View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

PERGAMON

Applied Mathematics Letters 14 (2001) 495–498

Applied Mathematics Letters

www.elsevier.nl/locate/aml

The Zero-Crossing Phase-Lock Loop: Results from Discrete Dynamical Theory

W. MAY
Applied Signal Technology

J. J. LEADER
Santa Clara University

(Received February 1999; revised and accepted February 2000)

Abstract—The zero-crossing discrete phase-lock loop (ZC-DPLL) is a key component of many digital receivers. Yet prior analyses have overlooked the impact of its nonlinearities. Dynamical systems theory immediately provides a more complete picture of ZC-DPLL operation. We also find that the ZC-DPLL displays unusual features derived from its odd symmetry and bimodality. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Zero-crossing discrete phase-lock loop, ZC-DPLL, Phase-lock loop, Maps of the interval.

1. INTRODUCTION

We describe some aspects of the discrete map

$$\psi_n = \psi_{n-1} - k \sin \psi_{n-1},$$

where $k \in [0, 4.6033]$ (see Figure 1). For this range of k, ψ maps the interval $(-\pi, \pi)$ into itself. ψ derives from an important component of digital communication receivers: it represents the response of a first order zero-crossing discrete phase-lock loop (ZC-DPLL) to an unmodulated carrier with a phase offset. The behavior of ψ with increasing k shows how the ZC-DPLL behaves with increasing signal strength.

The ZC-DPLL has received a great deal of attention in the engineering literature, including [1–4] (in fact, in another context, the ZC-DPLL map is the transcendental term in Kepler's equation). Prior work has focused on the conditions for convergence of the ZC-DPLL to the fixed point at 0, known as phase-lock. There are also observations that period 2 and period 4 orbits exist, and that a double period 2 orbit exists. Osborne [3] conjectures "that all orbits of even period exist".

Based on dynamical systems theory (see [5]), we expect to find that the prior results form a very incomplete description of ψ's behavior. For example, we easily find that orbits of all periods exist, showing that Osborne's conjecture is correct, but incomplete.

0893-9659/01/$ - see front matter © 2001 Elsevier Science Ltd. All rights reserved. Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
PII: S0893-9659(00)00183-X

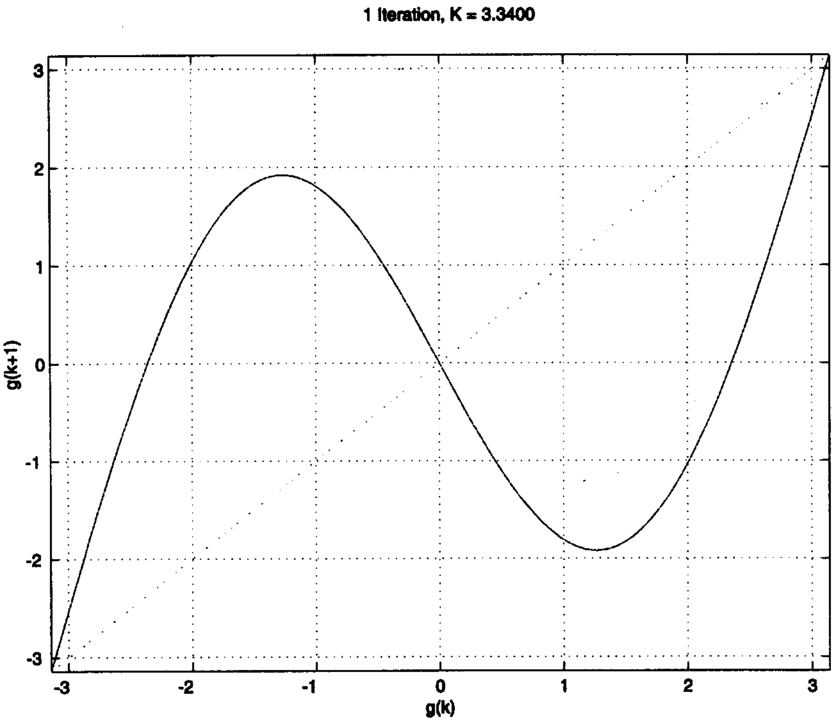

Figure 1. Function representing the ZC-DPLL.

2. TWIN ORBITS

ψ is a smooth map, an odd function, and it is bimodal. It is also easily seen that its Schwarzian derivative, defined as (see [5])

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is negative for $k > 1$. These characteristics drive ψ's interesting behavior. The discussion of Theorem 11.4 in [5] derives the relationship between the negative Schwarzian derivative and critical points of a smooth map in great detail. One result, for example, is that "periodic points with bounded stable sets must attract a critical point" (a negative Schwarzian derivative is assumed). From this we conclude that ψ can have two attracting periodic orbits for a given value of k, based on its bimodality and its negative Schwarzian derivative.

The double attracting orbits are evident in ψ's bifurcation diagram (see Figure 2). The black and gray orbits derive from two different initial conditions ($\pm.01$). We see the presence of two complete sets of period doubling cascades, beginning when $k = \pi$, evolving in tandem. The two sets of orbits are negatives of one another.

DEFINITION. *The twin of an orbit is formed by negating each point in the original orbit.*

LEMMA 1. *Given a periodic orbit, $\psi_0, \psi_1, \dots, \psi_{n-1}$, the orbit's twin is also a periodic orbit with the same period.*

PROOF. The twin orbit behavior follows immediately from ψ's odd symmetry. ∎

The bifurcation at $k = \pi$, where the double orbit behavior begins, is nonhyperbolic ($\psi' = 1$), suggesting a saddle node bifurcation. The second derivative is zero, showing that the bifurcation is an inflection point of ψ. This is a very nongeneric bifurcation.

As seen in the bifurcation diagram, all the orbits for $\pi < k < 3.5315$ are asymmetric. Hence, by Lemma 1 there are two distinct periodic orbits for each such value of k.

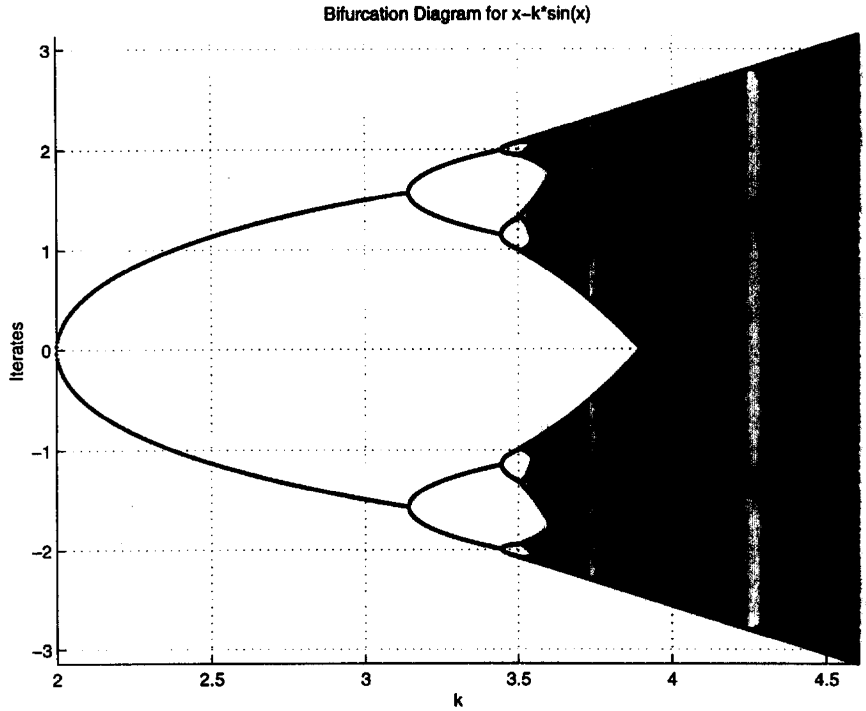

Figure 2. Bifurcation diagram for the ZC-DPLL.

3. BOUNDS FOR PERIODIC ORBITS

DEFINITION. *The high water mark is the farthest point from zero reached by a periodic orbit. The low water mark is the closest point to zero reached in a periodic orbit.*

It is easy to see (by graphical analysis) that the high water mark is bounded by the images of the two critical points. Points outside these bounds have backward orbits (preimages) that are single valued, and converge monotonically to the end points of the interval, $\pm\pi$. Therefore, such points cannot be on a periodic orbit.

The images of the critical points are $\pm(\sqrt{k^2-1} - \arccos 1/k)$. For high period orbits (periods higher than 4) the bound is quite close.

Once again, by graphical analysis we can show that for $k < 3.9014\ldots$ the low water mark is the image of the high water mark (i.e., the second iterate of the critical point). For k in this range the low water mark has the opposite sign of the high water mark.

The proof, which we will not present in detail, is based on the observation that a low water mark value that is closer to 0 than the image of the high water mark leads to a contradiction. Such a value can have several preimages, but all are either larger than the high water mark (so they cannot be on a periodic orbit), or are smaller than the low water mark (contradicting the assumption that our value is the low water mark).

For $k > 3.9014$ the image of the high water mark will have the same sign as the high water mark. It is then possible for the low water mark to have a preimage that is smaller than the high water mark. In this case, the bound on the low water mark is zero.

Using these bounds we can show that all periodic orbits for $k < 3.9014$ alternate in sign. For $k < 3.9014$ the high water marks are closer to zero than the zero crossings of ψ. Therefore, all positive values in a periodic orbit will map to negative values, and all negative values will map to positive values.

An obvious observation follows: since all periodic orbits alternate sign for $k < 3.9014$, only even period orbits are possible for this range of k. Osborne's conjecture is in fact correct for $k < 3.9014$.

4. SUMMARY

The results from dynamical systems theory provide a far more complete description of this system's behavior than previous analyses had suggested. Beyond this, we find that the special features (bimodality and odd symmetry) of ψ result in unique characteristics, such as twins for asymmetric periodic orbits.

The geometry of ψ also allows us to calculate precise bounds on the high and low water marks of periodic orbits for a large range of parameter values. This result can be used to characterize a wide range of undesirable states that have a cause that is intrinsic to the ZC-DPLL and the source signal, not noise or component failure.

REFERENCES

1. G.S. Gill and S.C. Gupta, First-order discrete phase-locked loop with applications to demodulation of angle-modulated carrier, *IEEE Trans. Comm.* **COM-20**, 454–462, (June 1972).
2. A. Weinberg and B. Liu, Discrete time analysis of nonuniform sampling first and second-order digital phase-locked loops, *IEEE Trans. Comm.* **COM-22**, 123–137, (February 1974).
3. H.C. Osborne, Theory of digital phase-locked loops for suppressed carrier recovery, Ph.D. Dissertation, University of Southern California, (February 1979).
4. F. Russo, Graphical analysis of a digital phase-locked loop, *IEEE Trans. Aerospace and Electronic Systems* **AES-15** (1), 88–94, (January 1979).
5. R.L. Devaney, *An Introduction to Chaotic Dynamical Systems, 2^{nd} Edition*, Addison-Wesley, Reading, MA, (1989).