{P}^i \subset \mathbb{P}$, and T_{rel} as in (3.3) is the relative tangent bundle,

(2) the subprojective bundles $\mathbb{P}^i \subset \mathbb{P}$ together generate a polystable subprojective bundle $\mathbb{P}_1$ over an open dense subset $U \subset X$ such that the complement $X \setminus U$ is of codimension at least two,

(3) the section σ constructed as in (3.2) for the subprojective bundle $\mathbb{P}_1 \subset \mathbb{P}|_U$ satisfies the condition

$$\text{degree}(\sigma^* T_{\text{rel}}) = 0, \tag{3.11}$$

where T_{rel} as in (3.3) is the relative tangent bundle,

(4) any stable subprojective bundle $\mathbb{P}' \subset \mathbb{P}$ defined over an open dense subset of X whose complement is of codimension at least two and satisfying (3.11) (for the subprojective bundle $\mathbb{P}' \subset \mathbb{P}$) is contained in $\mathbb{P}_1$.

Therefore, $\mathbb{P}_1$ is the unique maximal polystable subprojective bundle of $\mathbb{P}$, defined over an open dense subset whose complement is of codimension at least two, that satisfies the condition (3.11) (for the subprojective bundle $\mathbb{P}_1$).

Definition 3.2. For any semistable projective bundle $\mathbb{P}$ we will call the subprojective bundle $\mathbb{P}_1$ constructed above as the *socle* of $\mathbb{P}$.

4. STABILITY OF HOMOGENEOUS BUNDLES FOR CLASSICAL GROUPS

Let $Z(H) \subset H$ be the center of H, which, in general, is nonreduced. In other words, $Z(H)$ is a subgroup-scheme. The quotient $H/Z(H)$ is a product of simple groups.

Henceforth, we will assume that all the simple factors of $H/Z(H)$ are of classical type. Therefore, we have

$$H/Z(H) = \prod_{i=1}^{\ell_0} H_i, \tag{4.1}$$

where each H_i is either $\mathrm{PSL}(n_i, k)$ or $\mathrm{PSO}(n_i, k)$ or $\mathrm{PSp}(2n_i, k)$.

Theorem 4.1. *Assume that all the simple factors of $H/Z(H)$ are of classical type. Then the principal H-bundle $E_P(H)$ over G/P, defined in* (2.4), *is strongly stable with respect to any polarization on G/P.*

Proof. Consider the quotient group $H/Z(H)$. Let $E_P(H/Z(H))$ denote the principal $H/Z(H)$-bundle over G/P obtained by extending the structure group of the H-bundle $E_P(H)$ using the natural projection of H to $H/Z(H)$. From the definition of a stable principal bundle it follows that the $H/Z(H)$-bundle $E_P(H/Z(H))$ is strongly stable if and only if the H-bundle $E_P(H)$ is strongly stable (the parabolic subgroups of $H/Z(H)$ are in bijective correspondence with the parabolic subgroups of H with the correspondence constructed using the projection map $H \to H/Z(H)$); see Remark 2.2.

Consider a simple factor H_i of $H/Z(H)$ (see (4.1)). Let $E_P(H/Z(H))(H_i)$ denote the principal H_i-bundle over G/P obtained by extending the structure group of the principal $H/Z(H)$-bundle $E_P(H/Z(H))$ using the natural projection of $H/Z(H)$ to the simple group H_i. From the definition of a stable principal bundle it follows immediately that the $H/Z(H)$-bundle $E_P(H/Z(H))$ is strongly stable if and only if the H_i-bundle $E_P(H/Z(H))(H_i)$ is strongly stable for each $i \in [1, \ell_0]$ (any parabolic subgroup of $H/Z(H)$ is of the form $\prod_{i=1}^{\ell_0} P_i$, where P_i is a parabolic subgroup of H_i).

In view of the above observations, we can, and we will, assume that H is $\mathrm{PSL}(n, k)$ or $\mathrm{PSO}(n, k)$ or $\mathrm{PSp}(n, k)$. So henceforth H is either $\mathrm{PSL}(n, k)$ or $\mathrm{PSO}(n, k)$ or $\mathrm{PSp}(n, k)$.

Let

$$\delta : H \hookrightarrow \mathrm{PSL}(n, k) \tag{4.2}$$

be the natural inclusion (recall that H is either $\mathrm{PSL}(n, k)$ or $\mathrm{PSO}(n, k)$ or $\mathrm{PSp}(n, k)$ and hence it is a subgroup of $\mathrm{PSL}(n, k)$). Let $E_P(\mathrm{PSL}(n, k))$ denote the principal $\mathrm{PSL}(n, k)$-bundle over G/P obtained by extending the structure group of the principal H-bundle $E_P(H)$ using the homomorphism δ in (4.2). Note that $E_P(\mathrm{PSL}(n, k))$ is identified with the principal $\mathrm{PSL}(n, k)$-bundle obtained by extending the structure group of the principal P-bundle E_P using the homomorphism $\delta \circ \rho$, where ρ is the homomorphism in (2.3).

Let

$$E_P(\mathbb{P}) := E_P\big(\mathrm{PSL}(n, k)\big)\big(\mathbb{P}_k^{n-1}\big) \tag{4.3}$$

denote the projective bundle over G/P associated to the $\mathrm{PSL}(n,k)$-bundle $E_P(\mathrm{PSL}(n,k))$ for the standard action of $\mathrm{PSL}(n,k)$ on $\mathbb{P}^{n-1}_k$.

The principal H-bundle $E_P(H)$ is strongly semistable (see second part of Lemma 2.3). Therefore, from [8, p. 288, Theorem 3.23] it follows that the associated projective bundle $E_P(\mathbb{P})$ defined in (4.3) is semistable (if the characteristic of k is zero, then it directly follows from [8, p. 285, Theorem 3.18]).

The left translation action of G on itself is an action of G on the principal P-bundle E_P that lifts the left translation action of G on G/P (recall that the total space of E_P is G). This action of G on E_P induces an action of G on any fiber bundle over G/P associated to E_P; the induced action on such an associated bundle lifts the left translation action of G on G/P.

Let $\mathbb{P}_1$ be the socle of the semistable projective bundle $E_P(\mathbb{P})$ defined in (4.3) (see Definition 3.2). From the uniqueness of the socle it follows immediately that $\mathbb{P}_1$ is left invariant by the action of G on the total space of $E_P(\mathbb{P})$.

Therefore, the action of the isotropy subgroup P (the isotropy subgroup at $eP \in G/P$ for the action of G on G/P) on the fiber $E_P(\mathbb{P})_{eP}$ leaves the subspace

$$(\mathbb{P}_1)_{eP} \subset E_P(\mathbb{P})_{eP}$$

invariant. Recall that the homomorphism ρ in (2.3) has the property that $\rho(P)$ is not contained in any proper parabolic subgroup of H. From this property of ρ it follows immediately that for the action of P on $\mathbb{P}^{n-1}_k$ defined by $\delta \circ \rho$ (the homomorphism δ is defined in (4.2)) there is no proper linear subspace of $\mathbb{P}^{n-1}_k$ that is left invariant. Therefore, we conclude that $(\mathbb{P}_1)_{eP} = E_P(\mathbb{P})_{eP}$.

Consequently, we have $\mathbb{P}_1 = E_P(\mathbb{P})$. In other words, the projective bundle $E_P(\mathbb{P})$ is polystable. Therefore, the $\mathrm{PSL}(n,k)$-bundle $E_P(\mathrm{PSL}(n,k))$ over G/P is polystable.

We will next prove that $E_P(\mathrm{PSL}(n,k))$ is stable. For that it suffices to show that

$$H^0\big(G/P, \mathrm{ad}\big(E_P\big(\mathrm{PSL}(n,k)\big)\big)\big) = 0 \tag{4.4}$$

(see [4, p. 787, Claim]), where $\mathrm{ad}(E_P(\mathrm{PSL}(n,k)))$ is the adjoint bundle.

Let

$$\mathcal{W} \subseteq \mathrm{ad}\big(E_P\big(\mathrm{PSL}(n,k)\big)\big) \tag{4.5}$$

be the coherent subsheaf generated by the space of all global sections of the vector bundle $\mathrm{ad}(E_P(\mathrm{PSL}(n,k)))$. So the coherent sheaf $\mathcal{W}$ is globally generated.

The left translation action of the group G on E_P induces an action of G on the adjoint vector bundle $\mathrm{ad}(E_P(\mathrm{PSL}(n,k)))$. This action of G on $\mathrm{ad}(E_P(\mathrm{PSL}(n,k)))$ evidently leaves invariant the subsheaf $\mathcal{W}$ constructed in (4.5). From this it follows immediately that $\mathcal{W}$ is a subbundle of the vector bundle $\mathrm{ad}(E_P(\mathrm{PSL}(n,k)))$. More precisely, $\mathcal{W}$ is the vector bundle associated to the principal P-bundle E_P for the P-module $\mathcal{W}_{eP}$, where $\mathcal{W}_{eP}$ is the fiber of $\mathcal{W}$ over the point $eP \in G/P$. Note that the condition that G leaves invariant $\mathcal{W}$ implies that the subspace

$$\mathcal{W}_{eP} \subset \mathrm{ad}\big(E_P\big(\mathrm{PSL}(n,k)\big)\big)_{eP} \tag{4.6}$$

is left invariant by the action of the isotropy subgroup P at $eP \subset G$ (for the action of G on G/P); hence the fiber $\mathcal{W}_{eP}$ is a submodule of the P-module $\mathrm{PSL}(n, k)$. The module structure on $\mathrm{PSL}(n, k)$ is defined by $\delta \circ \rho$, where δ and ρ are as in (4.2) and (2.3) respectively.

Let

$$0 = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_\ell := \mathcal{W}_{eP} \tag{4.7}$$

be a filtration of left P-modules such that each successive quotient $\mathcal{U}_i/\mathcal{U}_{i-1}$, $i \in [1, \ell]$, is an irreducible P-module.

We will now need a couple of lemmata.

Lemma 4.2. *To prove* (4.4) *it is enough to show that the P-module $\mathcal{U}_i/\mathcal{U}_{i-1}$ in* (4.7) *is a trivial P-module for each $i \in [1, \ell]$.*

Proof. To prove the lemma we first recall that the homomorphism ρ in (2.3) factors through the homomorphism $\bar{\rho}$ in (2.11). Since $L(P)$ (the domain of $\bar{\rho}$) is reductive, there is no nontrivial homomorphism from $L(P)$ to a unipotent algebraic group.

Assume that $\mathcal{U}_i/\mathcal{U}_{i-1}$ is a trivial P-module for each $1 \leqslant i \leqslant \ell$. Consequently, the action of P on $\mathcal{W}_{eP}$ factors through a unipotent group. In view of the above remark it follows that $\mathcal{W}_{eP}$ is a trivial P-module.

Since the homomorphism ρ in (2.3) satisfies the condition that the image $\rho(P)$ is not contained in any proper parabolic subgroup of H, we conclude that the homomorphism $\delta \circ \rho$ has the property that adjoint action of P on the Lie algebra $\mathrm{PSL}(n, k)$ defined by $\delta \circ \rho$ does not have any nonzero invariants (the homomorphism δ is defined in (4.2)). This immediately implies that $\mathcal{W}_{eP} = 0$ (recall that $\mathcal{W}_{eP}$ is a trivial submodule of the P-module $\mathrm{PSL}(n, k)$). Therefore, we have $\mathcal{W} = 0$. From this it follows that (4.4) is valid. This completes the proof of the lemma. □

Lemma 4.3. *For each $i \in [1, \ell]$, the action of P on $\mathcal{U}_i/\mathcal{U}_{i-1}$ extends to an action of G on $\mathcal{U}_i/\mathcal{U}_{i-1}$.*

Proof. To prove this lemma, let $E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ be the vector bundle over G/P associated to the principal P-bundle E_P for the P-module $\mathcal{U}_\ell/\mathcal{U}_{i-1}$ (see (4.7)). We will first prove that $E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a trivial vector bundle.

Consider the adjoint action of P on the Lie algebra $\mathrm{PSL}(n, k)$ defined by $\delta \circ \rho$, where δ and ρ are as in (4.2) and (2.3) respectively. Since $\mathcal{U}_\ell/\mathcal{U}_{i-1}$, $i \in [1, \ell]$, is a subquotient of the P-module $\mathrm{PSL}(n, k)$ (see (4.6) and (4.7)), it can be shown that $\bigwedge^{\mathrm{top}}(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a trivial P-module of dimension one. To prove this first recall that the action of P on $\mathrm{PSL}(n, k)$ is defined by $\delta \circ \rho$, and the homomorphism ρ factors through $\bar{\rho}$ from $L(P)$; see (2.11). From (2.12) we know that the homomorphism $\bar{\rho}$ in (2.11) takes the connected component of the reduced center of $L(P)$ to $Z_0(H)$. Since H is either $\mathrm{PSL}(n, k)$ or $\mathrm{PSO}(n, k)$ or $\mathrm{PSp}(2n, k)$, the center $Z_0(H)$ is trivial. Therefore, the action of $L(P)$ on $\bigwedge^{\mathrm{top}}(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ factors through $L(P)/Z_0(L(P))$. Since $L(P)/Z_0(L(P))$ does not admit any nontrivial character we conclude that

$\bigwedge^{\mathrm{top}}(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a trivial $L(P)$-module, and hence $\bigwedge^{\mathrm{top}}(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a trivial P-module. Consequently, the line bundle $\bigwedge^{\mathrm{top}} E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$, which is associated to the principal P-bundle E_P for the character $\bigwedge^{\mathrm{top}}(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ of P, is a trivial line bundle over G/P.

Since $\mathcal{W}$ is globally generated and $E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a quotient bundle of $\mathcal{W}$, the vector bundle $E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is also globally generated.

A globally generated vector bundle E of rank n over G/P, such that $\bigwedge^n E$ is a trivial line bundle, is isomorphic to the trivial vector bundle of rank n [3, p. 137, Proposition 2.3]. Therefore, $E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1})$ is a trivial vector bundle, where $i \in [1, \ell]$.

For any $i \in [1, \ell]$, let $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})$ be the vector bundle over G/P associated to the principal P-bundle E_P for the P-module $\mathcal{U}_i/\mathcal{U}_{i-1}$. From the filtration (4.7) of P-modules we have the following exact sequence of P-modules:

$$0 \to \mathcal{U}_i/\mathcal{U}_{i-1} \to \mathcal{U}_\ell/\mathcal{U}_{i-1} \to \mathcal{U}_\ell/\mathcal{U}_i \to 0,$$

where $i \in [1, \ell]$. Therefore, the above defined vector bundle $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})$ fits in an exact sequence of vector bundles

$$0 \to E_P(\mathcal{U}_i/\mathcal{U}_{i-1}) \to E_P(\mathcal{U}_\ell/\mathcal{U}_{i-1}) \to E_P(\mathcal{U}_\ell/\mathcal{U}_i) \to 0$$

over G/P. Consider the dual of the above exact sequence of vector bundles:

$$0 \to E_P(\mathcal{U}_\ell/\mathcal{U}_i)^* \to E_P(\mathcal{U}_i/\mathcal{U}_{i-1})^* \to E_P(\mathcal{U}_i/\mathcal{U}_{i-1})^* \to 0. \tag{4.8}$$

Each $E_P(\mathcal{U}_\ell/\mathcal{U}_{j-1})^*$, $j \in [1, \ell]$, is a trivial vector bundle as $E_P(\mathcal{U}_\ell/\mathcal{U}_{j-1})$ is trivial. Therefore, from (4.8) it follows that $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})^*$ is a globally generated vector bundle with the top exterior product $\bigwedge^{\mathrm{top}} E_P(\mathcal{U}_i/\mathcal{U}_{i-1})^*$ a trivial line bundle. Consequently, $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})^*$ is a trivial vector bundle [3, p. 137, Proposition 2.3]. Hence $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})$ is a trivial vector bundle for each $i \in [1, \ell]$.

The action of G on the vector bundle $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})$ induces an action of G on the vector space

$$V_i := H^0\big(G/P, E_P(\mathcal{U}_i/\mathcal{U}_{i-1})\big). \tag{4.9}$$

Consider the evaluation map

$$\varphi : (G/P) \times V_i \to E_P(\mathcal{U}_i/\mathcal{U}_{i-1}),$$

where $(G/P) \times V_i$ is the trivial vector bundle over G/P with fiber V_i (defined in (4.9)).

The homomorphism φ commutes with the actions of G, and furthermore, φ is an isomorphism as $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})$ is a trivial vector bundle. Therefore, the action of G on V_i is an extension of the action of P on $V_i = E_P(\mathcal{U}_i/\mathcal{U}_{i-1})_{eP} = \mathcal{U}_i/\mathcal{U}_{i-1}$; note that since

$$E_P(\mathcal{U}_i/\mathcal{U}_{i-1}) = \big(G \times (\mathcal{U}_i/\mathcal{U}_{i-1})\big)/P,$$

sending any $v \in \mathcal{U}_i/\mathcal{U}_{i-1}$ to the image in $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})_{eP}$ of $(e, v) \in G \times (\mathcal{U}_i/\mathcal{U}_{i-1})$ we get an identification of the fiber $E_P(\mathcal{U}_i/\mathcal{U}_{i-1})_{eP}$ with $\mathcal{U}_i/\mathcal{U}_{i-1}$. This identification clearly commutes with the actions of P. This completes the proof of the lemma. □

Continuing with the proof of the theorem, the condition that $P \subset G$ is a parabolic subgroup without any simple factor implies that if V is a G-module of dimension at least two, then the action P on V preserves a nontrivial filtration of V. Consider the G-module $\mathcal{U}_i/\mathcal{U}_{i-1}$ (see Lemma 4.3). In view of the above observation, the given condition in (4.7) that $\mathcal{U}_i/\mathcal{U}_{i-1}$ as a P-module is irreducible implies that

(4.10) $\quad \dim \mathcal{U}_i/\mathcal{U}_{i-1} = 1.$

On the other hand, the action of P on the P-module $\mathrm{PSL}(n, k)$, of which $\mathcal{U}_i/\mathcal{U}_{i-1}$ is a subquotient, is defined by $\delta \circ \rho$, and ρ factors through the homomorphism $\bar{\rho}$, defined in (2.11), from $L(P)$. Also, from (2.12) we know that $\bar{\rho}(Z_0(L(P))) \subset Z_0(H)$, and we are given that $Z_0(H)$ is the trivial group (recall that H is either $\mathrm{PSL}(n, k)$ or $\mathrm{PSO}(n, k)$ or $\mathrm{PSp}(2n, k)$). Consequently, the action of P on $\mathcal{U}_i/\mathcal{U}_{i-1}$ factors through $L(P)/Z_0(L(P))$. Since $L(P)/Z_0(L(P))$ does not admit any nontrivial character, from (4.10) it follows immediately that $\mathcal{U}_i/\mathcal{U}_{i-1}$ is a trivial P-module. Now from Lemma 4.2 we conclude that (4.4) is valid. We noted earlier that (4.4) implies that the principal $\mathrm{PSL}(n, k)$-bundle $E_P(\mathrm{PSL}(n, k))$ over G/P is stable. Therefore, we have proved that the $\mathrm{PSL}(n, k)$-bundle $E_P(\mathrm{PSL}(n, k))$ is stable.

Since the homomorphism δ in (4.2) is an embedding of H, the above assertion that $E_P(\mathrm{PSL}(n, k))$ is stable implies that the principal H-bundle $E_P(H)$ is stable.

Let $F_H : H \to H$ be the Frobenius morphism of H; if the characteristic of the field k is zero, then we take F_H to be the identity morphism of H. It is easy to see that the composition $F_H \circ \rho$ also satisfies the condition that the image $F_H \circ \rho(P)$ is not contained in any proper parabolic subgroup of H, where ρ is the homomorphism in (2.3). As we noted earlier, the pulled back principal H-bundle $F^* E_P(H)$ over G/P is identified with the principal H-bundle obtained by extending the structure group of the principal P-bundle E_P using the homomorphism $F_H \circ \rho$, where F is the map in (2.6).

Therefore, replacing ρ by $F_H \circ \rho$ in the above argument proving that $E_P(H)$ is stable, and iterating it, we conclude that the H-bundle $E_P(H)$ is strongly stable. This completes the proof of Theorem 4.1. □

ACKNOWLEDGEMENT

The author is very grateful to the referee for providing helpful comments to improve the manuscript.

REFERENCES

[1] Anchouche B., Biswas I. – Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold, Amer. J. Math. **123** (2001) 207–228.

[2] Azad H., Biswas I. – On the principal bundles over a flag manifold, J. Lie Theory **14** (2004) 569–581.
[3] Biswas I. – On the stability of homogeneous vector bundles, J. Math. Sci. Univ. Tokyo **11** (2004) 133–140.
[4] Biswas I., Gómez T.L. – Restriction theorems for principal bundles, Math. Ann. **327** (2003) 773–792.
[5] Humphreys J.E. – Linear Algebraic Groups, Grad. Texts in Math., vol. 21, Springer-Verlag, New York, Heidelberg, Berlin, 1987.
[6] Mehta V.B., Ramanathan A. – Restriction of stable sheaves and representations of the fundamental group, Invent. Math. **77** (1984) 163–172.
[7] Ramanan S. – Holomorphic vector bundles on homogeneous spaces, Topology **5** (1966) 159–177.
[8] Ramanan S., Ramanathan A. – Some remarks on the instability flag, Tôhoku Math. J. **36** (1984) 269–291.
[9] Ramanathan A. – Stable principal bundles on a compact Riemann surface, Math. Ann. **213** (1975) 129–152.
[10] Ramanathan A., Subramanian S. – Einstein–Hermitian connections on principal bundles and stability, J. Reine Angew. Math. **390** (1988) 21–31.
[11] Umemura H. – On a theorem of Ramanan, Nagoya Math. J. **69** (1978) 131–138.

(Received June 2004)